Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/070204

A Novel Approach for Solving Time Fractional Nonlinear Fisher's Equation by Using Chebyshev Spectral Collocation Method

Allah Bakhsh Yazdani[1,*], *Morteza Mohammadnezhad Kiasari*[1] *and Hossein Jafari*[1,2]

[1] Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
[2] Department of Mathematical Sciences, University of South Africa, UNISA0003, South Africa

Received: 24 Oct. 2020, Revised: 11 Nov. 2020, Accepted: 11 Dec. 2020
Published online: 1 Apr. 2021

Abstract: In this paper, the Chebyshev spectral method is applied to solve the nonlinear Fisher fractional equation with initial boundary conditions. Here, the fractional derivative is considered in Caputo type. Then, using the Chebyshev spectral collocation method, the problem is transformed into an algebraic system. The results showed that this method is acceptable for numerical solution of the Fisher equation.

Keywords: Caputo derivative, Chebyshev spectral collocation method, time fractional nonlinear Fisher's equation.

1 Introduction

Spectral methods have an important significant role in approximate differential equations, which simplify makes it easy in treating many phenomena and models in physics, engineering, economic and many other fields. The most common distinguished feature for spectral methods is using them as a basis in forming polynomials or functions that are orthogonal with respect to the weight functions defined in bounded and unbounded domain [1–6]. Over the past century, fractional derivative equations been adopted in several areas such as physics, chemistry, engineering and even finance [7–12]. In addition to numerical solution, some new analytical methods, such as Adomian decomposition method [12, 13] homotopy perturbation method [14], Differential transform method(DTM) [15, 16], variational iteration method [17] and homotopy analysis method [11] exist. Spectral methods are a relatively accurate numerical method, which has recently gained considerable popularity [18–24, 26–28]. Now, we consider time fractional Fisher's equation

$$D_C^\alpha u - u_{xx} - u(u-1) = f, \qquad x \in [0,L], \qquad 0 \leqslant t \leqslant T, \tag{1}$$

$$u(x,0) = f_0(x), \qquad x \in [0,L], \tag{2}$$

$$u(0,t) = f_1(t), \qquad u(L,t) = f_2(t), \qquad 0 \leqslant t \leqslant T. \tag{3}$$

We use the floor function $\lfloor \alpha \rfloor$ to denote the largest integer less than or equal to α and the ceiling function $\lceil \alpha \rceil$ to denote the smallest integer greater than or equal to α. Here, $D_C^\alpha u$ which is the Caputo type fractional derivative of order $\alpha > 0$ is defined by

$$D_C^\alpha u(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} u^{(m)}(t)dt.$$

where $m = \lceil \alpha \rceil$.

2 Preliminaries

In this section, we review briefly the orthogonal Chebyshev polynomials and express some of their properties [19–22].

* Corresponding author e-mail: yazdani@umz.ac.ir

2.1 Chebyshev polynomials

We know well that Chebyshev's polynomial is defined in the interval $(-1,1)$. We introduce the shifted Chebyshev polynomial by presenting the following variable change $r=\frac{2t}{T}-1$, let

$$T_r(\frac{2t}{T}-1)=T_{T,r}(t), \qquad t\in(0,T).$$

Then, $T_{T,r}(t)$ can be obtained from the following recursive relation.

$$T_{T,r+1}(t)=2(\frac{2t}{T}-1)T_{T,r}(t)-T_{T,r-1}(t), \qquad r=1,2,..., \tag{4}$$

where $T_{T,0}(t)=1$ and $T_{T,1}(t)=\frac{2t}{T}-1$. Explicit form of the shifted Chebyshev polynomials $T_{T,r}(t)$ of degree r is

$$T_{T,r}(t)=r\sum_{k=0}^{r}(-1)^{r-k}\frac{(r+k-1)!2^{2k}}{(r-k)!(2k)!T^k}t^k, \tag{5}$$

where $T_{T,r}(0)=(-1)^r$ and $T_{T,r}(T)=1$. The orthogonality condition is

$$\int_0^T T_{T,j}(t)T_{T,k}(t)w_T(t)dt = \delta_{jk}\lambda_k, \tag{6}$$

where $w_T(x)=\frac{1}{\sqrt{Tx-x^2}}$ and $\lambda_k=\frac{b_k}{2}\pi$, with $b_0=2, b_i=1, i\geq 1$.

The approximation of u can be, as follows:

$$u_N(t)\simeq\sum_{i=0}^{N}c_iT_{T,i}(t). \tag{7}$$

2.2 Shifted Chebyshev-Gauss-quadratures

Let $\rho_0,\rho_1\cdots,\rho_N$ be $N+1$ Chebyshev-Gauss nodes in $[-1,1]$, and $\psi_i(t), i=0,\cdots,N$ be the corresponding Christoffel numbers dependent on these points. Let us assume t_i and ω_i, $i=0,1,...,N$ the nodes and Christoffel numbers of the (shifted) Chebyshev-Gauss-quadratures on the intervals $[0,T]$, then we have the following relations:

$$t_i = \frac{T}{2}(\rho_i+1), \tag{8}$$

$$\omega_i = \psi_i, \qquad 0\leq i\leq N. \tag{9}$$

We assume P_M is a set of polynomials of maximum degree M. According to Chebyshev-Gauss quadrature, we conclude for each $p\in P_{2M+1}$,

$$\int_0^T p(t)w_T(t)dt=\int_{-1}^{1}p(\frac{T}{2}(t+1))\frac{1}{\sqrt{1-t^2}}dt=\sum_{i=0}^{N}\psi_ip(\frac{T}{2}(\rho_i+1))=\sum_{i=0}^{N}\omega_ip(t_i). \tag{10}$$

You can refer to [1–3] for more information.

2.3 The fractional derivatives of shifted Chebyshev polynomials [18]

The fractional derivative of order α in the Caputo type for the Chebyshev polynomials is

$$D_C^\alpha T_{T,r}=0, \qquad r=0,1,\cdots,\lceil\alpha\rceil-1, \qquad \alpha\geq 0, \tag{11}$$

$$D_C^\alpha T_{T,r}=\sum_{s=0}^{\infty}\Omega_\alpha(r,s)T_{T,s}(t), \qquad r=\lceil\alpha\rceil,\lceil\alpha\rceil+1,..., \tag{12}$$

$$\Omega_\alpha(r,s)=\sum_{k=\lceil\alpha\rceil}^{r}\frac{(-1)^{r-k}2r(r+k-1)!\Gamma(k-\alpha+\frac{1}{2})}{b_sT^\alpha\Gamma(k+\frac{1}{2})(r-k)!\Gamma(k-\alpha-s+1)\Gamma(k+s-\alpha+1)}.$$

3 Description of the method

We consider the numerical approximation, as follows:

$$u_N(x,t) = \sum_{r=0}^{N} a_r(x) T_{T,r}(t), \tag{13}$$

$$a_r(x) = \frac{1}{\lambda_r} \sum_{s=0}^{N} T_{T,r}(t_s) u_N(x,t_s) \omega_s, \qquad r = 0,1,\cdots,N. \tag{14}$$

where $a_r(x)$ is unmown function and t_s is the Chebyshev-Gauss nodes. In the first step, a set of appropriate points must be selected. We use the Chebyshev-Gauss nodes (8). Finally, we approximate the unknown solution with respect to these points. Putting $a_r(x)$ in the equation of (13), we have

$$\begin{aligned} u_N(x,t) &= \sum_{r=0}^{N} \left(\frac{1}{\lambda_r} \sum_{s=0}^{N} T_{T,r}(t_s) u_N(x,t_s) \omega_s \right) T_{T,r}(t) \\ &= \sum_{r=0}^{N} \left(\sum_{s=0}^{N} \frac{1}{\lambda_r} T_{T,r}(t_s) u_N(x,t_s) \omega_s \right) T_{T,r}(t) \\ &= \sum_{s=0}^{N} \left(\sum_{r=0}^{N} \frac{1}{\lambda_r} T_{T,r}(t_s) T_{T,r}(t) \omega_s \right) u_N(x,t_s). \end{aligned}$$

Then, we have

$$u_N(x,t) = \sum_{s=0}^{N} \left(\sum_{r=0}^{N} \frac{1}{\lambda_r} T_{T,r}(t_s) T_{T,r}(t) \omega_s \right) u_N(x,t_s). \tag{15}$$

We take τ as the step-size in variable x, $x_h = h\tau, h = 0,1,\cdots,m; m = [\frac{L}{\tau}], u^h = u(x_h,t)$.
If we rewrite equation (1) in terms of the discretization of the posterior, we have

$$D_C^{\alpha} u^h - \frac{u^h - 2u^{h-1} + u^{h-2}}{\tau^2} + u^h - (u^{h-1})^2 = f^{h-1}, \qquad h = 2,\cdots,m \tag{16}$$

If we apply Chebyshev spectral collocation method, we have

$$D_C^{\alpha} u^h(t_n) - \frac{u^h(t_n) - 2u^{h-1}(t_n) + u^{h-2}(t_n)}{\tau^2} + u^h(t_n) - (u^{h-1})^2(t_n) = f^{h-1}(t_n), \qquad n = 1,\cdots,N.$$

where

$$u^h(t_n) = u_N(x_h,t_n) = \sum_{j=0}^{N} \left(\sum_{i=0}^{N} \frac{1}{\lambda_i} T_{T,i}(t_j) T_{T,i}(t_n) \omega_j \right) u_N(x_h,t_j).$$

now we put

$$\begin{aligned} G_n^h &= (u^{h-1})^2(t_n) + f^{h-1}(t_n) + \frac{2u^{h-1}(t_n) - u^{h-2}(t_n)}{\tau^2}, \\ Q_{ns} &= \sum_{r=0}^{N} \frac{1}{\lambda_r} T_{T,r}(t_s) T_{T,r}(t_n), \\ M_{ns} &= \sum_{r=0}^{N} \frac{1}{\lambda_r} T_{T,r}(t_s) D_C^{\alpha} T_{T,r}(t_n), \\ \beta &= (\frac{1}{\tau^2} - 1). \end{aligned}$$

where the matrix elements of M_{ns} and Q_{ns} are easily computable with respect to (5), (12) and (11).

At the end, at each time step, we must solve the following algebraic system

$$\sum_{s=0}^{N} [M_{ns} - \beta Q_{ns}] \omega_s u_N(x_h,t_s) = G_n^h, \qquad n = 1,\cdots,N,$$

$$\sum_{s=0}^{N} \left(\sum_{r=0}^{N} \frac{(-1)^r}{\lambda_r} T_{T,r}(t_s) \omega_s \right) u_N(x_h,t_s) = f_0(x_h).$$

4 Numerical example

Example 1. We consider equation (1), as follows:

$$\alpha = 0.8, [0,L] = [0,1],$$
$$f(x,t) = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} xt(1-x) + t^2x^4 - 2t^2x^3 + t^2x^2 + tx^2 - tx + 2t,$$
$$u(x,0) = 0,\ f_1(t) = 0,\ f_2(t) = 0.$$

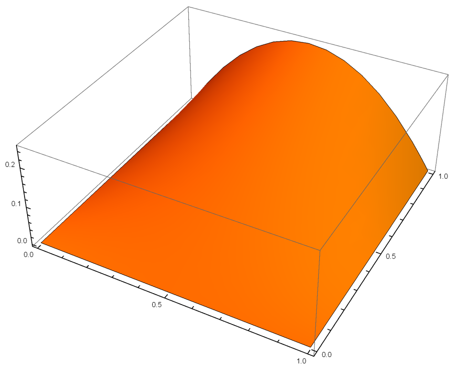

Fig. 1: Plots of exact solution of Example 1

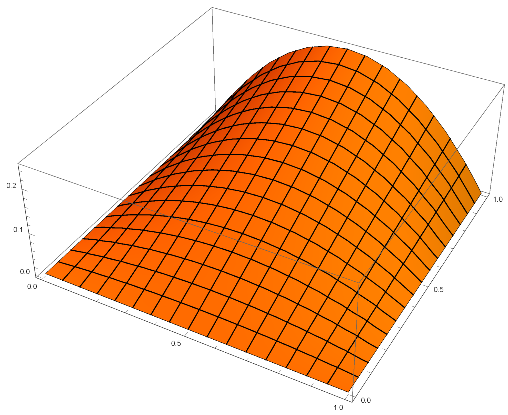

Fig. 2: Plots of our numerical solution of Example 1

The exact solution is

$$u(x,t) = tx(1-x).$$

We present the numerical solution for (1) with $N = 20$ in some values of x and $\tau = 0.1$ as shown in Fig.1, and Fig.2. From the comparison, an almost good approximation is observable.

Example 2. Consider the following Fisher equation with the initial condition

$$D_C^{\alpha} u = u_{xx} + 6u(1-u), \tag{17}$$
$$u(x,0) = \frac{1}{(1+e^x)^2}. \tag{18}$$

For the special case $\alpha = 1$, the solution will be, as follows:

$$u(x,t) = \frac{1}{(1+e^{x-5t})^2},$$

which is an exact solution of the classical Fisher equation([15]).

We present the numerical solution for (2) with $N = 20$, $\alpha = 0.98$ and $\tau = 0.1$ as shown in Fig.3, and Fig.4 From the comparison between our method and DTM([15]), an almost good approximation is observable.

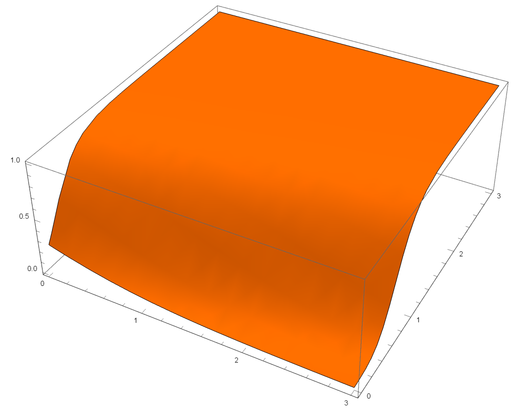

Fig. 3: Plots of exact solution of Example 2, $\alpha = 1$

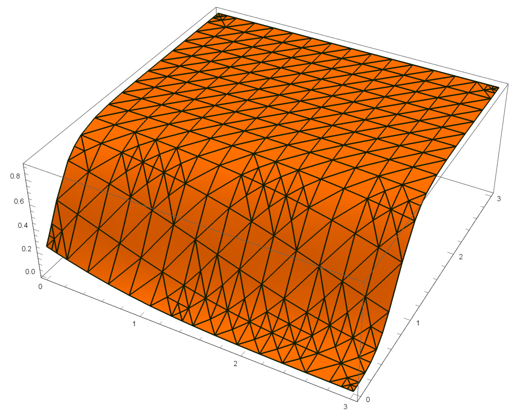

Fig. 4: Plots of our numerical solution of Example 2

5 Conclusion

In this paper, the Chebyshev spectral method has been applied to solve the nonlinear fractional order Fisher equation with initial boundary conditions. The numerical results showed the validity of our method. We emphasize that this technique is also applicable to other Jacobi polynomials. *Mathematica* has been used for computation.

References

[1] J. Shen, T. Tanga and L. Wang, *Spectral methods, algorithms, analysis and applications*, Springer Series in Computational Mathematics.
[2] J. S. Hesthaven, S. Gottlieb and D. Gottlieb, *Spectral methods for time-dependent problems*, Cambridge Monographs on Applied and Computational Mathematics.
[3] D. Gottlieb and S. A. Orszag, *Numerical analysis of spectral methods: theory and applications*, SIAM–CBMS, Philadelphia, 1977.
[4] C. Canuto, M.Y. Hussaini, A. Quarteroni and T. A. Zang, *Spectral methods in fluid dynamics*, Springer-Verlag, 1987.
[5] M. A. Ramadan, D. Baleanu and M. A. G. Nassar, Highly accurate numerical technique for population models via rational Chebyshev collocation method,*Mathematics* **7**(10) , Article Number: 913 (2019).
[6] R. Baltensperger and M. R. Trummer, Spectral differencing with a twist, *SIAM J. Sci. Comput*. **24**(5), 1465–1487 (2003).
[7] R. Hilfer, *Applications of fractional calculus in physics*, Singapore, World Scientific, 2000.
[8] J. A. T. Machado, V. Kiryakova and F. Mainardi, Recent history of fractional calculus, *Commun. Nonlin. Sci. Numer. Simul.* **16**(3), 1140–1153 (2011).
[9] K. B. Oldham and J. Spanier, *The fractional calculus*, New York, Academic Press, 1974.
[10] K. Diethelm, *The analysis of fractional differential equations*, Berlin, Springer-Verlag, 2010.
[11] Z. Odibat, S. Momani and H. Xu, A reliable algorithm of homotopy analysis method for solving nonlinear fractional differential equations, *Appl. Math. Model.* **34**, 593–600 (2010).
[12] C. Li and C. Taoa, On the fractional Adams method, *Comput. Math. Appl.* **58**, 1573–1588 (2009).
[13] S. S. Ray and R. K. Bera, Solution of an extraordinary differential equation by Adomian decomposition method, *J. Appl. Math.* **4**, 331–338 (2004).
[14] O. Abdulaziz, I. Hashim and S. Momani, Application of homotopy-perturbation method to fractional IVPs, *J. Comput. Appl. Math.* **216**, 574–584 (2008).
[15] M. Mmirzazadeh, A novel approach for solving fractional Fisher equation using differential transform method, *M. Pramana J. Phys.* **86**, 957-968 (2016).
[16] S. Momani and Z. Odibat, A novel method for nonlinear fractional partial differential equations: Combination of DTM and generalized Taylor's formula, *J. Comput. Appl. Math.* **220**, 85–95 (2008).
[17] S. Yanga, A. Xiao and H. Su, Convergence of the variational iteration method for solving multi-order fractional differential equations, *Comput. Math. Appl.* **60**, 2871–2879 (2010).
[18] E. H. Doha, A. H. Bhrawy and S. S. Ezz-Eldien, Efficient Chebyshev spectral methods for solving multi-term fractional orders differential equations, *Appl. Math. Model.* **35**(12), (2011) 5672-5662.
[19] L. Fox and I. B. Parker, *Chebyshev polynomials in numerical analysis*, Oxford U. Press, 1968.
[20] D. Funaro, *Polynomial approximation of differential equations*, Berlin, Springer-Verlag, 1992.
[21] W. Gautschi, *Orthogonal polynomials: computation and approximation*, New York, Oxford University Press, 2004.
[22] G. Szego, *Orthogonal polynomials*, Colloquium Publications, XXIII, American Mathematical Society, 1939.
[23] U. Farooq, H. Khan, D. Baleanu and M. Arif, Numerical solutions of fractional delay differential equations using Chebyshev wavelet method, *Comput. Appl. Math.* **38**(4), Article Number: UNSP 195 (2019).
[24] A. M. Mishra, D. Baleanu, F. Tchier and S. D. Purohit, Certain results comprising the weighted Chebyshev function using pathway fractional integrals,*Mathematics* **7**(10), Article Number: 896 (2019).
[25] M. Abdelhakem, H. Moussa, D. Baleanu and M. El-Kady, Shifted Chebyshev schemes for solving fractional optimal control problems, *J. Vibr. Contr.* **25**(15), 2143–2150 (2019).
[26] K. M. Saad, M. M. Khader, J. F. Gomez-Aguilar and D. Baleanu, Numerical solutions of the fractional Fisher's type equations with Atangana-Baleanu fractional derivative by using spectral collocation methods, *Chaos* **29**(2), Article Number: 023116 (2019).
[27] E. H. Doha, M. A. Abdelkawy, A. Z. M. Amin and D. Baleanu, Approximate solutions for solving nonlinear variable-order fractional Riccati differential equations, *Nonlin. Anal. Model. Contr.* **24**(2), 176–188 (2019).
[28] D. Baleanu, B. Shiri, H. M. Srivastava and M. Al Qurashi, A Chebyshev spectral method based on operational matrix for fractional differential equations involving non-singular Mittag-Leffler kernel, *Adv. Differ. Equ.* Article Number: 353 (2018).