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Numerics of Fractional Langevin Equation Driven by Fractional Brownian Motion Using Non-Singular Fractional Derivative

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Abstract: In this work, we presented the numerical investigation on the dynamics of fractional Langevin equation which is driven by a fractional Brownian motion and Caputo-Fabrizio fractional derivative operator were utilized. The order of fractional derivative was considered to be $\nu = 2 - 2H$, where $H \in (1/2, 1)$ is the Hurst's index. In the context of numerical schemes, we present different numerical approaches such as the discrete sequence of finite difference, to simplify the second-order ordinary derivative, while for the fractional derivative term, we presented the discrete approximation using simple quadrature formula. Additionally, for overdamped case (without inertial term), we used the Adams-Bashforth method corresponding to the Caputo-Fabrizio fractional derivative. The convergence and stability analysis of the obtained numerical solution were established in this study.

Keywords: Fractional Langevin equation, fractional Brownian motion, Caputo-Fabrizio fractional derivative, simple quadrature formula, Adams-Bashforth method, fluctuation-dissipation theorem.

1 Introduction, motivation and preliminaries

Application of fractional calculus has been widely studied over these past years [1–4]. However, dealing with more complicated systems using fractional differential equations in modeling real-world phenomena cannot always be handled using analytic method sometimes due to its complexity. To address these problems, many scientists and researchers relies on the numerical approximation methods [5–9].
In this work, we will study the dynamics of the fractional Langevin equation in Caputo-Fabrizio fractional derivative sense via a numerical method. The first stochastic differential equation was introduced by Paul Langevin, back in 1908, to provide a mathematical interpretation of Robert Brown's observation on the random motion of pollen grains and dust. Langevin's model can be written in mathematical form as

$$m\frac{d^2x(t)}{dt^2} = -\gamma\frac{dx(t)}{dt} + F(x) + \zeta(t), \tag{1}$$

where m is the particle's mass, γ is the friction coefficient, while $F(x)$ is the applied external force. The term $\zeta(t)$ represents the random force due to thermal fluctuation on the particle's interaction on the molecules of its environment. We can actually write it as $\zeta(t) = \frac{dB(t)}{dt}$, where $B(t)$ is the known classical Brownian motion. Equation (1) has been extensively applied on different systems these past years such as the dynamics of protein-protein encounter [10], photoelectrons counting [11], stock market analysis [12], subdiffusion within proteins [13], single-file diffusion [14], etc. However, in this study, we fixed our focus on the fractional Langevin equation of the form

$$m\frac{d^2x(t)}{dt^2} = -\gamma(1-\nu)\,{}_{CF}D_0^{\nu}x(t) + F(x) + \epsilon\zeta^H(t), \tag{2}$$

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where ${}_{CF}D_0^{\nu}$ is Caputo-Fabrizio fractional derivative operator [15]. $\zeta^H(t)$ is the known fractional Gaussian noise and ϵ is the noise noise strength, which satisfies the fluctuation-dissipation theorem [16], such that $\epsilon = \sqrt{\frac{k_B T \gamma \Gamma(1/2-H)\Gamma(1/2+H)}{(2H-1)\Gamma(2-2H)}}$ with $H \in (1/2, 1)$, k_B the Boltzmann's constant, and T the temperature. In contrast to classical Langevin equation (1), the fractional Langevin equation (2) can capture another feature due to the applied kernel by the fractional derivative, which can not be obtained in classical Langevin equation. Numerical studies have been conducted to investigate the dynamics of fractional Langevin equation [17, 18] where Caputo derivative was utilized. In contrast, this work has an extended discussion on fractional Langevin equation of motion where underdamped and overdamped system were explored together with the inclusion of linear force.
We organized this study as follows: In section 2, we presented the definition of Caputo-Fabrizio fractional derivative with some known properties, together with the introduction of fractional Brownian motion (FBM). In section 3, we solved the numerical solution of fractional Langevin equation using Caputo-Fabrizio derivative for the underdamped and overdamped case with the presence of external force $F(x) = -kx(t)$, where k is the force stiffness. Lastly, section 4 concludes this study.

2 Preliminaries

In this section, we give some basic definition of the recently introduced fractional derivative together with its some basic properties. In addition, we also review the basic definition of fractional Brownian motion and fractional Gaussian noise.

2.1 Caputo-Fabrizio fractional derivative

One of the most recently introduced fractional derivatives is the Caputo-Fabrizio fractional derivative [15] which encompasses a wide encouraging property compare to the previous version. This new fractional derivative was believed to be able to describe heterogeneities and structures at different scales, which can not be described by the classical derivatives. Over these past recent years, the application of this interesting fractional derivative has been studied in different areas [19–25]. In this study, we investigate the stability, and convergence of the Caputo-Fabrizio fractional derivative on fractional Langevin equation, numerically. We present the definition of Caputo-Fabrizio fractional derivative in the subsequent definitions.

Definition 1.*Recall that the known Caputo fractional time derivative of order ν is given by*

$$D_t^{\nu} x(t) = \frac{1}{\Gamma(1-\nu)} \int_a^t \frac{x'(s)}{(t-s)^{\nu}} ds \tag{3}$$

with $\nu \in [0, 1]$ and $a \in (-\infty, 0)$.

Definition 2.*Let $f \in H^1(a, b), b > a, \nu \in (0, 1)$ then the Caputo-Fabrizio fractional derivative is defined as*

$$^{CF}D_t^{\nu} f(t) = \frac{C(\nu)}{1-\nu} \int_a^t f'(s) \exp\left\{-\nu \frac{t-s}{1-\nu}\right\} ds, \tag{4}$$

where $C(\nu)$ is a normalization function such that $C(0) = C(1) = 1$. If the function that does not belong to $H^1(a, b)$, then the Caputo-Fabrizio fractional derivative can be reformulated as

$$^{CF}D_t^{\nu} f(t) = \frac{\nu C(\nu)}{1-\nu} \int_a^t (f(t) - f(s)) \exp\left\{-\nu \frac{t-s}{1-\nu}\right\} ds. \tag{5}$$

However, Losada and Neito [26] rewrite the definition of Caputo-Fabrizio fractional derivative in the form

$$^{CF}D_t^{\nu} f(t) = \frac{(2-\nu)C(\nu)}{2(1-\nu)} \int_a^t f'(s) \exp\left\{-\nu \frac{t-s}{1-\nu}\right\} ds \tag{6}$$

to solve the normalization function $C(\nu)$, they applied the fundamental theorem of calculus vi Laplace transformation to yield

$$C(\nu) = \frac{2}{2-\nu}, \quad 0 \le \nu \le 1.$$

Definition 3. *The Caputo-Fabrizio fractional derivative of order ν of a function f is given by*

$$ {}^{CF}D_t^{\nu} f(t) = \frac{1}{1-\nu}\int_a^t f'(s)\exp\left\{-\nu\frac{t-s}{1-\nu}\right\} ds. \tag{7}$$

Property 1. If the Caputo-Fabrizio fractional derivative is of order $(\nu+n)$ such that $n \geq 1$, then we have

$$ {}^{CF}D_t^{(\nu+n)} f(t) = {}^{CF}D_t^{\nu}\left(f^n(t)\right).$$

2.2 Brief review of fractional Brownian motion and fractional Gaussian noise

In the classical Brownian motion theory, it is an established fact that $\xi(t) = \frac{dB(t)}{dt}$, where $\xi(t)$ is the Gaussian white noise. The idea of fractional Brownian motion within Hilbert space was first studied by A. N. Kolmogorov [27]. However, Mandelbort and Van Ness [28] developed the fractional Brownian motion in terms of a stochastic integral representation given by

$$ B^H(t) = \frac{1}{\Gamma(H+\frac{1}{2})}\left(\int_{-\infty}^0 \left\{(t-s)^{H-1/2} - (-s)^{H-1/2}\right\} dB(s) + \int_0^t (t-s)^{H-1/2} dB(s)\right), \tag{8}$$

where $\Gamma(\cdot)$ is the known Gamma function. In the work of A.W. Lo [29] they introduced an alternative representation of the fractional Brownian motion which is the basis of their model as

$$ \tilde{B}^H(t) = \frac{1}{\Gamma(H+\frac{1}{2})}\int_0^t (t-s)^{H-1/2} dB(s). \tag{9}$$

This form of the fractional Brownian motion is *truncated* in the sense that the limit of integration from negative infinity to zero in equation (8) is truncated.

Definition 4. *A Gaussian process $B^H(t), t > 0$ with a continuous path is a standard fractional Brownian motion with Hurst parameter $H \in (0,1)$, if it satisfies:*

–$\langle B^H(t)\rangle$=0.
–$\langle B^H(t)B^H(s)\rangle$=$\frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$, $\forall s,t \in R^+$.

Property 2. The fractional Brownian has the following properties:

1. If $H = 1/2$, $W_t^{1/2}$ becomes a Brownian motion.
2. $W_0^H = 0$, $\langle W_t^H\rangle = 0$, $\langle (W_t^H)^2\rangle = t^{2H}, t \geq 0$,
3. It has a stationary increments, $\langle W_t^H - W_s^H\rangle^2 = |t-s|^{2H}$,
4. It has a self-similarity property $\left(W_{\nu t}^H, t \geq 0\right) = \left(\nu^H W_t^H\right)$,
5. W_t^H has a continuous trajectories.

From the given definitions and properties above, we deduce that fractional Brownian motion is a continuous Gaussian process, self-similar, and has a stationary increments.
The fractional Gaussian noise is similar to the white noise such as

$$ \zeta^H(t) = \frac{dB^H(t)}{dt}. \tag{10}$$

In addition, fractional Gaussian noise has zero mean and having a covariance of the form

$$ \langle \zeta^H(t)\zeta^H(s)\rangle = |t-s|^{2H-2}. \tag{11}$$

From equation (11), we can see that the values $H \in (1/2,1)$ corresponds to long-term positive autocorrelation or a series of high valued dependency. On the other hand, for the values $H \in (0,1/2)$ corresponds to a time series with alternating or switching between high values and low values or negatively correlated. In addition, if $H = 1/2$ corresponds to uncorrelated series because of the absolute values of the autocorrelation decay algebraically to zero.

3 Numerical solution and algorithm

In this study, we use the recent numerical algorithm to solve for the fractional Langevin equation driven by fractional Brownian motion in Caputo-Fabrizio fractional derivative sense, which is the main contribution of this work. Moreover, two cases of fractional Langevin equation (underdamped and overdamped) were considered in this study with the presence of a linear external force, $F(x) = -kx(t)$. Furthermore, the convergence and stability of the numerical solution were also investigated to show the existence of the numerical solution.

3.1 Underdamped fractional Langevin equation driven by fractional Brownian motion

In equation (2), we have

$$m\frac{d^2x(t)}{dt^2} = -\gamma(1-\nu)\,_{CF}D_0^{\nu}x(t) - kx(t) + \epsilon\zeta^H(t) \tag{12}$$

observe that if $k = 0$, we have the free motion. Now, using finite difference formula for the second-order derivative on the left side of equation (12) at $0 \le t \le T$ with grid points $[0,T]$ such that $t_n = nh$, $n = 1,2,3,...,TN$, N is the grid size; we have

$$\frac{d^2x(t)}{dt^2} \approx \frac{x'(t_n) - x'(0)}{h}, \tag{13}$$

where h is the time increment. Substituting (13) to (12), we have

$$x'(t_n) - x'(0) = -\frac{\gamma}{m}h(2H-1)\,_{CF}D_0^{2-2H}x(t_n) - \frac{k}{m}hx(t_n) + \frac{\epsilon}{m}\left(B^H(t_n) - B^H(0)\right). \tag{14}$$

The first term of the right-hand side of equation (14) has the Caputo-Fabrizio fractional derivative. To approximate this, we can use the simple quadrature formula at point t_n given by

$$_{CF}D_0^{2-2H}x(t_n) = \frac{1}{2H-1}\int_0^{t_n} x'(\tau)\exp\left(-\frac{2-2H}{2H-1}(t_n-\tau)\right)d\tau \tag{15}$$

which can also be modified by first order approximation

$$_{CF}D_0^{2-2H}x(t_j) = \frac{1}{2H-1}\sum_{j=1}^{n}\int_{(j-1)h}^{jh}\left(\frac{x_{j+1}-x_j}{h} + O(h)\right)\exp\left(-\frac{2-2H}{2H-1}(t_j-\tau)\right)d\tau. \tag{16}$$

Now, integrating (16), we have the expression

$$_{CF}D_0^{2-2H}x(t_j) = \frac{1}{2-2H}\sum_{j=1}^{n}\left(\frac{x_{j+1}-x_j}{h} + O(h)\right)D_j, \tag{17}$$

where

$$D_j = \exp\left(-\frac{2-2H}{2H-1}(n-j+1)\right) - \exp\left(-\frac{2-2H}{2H-1}(n-j).\right) \tag{18}$$

We can finally obtain the numerical approximation for the fractional derivative term of equation (12) as

$$_{CF}D_0^{2-2H}x(t_j) = \frac{1}{2-2H}\sum_{j=1}^{n}\left(\frac{x_{j+1}-x_j}{h}\right)D_j + \frac{1}{2-2H}\sum_{j=1}^{n}D_jO(h). \tag{19}$$

Now, to obtain the numerical solution of equation (2), we apply again the finite difference on the left-hand side of equation (12) to get

$$\begin{aligned} x_n =& v_o h - \frac{\gamma}{m}(2H-1)h^2\left(\frac{1}{2-2H}\sum_{j=1}^{n}\left(\frac{x_{j+1}-x_j}{h}\right)D_j + \frac{1}{2-2H}\sum_{j=1}^{n}D_jO(h)\right) \\ &+ \left(1 - \frac{k}{m}h^2\right)x_{n-1} + \frac{\epsilon}{m}h\left(B^H(t_n)\right), \end{aligned} \tag{20}$$

where $v_o = x'(0)$ is interpreted as the initial velocity. The following theorem presents the convergence of the numerical result (20).

Theorem 1. *Let $x(t) = x(t_n)$ be a continuous function on $L^2[a,b]$ such that $t_n = nh$ and let the order of fractional derivative of Caputo-Fabrizio derivative be $\nu = 2-2H$, $H \in \left(\frac{1}{2}, 1\right)$. Then, we have the truncation term*

$$O(h^4) = h^3 nO(h). \tag{21}$$

Proof. From equations (17) and (20), we have

$$\begin{aligned} R(H,h) &= \frac{2H-1}{2-2H} h^2 \sum_{j=1}^{n} D_j O(h) \\ &= \frac{2H-1}{2-2H} h^2 \sum_{j=1}^{n} \left[\exp\left(-\frac{2-2H}{2H-1}(n-j+1) \right) - \exp\left(-\frac{2-2H}{2H-1}(n-j) \right) \right] O(h) \end{aligned}$$

we can write

$$\sum_{j=1}^{n} \left[\exp\left(-\frac{2-2H}{2H-1}(n-j+1) \right) - \exp\left(-\frac{2-2H}{2H-1}(n-j) \right) \right] = \exp\left(-\frac{2-2H}{2H-1} hn \right) - 1$$

and taking the first-order expansion of the exponential function, we have

$$\exp\left(-\frac{2-2H}{2H-1} hn \right) \approx 1 - \frac{2-2H}{2H-1} hn$$

Then the above expression can be written into

$$\begin{aligned} R(H,h) &= h^3 nO(h) \\ \Longrightarrow\ R(h) &= O(h^4) \end{aligned}$$

Hence, $O(h^4) = h^3 nO(h)$.

Theorem 3.1.1 shows that the numerical solution for the underdamped case of fractional Langevin equation in Caputo-Fabrizio fractional derivative sense has strong convergence, in which the truncation term is a function of fourth order of the time increment. To test the stability of the numerical solution given in equation (20), we present the next theorem.

Theorem 2. *Suppose, $x(t_n) = x(t)$ be a continuous function of discrete time $t_n = nh$ on $L^2[a,b]$. Then for every $n \in \mathcal{N}$,*

$$||x_n - Cx_{n-1}||_\infty < K \left|\left| \sum_{j=1}^{n} (x_{j+1} - x_j) D_j \right|\right|_\infty + A$$

such that if $||D_j||_\infty \to 0$ as $n \to \infty$, then $||x_n - Cx_{n-1}||_\infty \to 0$ as $n \to \infty$.

Proof.

$$\begin{aligned} ||x_n - Cx_{n-1}||_\infty &= -\frac{\gamma}{m}(2H-1)h^2 \left(\frac{1}{2-2H} \left|\left| \sum_{j=1}^{n} \left(\frac{x_{j+1} - x_j}{h} \right) D_j + \frac{1}{2-2H} \sum_{j=1}^{n} D_j O(h) \right|\right|_\infty \right) \\ &\quad + \left|\left| v_o h + \frac{\epsilon}{m} h B^H(t_n) \right|\right|_\infty \\ &\le -\frac{\gamma(2H-1)}{m(2-2H)} h \left|\left| \sum_{j=1}^{n} (x_{j+1} - x_j) D_j \right|\right|_\infty + \left|\left| h^3 nO(h) \right|\right|_\infty + ||A||_\infty \\ &< -\frac{\gamma(2H-1)}{m(2-2H)} h \left|\left| \sum_{j=1}^{n} (x_{j+1} - x_j) D_j \right|\right|_\infty + ||A||_\infty. \end{aligned}$$

Note that the fractional Brownian motion $B^H(t_n)$ is a long-range dependent and

$$\left\|\sum_{j=1}^{n}\left(x_{j+1}-x_j\right)D_j\right\|_\infty = \left\|\sum_{j=1}^{n}\left(x_{j+1}-x_j\right)\left[\exp\left(-\frac{2-2H}{2H-1}hn\right)-1\right]\right\|_\infty$$

So,

$$\begin{aligned}||x_n - Cx_{n-1}||_\infty &< K\left\|\sum_{j=1}^{n}\left(x_{j+1}-x_j\right)D_j\right\|_\infty + A\\ &< K\left\|\sum_{j=1}^{n}\left(x_{j+1}-x_j\right)\left[\exp\left(-\frac{2-2H}{2H-1}hn\right)-1\right]\right\|_\infty + A.\end{aligned}$$

Hence,

$$\left\|\sum_{j=1}^{n}\left(x_{j+1}-x_j\right)\left[\exp\left(-\frac{2-2H}{2H-1}hn\right)-1\right]\right\|_\infty \to 0$$

as $n \to \infty$. This completes the proof.

We have shown that the underdamped fractional Langevin equation in Caputo-Fabrizio fractional derivative sense have stability in numerical schemes.

3.2 Overdamped fractional Langevin equation driven by fractional Brownian motion

Dropping the inertial term ($m = 0$) of equation (2), we have the overdamped fractional Langevin equation

$${}_{CF}D^{2-2H}x(t) = -\frac{k}{\gamma}\frac{1}{2H-1}x(t) + \frac{\epsilon}{\gamma}\frac{1}{2H-1}\zeta^H(t). \tag{22}$$

Let

$$f(t,x(t)) = -\frac{k}{\gamma}\frac{1}{2H-1}x(t) + \frac{\epsilon}{\gamma}\frac{1}{2H-1}\zeta^H(t). \tag{23}$$

Then,

$$\frac{1}{2H-1}\int_0^t x'(s)\exp\left(-\frac{2-2H}{2H-1}(t-s)\right)ds = f(t,x(t)). \tag{24}$$

Applying Laplace transform and solving for $x(t)$, we have

$$x(t) - x(0) = (2H-1)f(t,x(t)) + (2-2H)\int_0^t f(s,x(s))ds. \tag{25}$$

If we have grid size N, then we have the discrete grid points $t_n = nh$, $n \in \mathcal{N}$. Descritizing equation (25), we have

$$x(t_n) - x(0) = (2H-1)f(t_{n-1},x(t_{n-1})) + (2-2H)\int_0^{t_n} f(s,x(s))ds \tag{26}$$

and

$$x(t_{n-1}) - x(0) = (2H-1)f(t_{n-2},x(t_{n-2})) + (2-2H)\int_0^{t_{n-1}} f(s,x(s))ds. \tag{27}$$

Now, subtracting equation (27) to (26), yields

$$x(t_n) - x(t_{n-1}) = (2H-1)\left[f(t_n,x(t_n)) - f(t_{n-2},x(t_{n-2}))\right] + (2-2H)\int_{t_{n-1}}^{t_n} f(s,x(s))ds, \tag{28}$$

where

$$\int_{t_{n-1}}^{t_n} f(s,x(s))ds = \int_{t_{n-1}}^{t_n} \left[\frac{f(t_{n-1},x(t_{n-1}))}{h}(s-t_{n-2}) - \frac{f(t_{n-2},x(t_{n-2}))}{h}(s-t_{n-1})\right]$$
$$= \frac{3}{2}hf(t_{n-1},x(t_{n-1})) - \frac{1}{2}hf(t_{n-2},x(t_{n-2})). \quad (29)$$

Putting back equation (29) to (28) and simplifying further to have

$$x(t_n) - x(t_{n-1}) = \left\{(2H-1) + \frac{3(2-2H)h}{2}\right\} f(t_{n-1},x(t_{n-1}))$$
$$+ \left\{(2H-1) + \frac{(2-2H)h}{2}\right\} f(t_{n-2},x(t_{n-2})). \quad (30)$$

which is the two-step Adams-Bashforth method for Caputo-Fabrizio fractional derivative. Observe that for $H = 1/2$, the classical Adams-Bashforth method is recovered. Hence, the numerical solution for the overdamped fractional Langevin equation driven by fractional Brownian motion is given by

$$x(t_n) = x(t_{n-1}) + \left\{(2H-1) + \frac{3(2-2H)h}{2}\right\} f(t_{n-1},x(t_{n-1}))$$
$$+ \left\{(2H-1) + \frac{(2-2H)h}{2}\right\} f(t_{n-2},x(t_{n-2})), \quad n \geq 2, \quad (31)$$

where

$$f(t_{n-1},x(t_{n-1})) = -\frac{k}{\gamma}\frac{1}{2H-1}x(t_{n-1}) + \frac{\epsilon}{\gamma}\frac{1}{2H-1}\frac{B^H(t_{n-1})}{h}, \quad (32)$$
$$f(t_{n-2},x(t_{n-2})) = -\frac{k}{\gamma}\frac{1}{2H-1}x(t_{n-2}) + \frac{\epsilon}{\gamma}\frac{1}{2H-1}\frac{B^H(t_{n-2})}{h}. \quad (33)$$

To verify the convergence of the numerical solution (31), we present the following theorem

Theorem 3.*If $x(t) = x(t_n)$ is the solution of fractional Langevin equation under the Caputo-Fabrizio fractional derivative with fractional order $\nu = 2-2H$, such that $x(t)$ is continuous at $0 \leq t \leq T$. Then,*

$$x(t_n) = x(t_{n-1}) + \left\{(2H-1) + \frac{3(2-2H)h}{2}\right\} f(t_{n-1},x(t_{n-1})),$$
$$+ \left\{(2H-1) + \frac{(2-2H)h}{2}\right\} f(t_{n-2},x(t_{n-2})) + O_n^H(h), \quad (34)$$

where $||O_n^H(h)||_\infty < X$.

*Proof.*Using equation (25)-(28), we have

$$x(t_n) - x(t_{n-1}) = (2H-1)\left[f(t_{n-1},x(t_{n-1})) - f(t_{n-2},x(t_{n-2}))\right] + (2-2H)\int_{t_{n-1}}^{t_n} f(s,x(s))ds,$$
$$= (2H-1)\left[f(t_{n-1},x(t_{n-1})) - f(t_{n-2},x(t_{n-2}))\right] + (2-2H),$$
$$\times \int_{t_{n-1}}^{t_n} \left\{\frac{f(t_{n-1},x(t_{n-1}))}{h}(s-t_{n-2}) - \frac{f(t_{n-2},x(t_{n-2}))}{h}(s-t_{n-1})\right\} ds,$$
$$+ (2-2H)\int_{t_{n-1}}^{t_n} \sum_{j=2}^{n-1}\prod_{j=2}^{n-1}(-1)^j\left(\frac{s-t_j}{h}\right) f(t_j,x(t_j))ds, \quad (35)$$

then we have

$$x(t_n) = x(t_{n-1}) + \left\{(2H-1) + \frac{3(2-2H)h}{2}\right\} f(t_{n-1}, x(t_{n-1})),$$
$$+ \left\{(2H-1) + \frac{(2-2H)h}{2}\right\} f(t_{n-2}, x(t_{n-2})),$$
$$\underbrace{+ (2-2H)\int_{t_{n-1}}^{t_n} \sum_{j=2}^{n-1}\prod_{j=2}^{n-1}(-1)^j \left(\frac{s-t_j}{h}\right) f(t_j, x(t_j))ds}_{O_n^H(h)} . \quad (36)$$

Or, we can write the truncation term

$$O_n^H(h) = (2-2H)\int_{t_{n-1}}^{t_n} \sum_{j=2}^{n-1}\prod_{j=2}^{n-1}(-1)^j \left(\frac{s-t_j}{h}\right) f(t_j, x(t_j))ds. \quad (37)$$

Hence,

$$||O_n^H(h)||_\infty = (2-2H)\left\|\int_{t_{n-1}}^{t_n} \sum_{j=2}^{n-1}\prod_{j=2}^{n-1}(-1)^j \left(\frac{s-t_j}{h}\right) f(t_j, x(t_j))ds\right\|_\infty$$
$$\leq (2-2H)\int_{t_{n-1}}^{t_n} \left\|\sum_{j=2}^{n-1}\prod_{j=2}^{n-1}(-1)^j \left(\frac{s-t_j}{h}\right) f(t_j, x(t_j))ds\right\|_\infty$$
$$\leq (2-2H)\int_{t_{n-1}}^{t_n} \sum_{j=2}^{n-1}\prod_{j=2}^{n-1}\left|(-1)^j \left(\frac{s-t_j}{h}\right)\right| ||f(t_j, x(t_j))ds||_\infty$$
$$< (2-2H)\int_{t_{n-1}}^{t_n} \sum_{j=2}^{n-1}\prod_{j=2}^{n-1}\left|\left(\frac{s-t_j}{h}\right)\right| \sup\{\max|f(t_j, x(t_j))|\}\, ds$$
$$< (2-2H)\frac{n!h^n}{4}X. \quad (38)$$

Therefore,

$$||O_n^H(h)||_\infty < (2-2H)n!h^n X.$$

Theorem 3.2.1 shows that the overdamped case of fractional Langevin equation has lesser convergence compare to the underdamped case of fractional Langevin equation. This result indicates that the underdamped and overdamped cases have different nature in the framework of Caputo-Fabrizio fractional derivative. One logical reason is that this speculation is valid due to the non-locality of the Caputo-Fabrizio fractional derivative [15]. However, Theorem 3.2.1 is the mathematical proof of the difference between the two cases.

Proposition 1. *The underdamped case and overdamped case of fractional Langevin equation have different stability due to their convergency along the given space with grid points $t_n = nh$, $n \in \mathcal{N}$, in Caputo-Fabrizio fractional derivative sense.*

Theorem 4. *If $x(t) = x(t_n)$ is the solution of fractional Langevin equation under the Caputo-Fabrizio fractional derivative with fractional order $\nu = 2-2H$, for every $n \in \mathcal{N}$*

$$||x(t_n) - x(t_{n-1})||_\infty < (2H-1)||f(t_n, x(t_n)) - f(t_{n-2}, x(t_{n-2}))||_\infty + (2-2H)\frac{n!h^n}{4}$$

such that if $||f(t_n, x(t_n)) - f(t_{n-2}, x(t_{n-2}))||_\infty \to 0$ as $n \to \infty$, then $||x(t_n) - x(t_{n-1})||_\infty \to 0$ as $n \to \infty$.

Proof.

$$\begin{aligned}||x(t_n)-x(t_{n-1})||_\infty &= \left|\left|(2H-1)\left[f(t_n,x(t_n))-f(t_{n-2},x(t_{n-2}))\right]+(2-2H)\int_{t_{n-1}}^{t_n} f(s,x(s))ds\right|\right|_\infty \\ &\le (2H-1)||f(t_n,x(t_n))-f(t_{n-2},x(t_{n-2}))||_\infty+(2-2H)\left|\left|\int_{t_{n-1}}^{t_n} f(s,x(s))ds\right|\right|_\infty \\ &\le (2H-1)||f(t_n,x(t_n))-f(t_{n-2},x(t_{n-1}))||_\infty+(2-2H)\int_{t_{n-1}}^{t_n} ds\left|\sum_{j=0}^{n-1}\prod_{j=0}^{n-2}\frac{s-t_j}{h}\right|. \\ &< (2H-1)||f(t_n,x(t_n))-f(t_{n-2},x(t_{n-1}))||_\infty+(2-2H)\frac{n!h^n}{4} \end{aligned} \tag{39}$$

This completes the proof.

It has been shown that the overdamped case of fractional Langevin equation has also an stability for long range values.

4 Conclusion

In this work, we present the numerical solution and algorithm of the dynamics of the fractional Langevin equation in Caputo-Fabrizio fractional derivative sense. The Convergence and stability analysis of the two cases were studied and established in this work, where it shows that the underdamped and overdamped cases of fractional Langevin equation have different nature in terms of mathematical formulation and description. More importantly, the numerical solution exists for both cases. In addition, the proposed numerical algorithm displays an important characteristic of the fractional Langevin equation, which is different from the known method found in literature.

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