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A Generalized Transmuted Fréchet Distribution

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Abstract: We introduce a generalized version of the quadratic rank transmuted Fréchet distribution that generalizes the standard Fréchet model by incorporating extra shape parameters into its distribution functions. We study the main mathematical and statistical properties of the proposed generalized transmuted Fréchet model, including its hazard rate function, moments, moment-generating function, quantile function, order statistics, moments of order statistics, probability weighted moment, L-moments and finally maximum likelihood estimator.

Keywords: Fréchet distribution, transmutation, entropy, order statistics, moments of order statistics

1 Introduction

The Fréchet probability distribution, named after the French mathematician Maurice Fréchet who developed it in 1927. It is also known as the inverse Weibull distribution ([1]). It is a special case of the generalized extreme value distribution. The Fréchet probability model is used to model maximum values in a data set. It is used to model a wide range of phenomena like flood analysis, horse racing, human lifespans, maximum rainfalls and river discharges in hydrology. In particular, the Fréchet distribution is used in engineering reliability. It also can be used to model a variety of failure characteristics such as infant mortality, useful life, and wear-out periods.

The cumulative distribution function (cdf) of the two-parameter Fréchet distribution is given by

$$G_{\alpha,\beta}(x) = e^{-\beta x^{-\alpha}}, \quad x \geq 0, \tag{1}$$

where $\alpha > 0$ is the shape parameter and $\beta > 0$ is the scale parameter.

The corresponding probability density function (pdf) of the two-parameter Fréchet distribution is given as

$$g_{\alpha,\beta}(x) = \alpha\beta x^{-\alpha-1} e^{\beta\left(-x^{-\alpha}\right)}, \quad \alpha > 0, \beta > 0, x > 0. \tag{2}$$

A random variable X is said to have a (quadratic rank) transmuted distribution ([2]) if its pdf and cdf can be respectively written as:

$$\begin{aligned} f_X(x) =& g(x)\left[1+\lambda-2\lambda G(x)\right], \\ F_X(x) =& (1+\lambda)G(x) - \lambda G^2(x), \quad -1 \leq \lambda \leq 1. \end{aligned}$$

Many authors proposed several extensions of the Fréchet distribution. For example, [3] proposed the exponentiated Fréchet distribution, [4] studied the beta Fréchet distribution, [5] introduced the Marshall-Olkin Fréchet distribution, [6] proposed the Kumaraswamy Fréchet distribution, [7] introduced the transmuted Marshall-Olkin Fréchet distribution, and [8] introduced the exponentiated-transmuted Fréchet distribution.

Also, [9] proposed a three-paramter extended Fréchet model and named it the modified Fréchet distribution.

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Recently, many authors studied the (quadratic transmutation) of the Fréchet distribution. For example, [10] introduced the transmuted Fréchet distribution and studied its statistical properties. [11] also introduced the transmuted Fréchet distribution and studied its properties and applications.

In this paper, a generalized transmuted Fréchet distribution is proposed as in the following.

2 The generalized transmuted Fréchet distribution

Given a baseline distribution X with cdf $G(x)$, the cdf of a generalized transmuted X can be given by the equation (See [12])

$$F(x) = G(x)^a \left[(\lambda+1) - \lambda G(x)^b\right], \quad -1 \leq \lambda \leq 1, a>0, b>0. \tag{3}$$

Based on (3), the cdf of the generalized transmuted Fréchet (GTF) distribution is given by

$$F_{a,b,\alpha,\beta,\lambda}(x) = G_{\alpha,\beta}(x)^a \left[(\lambda+1) - \lambda G_{\alpha,\beta}(x)^b\right], -1 \leq \lambda \leq 1, a>0, b>0, x \geq 0. \tag{4}$$

Substituting from (1) in (4) and simplifying, the cdf of the GTF distribution becomes

$$F_X(x;\Psi) = e^{-a\beta x^{-\alpha}} \left[1+\lambda\left(1-e^{-b\beta x^{-\alpha}}\right)\right], \tag{5}$$

where Ψ is the vector $(\alpha,\beta,\lambda,a,b)$, $a>0,\ b>0$ and $|\lambda| \leq 1$.

The pdf of the GTF distribution is given by

$$f_X(x;\Psi) = \alpha\beta\, x^{-(\alpha+1)}\left[a(1+\lambda)\, e^{-\beta a x^{-\alpha}} - (a+b)\lambda\, e^{-\beta(a+b)x^{-\alpha}}\right], \ x>0. \tag{6}$$

3 Shapes of the density and hazard rate functions

The reliability function of the cdf $F(x)$ of distribution is defined by $R(x) = 1 - F(x)$.

For the Generalized Transmuted Frechet (GTF) distribution, the reliability function is given as,

$$R(x) = 1 - \left[(1+\lambda)e^{-\beta a x^{-\alpha}} - \lambda\, e^{-\beta(a+b)x^{-\alpha}}\right].$$

The hazard rate function can be written as the ratio of the pdf $f(x)$ and the reliability function $R(x) = 1 - F(x)$. That is,

$$h(x) = \frac{f(x)}{R(x)},$$

then we can find the hazard rate function of GTF distribution by (5) and (6):

$$h(x) = \frac{\alpha\beta a(1+\lambda)\, x^{-(\alpha+1)}\, e^{-\beta a x^{-\alpha}} - \alpha\beta(a+b)\lambda\, x^{-(\alpha+1)}\, e^{-\beta(a+b)x^{-\alpha}}}{1-\left[(1+\lambda)e^{-\beta a x^{-\alpha}} - \lambda\, e^{-\beta(a+b)x^{-\alpha}}\right]}.$$

The cumulative hazard function is defined by

$$H(x) = -\ln R(x),$$

so the cumulative hazard function of the GTF distribution is

$$H(x) = -\ln\left\{1-\left[(1+\lambda)e^{-\beta a x^{-\alpha}} - \lambda\, e^{-\beta(a+b)x^{-\alpha}}\right]\right\}. \tag{7}$$

The reverse hazard function is

$$r(x) = \frac{f(x)}{F(x)}. \tag{8}$$

Using (8), we can write the reverse hazard function of GTF distribution as

$$r(x) = \frac{\alpha\beta x^{-(\alpha+1)}\left[a(1+\lambda)-(a+b)\lambda\, e^{-\beta b x^{-\alpha}}\right]}{1+\lambda-\lambda\, e^{-\beta b x^{-\alpha}}}. \tag{9}$$

The Odd function of a distribution with cdf $F(x)$ is defined as

$$O(x) = \frac{F(x)}{1-F(x)}. \tag{10}$$

Then the Odd function of the GTF distribution is given as

$$O(x) = \left\{\left[(1+\lambda)e^{-\beta a x^{-\alpha}} - \lambda\, e^{-\beta(a+b)x^{-\alpha}}\right]^{-1} - 1\right\}^{-1}. \tag{11}$$

4 Means

4.1 Harmonic Mean

The Harmonic Mean (see [13]) is defined as

$$\frac{1}{H} = \int \frac{1}{x}\, f(x)dx. \tag{12}$$

Theorem 1. Let $X \sim GTF(\Psi)$. Then the harmonic mean of X is given by

$$H = \frac{\beta^{\frac{1}{\alpha}}}{\Gamma(1+\frac{1}{\alpha})}\left(\frac{1+\lambda}{a^{\frac{1}{\alpha}}} - \frac{\lambda}{(a+b)^{\frac{1}{\alpha}}}\right)^{-1}. \tag{13}$$

proof. Substitute (6) in (12) to get $\frac{1}{H} = (1+\lambda)I_1 - \lambda I_2$, where

$$I_1 = \int_0^\infty \alpha\beta a\, x^{-(\alpha+1)}\, x^{-1}\, e^{-\beta a x^{-\alpha}} dx,$$
$$I_2 = \int_0^\infty \alpha\beta(a+b)\, x^{-(\alpha+1)}\, x^{-1}\, e^{-\beta(a+b)x^{-\alpha}} dx. \tag{14}$$

To compute I_1. Let $u = \beta a x^{-\alpha}$, so $-du = \alpha\beta a\, x^{-(\alpha+1)}dx$. Hence,

$$I_1 = \frac{1}{(\beta a)^{\frac{1}{\alpha}}} \int_0^\infty e^{-u}\, u^{(1+\frac{1}{\alpha})-1}\, du = \frac{\Gamma(1+\frac{1}{\alpha})}{(\beta a)^{\frac{1}{\alpha}}}. \tag{15}$$

Similarly,

$$I_2 = \frac{\Gamma(1+\frac{1}{\alpha})}{[\beta(a+b)]^{\frac{1}{\alpha}}}\,.$$

4.2 Geometric Mean

The Geometric Mean (see [13]) is defined as

$$\ln G = \int (\ln x)\, f(x) dx. \tag{16}$$

Theorem 2. Let $X \sim GTF(\Psi)$. Then the geometric mean of X is given by

$$G = \exp\left\{ \frac{1}{\alpha}\left[(1+\lambda)\Big(\ln(\beta a) + \gamma\Big) - \lambda\Big(\ln[\beta(a+b) + \gamma]\Big) \right] \right\}, \tag{17}$$

where γ is Euler-Mascheronic constant.
Proof. Using the Euler-Mascheroni constant $\gamma = -\int_0^\infty e^{-x}\ \ln x\, dx$.

4.3 Mean Residual Lifetime

The mean residual life (MRL) function has many applications, for example, in insurance, maintenance and product quality control, economics and social studies [14].
The mean residual lifetime is defined as,

$$MRLT = \mathrm{E}\big[X - x \mid X > x\big] = \frac{1}{1-F(x)} \int_x^\infty t\, f(t)\, dt - x. \tag{18}$$

Theorem 3. Let $X \sim GTF(\Psi)$. Then the mean residual lifetime of X is given by

$$MRL = \frac{\beta^{\frac{1}{\alpha}}\left[(1+\lambda) a^{\frac{1}{\alpha}} \gamma\big(1-\frac{1}{\alpha}, \beta a x^{-\alpha}\big) - \lambda (a+b)^{\frac{1}{\alpha}} \gamma\big(1-\frac{1}{\alpha}, \beta(a+b)x^{-\alpha}\big)\right]}{1 - \left[(1+\lambda) e^{-\beta a x^{-\alpha}} - \lambda\, e^{-\beta(a+b)x^{-\alpha}} \right]} - x. \tag{19}$$

where $\gamma(a,x) = \int_0^x t^{a-1}\, e^{-t} dt$.

4.4 Mean Past Lifetime

The mean past lifetime is defined as

$$k(x) = \mathrm{E}\big[x - X \mid X \le x\big] = \frac{\int_0^x F(t)\, dt}{F(x)} = x - \frac{\int_0^x t\, f(t)\, dt}{F(x)}. \tag{20}$$

Theorem 4. Let $X \sim GTF(\Psi)$. Then the mean past lifetime of X is given by

$$k(x) = x - \frac{\beta^{\frac{1}{\alpha}}\left[(1+\lambda) a^{\frac{1}{\alpha}} \Gamma\big(1-\frac{1}{\alpha}, \beta a x^{-\alpha}\big) - \lambda (a+b)^{\frac{1}{\alpha}} \Gamma\big(1-\frac{1}{\alpha}, \beta(a+b)x^{-\alpha}\big)\right]}{(1+\lambda) e^{-\beta a x^{-\alpha}} - \lambda\, e^{-\beta(a+b)x^{-\alpha}}}. \tag{21}$$

where $\Gamma(a,x) = \int_x^\infty t^{a-1}\, e^{-t} dt$.

5 Quantile Function

Theorem 5. Let $X \sim GTF(\Psi)$. Then the quantile function of X, is given by

$$x_q = \left[-\frac{\beta a}{\ln B(q,\lambda)} \right]^{\frac{1}{\alpha}}. \tag{22}$$

Proof. Assume that $a=b$. To compute the quantile function of the GTF distribution, we replace about x by x_q and about $F(x)$ by q in (5) to get the equation

$$q=(1+\lambda)e^{-\beta a x^{-\alpha}}-\lambda\, e^{-2\beta a x^{-\alpha}}. \tag{23}$$

Let $y=\exp\{-\beta a x^{-\alpha}\}$. Then, the solution of the equation $(1+\lambda)y-\lambda y^2=q$ is given as

$$y=e^{-\beta a x^{-\alpha}}=\frac{(1+\lambda)+\sqrt{(1+\lambda)^2-4aq}}{2\lambda}. \tag{24}$$

Now, let the function $B(q,\lambda)$ be defined by

$$B(q,\lambda)=\frac{(1+\lambda)+\sqrt{(1+\lambda)^2-4aq}}{2\lambda}.$$

Therefore, the solution of (24) is

$$x_q=\left[-\frac{\beta a}{\ln B(q,\lambda)}\right]^{\frac{1}{\alpha}}.$$

6 Moments and moment-generating function

Theorem 6. Let $X\sim GTF(\Psi)$. Then the rth moment of X is given by

$$\mathrm{E}(X^r)=\beta^{\frac{r}{\alpha}}\Gamma\big(1-\frac{r}{\alpha}\big)\Big[(1+\lambda)a^{\frac{r}{\alpha}}-\lambda(a+b)^{\frac{r}{\alpha}}\Big],\quad 0<r<\alpha. \tag{25}$$

Theorem 7. The moment-generating function of X is given by

$$\mathrm{M}_X(t)=\sum_{i=0}^{k}\frac{t^i}{i!}\Gamma\big(1-\frac{i}{\alpha}\big)\Big[(1+\lambda)(\beta a)^{\frac{i}{\alpha}}-\lambda[\beta(a+b)]^{\frac{i}{\alpha}}\Big]. \tag{26}$$

7 Order Statistic

Let $X_1,\cdots,X_n$ be a random sample of size n from the GTF distribution with parameters $\alpha>0,\beta>0,a>0,b>o$. Let $X_{1:n},X_{2:n},\cdots,X_{n:n}$ be the corresponding order statistics obtained by arranging $X_i, i=1,2,\ldots,n$, in non-decreasing order of magnitude. The i^{th} element of this sequence, $X_{i:n}$, is called the i^{th} order statistic.

From ([15], Page 232), the pdf of the i^{th} order statistics is obtained by

$$f_{X_{i:n}}(x)=i\binom{n}{i}f(x)[F(x)]^{i-1}[1-F(x)]^{n-i}. \tag{27}$$

Below, we will compute the r^{th} moment of the i^{th} order statistics in three cases according to the value of λ .

7.1 *Order Statistic from GTF when $\lambda\neq 0$ and $\lambda\neq -1$*

Proposition 1. Let $X_{i:n}$ be the i^{th} order statistic from $X\sim GTF(\Psi)$ with $\lambda\neq 0$ and $\lambda\neq -1$. Then, the pdf of the ith order statistic is given by

$$f_{X_{i:n}}(x)=\alpha\beta\, x^{-(\alpha+1)}\sum_{j=0}^{n-i}\sum_{k=0}^{h}\phi(i,j,k,n)(1+\lambda)^{h-k}\lambda^k\Big[a(1+\lambda)e^{wx^{-\alpha}}-\lambda(a+b)e^{(w+b)x^{-\alpha}}\Big], \tag{28}$$

where $h=i+j-1$, $w=[bk+a(i+j)]$ and $\phi(i,j,k,n)=(-1)^{j+k}\, i\binom{n}{i}\binom{n-i}{j}\binom{h}{k}$.

Theorem 8. Let $X_{i:n}$ be the i^{th} order statistic from $X \sim GTF(\Psi)$ with $\lambda \neq 0$ and $\lambda \neq -1$. Then the r^{th} moment of $X_{i:n}$ is given by

$$E[X_{i:n}^r] = \beta^{\frac{r}{\alpha}}\, \Gamma\big(1-\frac{r}{\alpha}\big) \sum_{j=0}^{n-i}\sum_{k=0}^{h} \phi(i,j,k,n)(1+\lambda)^{h-k}\lambda^k \Big[a(1+\lambda)w^{(\frac{r}{\alpha}-1)} - \lambda(a+b)(w+b)^{(\frac{r}{\alpha}-1)}\Big], \tag{29}$$

where $h=i+j-1$, $w=[bk+a(i+j)]$ and $\phi(i,j,k,n)=(-1)^{j+k}\, i\binom{n}{i}\binom{n-i}{j}\binom{h}{k}$.

7.2 Order Statistic from GTF when $\lambda = 0$

Proposition 2. Let $X_{i:n}$ be the i^{th} order statistic from $X \sim GTF(\Psi)$ with $\lambda = 0$. Then, the pdf of the i^{th} order statistic is given by

$$f_{X_{i:n}}(x) = \alpha\beta a\, x^{-(\alpha+1)} \sum_{j=0}^{n-i} (-1)^{j+k}\, i\binom{n}{i}\binom{n-i}{j} e^{-\beta a(i+j)x^{-\alpha}}. \tag{30}$$

Theorem 9. Let $X_{i:n}$ be the i^{th} order statistic from $X \sim GTF(\Psi)$ with $\lambda = 0$. Then the r^{th} moment of $X_{i:n}$ is given by

$$E[X_{i:n}^r] = (\beta a)^{\frac{r}{\alpha}}\, \Gamma\big(1-\frac{r}{\alpha}\big) \sum_{j=0}^{n-i} i\binom{n}{i}\binom{n-i}{j}(i+j)^{(\frac{r}{\alpha}-1)}. \tag{31}$$

Corollary 1. Let $X_{i:n}$ be the i^{th} order statistic from $X \sim GTF(\Psi)$ with $\lambda = 0$, then

$$E[X_{i:n}^r] = E(X^r) \sum_{j=0}^{n-i} i\binom{n}{i}\binom{n-i}{j}(i+j)^{(\frac{r}{\alpha}-1)}. \tag{32}$$

7.3 Order Statistic from GTF when $\lambda = -1$

Proposition 3. Let $X_{i:n}$ be the i^{th} order statistic from $X \sim GTF(\Psi)$ with $\lambda = -1$. Then, the pdf of the i^{th} order statistic is given by

$$f_{X_{i:n}}(x) = \alpha\beta(a+b)\, x^{-(\alpha+1)} \sum_{j=0}^{n-i} (-1)^{j+k}\, i\binom{n}{i}\binom{n-i}{j} e^{-\beta(a+b)(i+j)x^{-\alpha}}. \tag{33}$$

Theorem 10. Let $X_{i:n}$ be the i^{th} order statistic from $X \sim GTF(\Psi)$ with $\lambda = -1$. Then the r^{th} moment of $X_{i:n}$ is given by

$$E[X_{i:n}^r] = [\beta(a+b)]^{\frac{r}{\alpha}}\, \Gamma\big(1-\frac{r}{\alpha}\big) \sum_{j=0}^{n-i} i\binom{n}{i}\binom{n-i}{j}(i+j)^{(\frac{r}{\alpha}-1)}. \tag{34}$$

Corollary 2. Let $X_{i:n}$ be the i^{th} order statistic from $X \sim GTF(\Psi)$ with $\lambda = 0$, then

$$E[X_{i:n}^r] = E(X^r) \sum_{j=0}^{n-i} i\binom{n}{i}\binom{n-i}{j}(i+j)^{(\frac{r}{\alpha}-1)}. \tag{35}$$

8 L-moments and Probability Weighted Moments

Here we are about to have a tool by which we can easily find the L-moments for any distribution.
The probability weighted moments (PWMs) of a random variable X with (cdf) $F(X)$ and (pdf) $f(x)$ is the quantities (see [16])

$$M_{p,r,s} = E\left[X^p F(X)^r (1-F(X))^s\right] = \int_0^1 X^p F(X)^r (1-F(X))^s dF, \qquad r = 0, 1, \ldots.$$

Two particular useful special cases are the probability weighted moments $\alpha_r = M_{1,0,r}$ and $\beta_r = M_{1,r,0}$. For a distribution that has cdf $F(x)$ and pdf $f(x)$

$$\alpha_r = \int_0^1 x(1-F(x))^r dF = \int_{-\infty}^{\infty} x f(x) \big(1-F(x)\big)^r \, dx, \tag{36}$$

$$\beta_r = \int_0^1 x F(x)^r dF(x) = \int_{-\infty}^{\infty} x f(x) F^r(x) \, dx. \tag{37}$$

L-moments are a linear combination of probability-weighed moments (see [17]),

$$\lambda_{r+1} = \sum_{m=0}^{r} p^*_{r,m} \, \beta_m = (-1)^r \sum_{m=0}^{r} p^*_{r,m} \, \alpha_m, \tag{38}$$

where $p^*_{r,m} = (-1)^{r-m}\binom{r}{m}\binom{r+m}{m}$.
For example, the first four L-moments are related to the PWMs as follows :

$$\begin{aligned} \lambda_1 &= \beta_0 = \alpha_0, \\ \lambda_2 &= 2\beta_1 - \beta_0 = \alpha_0 - 2\alpha_1, \\ \lambda_3 &= 6\beta_2 - 6\beta_1 + \beta_0 = \alpha_0 - 6\alpha_1 + 6\alpha_2, \\ \lambda_4 &= 20\beta_3 - 30\beta_2 + 12\beta_1 + \beta_0 = \alpha_0 - 12\alpha_1 + 30\alpha_2 - 20\alpha_3. \end{aligned} \tag{39}$$

Below, we will compute the β_m (PWMs) and L-moments of $CTFr(\Psi)$ distribution in three cases according to the value of λ .

8.1 L-moments and Probability Weighted Moments of $GTF(\Psi)$ distribution when $\lambda \neq 0$ and $\lambda \neq -1$

Theorem 11. Let $X \sim GTF(\Psi)$ with $\lambda \neq -1$ and $\lambda \neq 0$. Then the probability weighted moments β_m of a random variable X is

$$\beta_m = \beta^{\frac{1}{\alpha}} \, \Gamma\big(1-\frac{1}{\alpha}\big) \sum_{k=0}^{m} (-1)^k \binom{m}{k} (1+\lambda)^{m-k} \, \lambda^k \left[a(1+\lambda) w^{(\frac{1}{\alpha}-1)} - \lambda(a+b)(w+b)^{(\frac{1}{\alpha}-1)}\right], \tag{40}$$

where $w = [bk + a(i+j)]$.
Theorem 12. Let $X \sim GTF(\Psi)$ with $\lambda \neq -1$ and $\lambda \neq 0$. Then the probability weighted moments β_r of a random variable X is

$$L_{r+1} = \beta^{\frac{1}{\alpha}} \, \Gamma\big(1-\frac{1}{\alpha}\big) \sum_{m=0}^{r} \sum_{k=0}^{m} \psi(r,m,k)(1+\lambda)^{m-k} \, \lambda^k \left[a(1+\lambda) w^{(\frac{1}{\alpha}-1)} - \lambda(a+b)(w+b)^{(\frac{1}{\alpha}-1)}\right], \tag{41}$$

where $\psi(r,m,k) = (-1)^{r+k-m}\binom{m}{k}\binom{r}{m}\binom{r+m}{m}$ and $w = [bk + a(i+j)]$.

8.2 L-moments and Probability Weighted Moments of $GTF(\Psi)$ distribution when $\lambda = 0$

Theorem 13. Let $X \sim GTF(\Psi)$ with $\lambda = 0$. Then the probability weighted moments β_m of a random variable X is

$$\beta_m = (\beta a)^{\frac{1}{\alpha}} \Gamma\big(1-\frac{1}{\alpha}\big)(m+1)^{(\frac{1}{\alpha}-1)}. \tag{42}$$

Theorem 14. Let $X \sim GTF(\Psi)$ with $\lambda = 0$. Then the probability weighted moments β_m of a random variable X is

$$L_{r+1} = (\beta a)^{\frac{1}{\alpha}} \Gamma\big(1-\frac{1}{\alpha}\big) \sum_{m=0}^{r} (-1)^{r-m} \binom{r}{m} \binom{r+m}{m} (m+1)^{(\frac{1}{\alpha}-1)}. \tag{43}$$

8.3 L-moments and Probability Weighted Moments of $GTF(\Psi)$ distribution when $\lambda = -1$

Theorem 15. Let $X \sim GTF(\Psi)$ with $\lambda = -1$. Then the probability weighted moments β_m of a random variable X is

$$\beta_m = [\beta(a+b)]^{\frac{1}{\alpha}} \Gamma(1-\frac{1}{\alpha})(m+1)^{(\frac{1}{\alpha}-1)}. \tag{44}$$

Theorem 16. Let $X \sim GTF(\Psi)$ with $\lambda = -1$. Then the L-moments of a random variable X is

$$L_{r+1} = [\beta(a+b)]^{\frac{1}{\alpha}} \Gamma(1-\frac{1}{\alpha}) \sum_{m=0}^{r} (-1)^{r-m} \binom{r}{m} \binom{r+m}{m} (m+1)^{(\frac{1}{\alpha}-1)}. \tag{45}$$

9 Maximum Likelihood Estimator

Let $X_1, X_2, \ldots, X_n$ be a random variable GTF distribution [see(6)], the likelihood function of vector of parameters $\Psi = (\alpha, \beta, \lambda, a, b)$ as

$$\begin{aligned} L(\Psi) &= \prod_{i=1}^{n} f(x_i) \\ &= \alpha^n \beta^n \left[\prod_{i=1}^{n} x_i^{-(\alpha+1)}\right] \prod_{i=1}^{n} \left[a(1+\lambda)\ e^{-\beta a x^{-\alpha}} - (a+b)\lambda e^{-\beta(a+b)x^{-\alpha}}\right]. \end{aligned}$$

Then its log-likelihood function is

$$\begin{aligned} \log L(\Psi) &= n\log\alpha + n\log\beta - (\alpha+1)\sum_{i=1}^{n}\log x_i \\ &+ \sum_{i=1}^{n} \log\left[a(1+\lambda)\, e^{-\beta a x_i^{-\alpha}} - (a+b)\lambda e^{-\beta(a+b)x_i^{-\alpha}}\right]. \end{aligned} \tag{46}$$

Then differentiated with respect to $\alpha, \beta, \lambda, a, b$ respectively, we get :

$$\sum_{i=1}^{n} \log x_i = \frac{n}{\alpha} + \beta \sum_{i=1}^{n} \frac{x_i^{-\alpha} \log x_i \left[(1+\lambda)a^2\, e^{-\beta a x_i^{-\alpha}} - \lambda(a+b)e^{-\beta(a+b)x_i^{-\alpha}}\right]}{a(1+\lambda)\, e^{-\beta a x_i^{-\alpha}} - (a+b)\lambda e^{-\beta(a+b)x_i^{-\alpha}}}. \tag{47}$$

$$\sum_{i=1}^{n} \frac{a\, e^{-\beta a x_i^{-\alpha}} - (a+b)e^{-\beta(a+b)x_i^{-\alpha}}}{a(1+\lambda)\, e^{-\beta a x_i^{-\alpha}} - (a+b)\lambda e^{-\beta(a+b)x_i^{-\alpha}}} = 0. \tag{48}$$

$$\sum_{i=1}^{n} \frac{e^{-\beta a x_i^{-\alpha}} \left[\beta(b\lambda - a)x_i^{-\alpha} + 1\right]}{a(1+\lambda)\, e^{-\beta a x_i^{-\alpha}} - (a+b)\lambda e^{-\beta(a+b)x_i^{-\alpha}}} = 0. \tag{49}$$

$$\lambda \sum_{i=1}^{n} \frac{e^{-\beta a x_i^{-\alpha}} \left[\beta(a+b)\lambda x_i^{-\alpha} - 1\right]}{a(1+\lambda)\, e^{-\beta a x_i^{-\alpha}} - (a+b)\lambda e^{-\beta(a+b)x_i^{-\alpha}}} = 0. \tag{50}$$

10 Application

Here, we fit a data set presented by [18] using our proposed Fréchet model. The data was collected from a group of 46 patients, per years, upon the recurrence of leukemia whom received autologous marrow. The data set is listed below which is about leukemia free-survival times (in years) for the 46 autologous transplant patients:

0.0301	0.0384	0.063	0.0849	0.0877	0.0959	0.1397	0.1616	0.1699	0.2137
0.2137	0.2164	0.2384	0.2712	0.274	0.3863	0.4384	0.4548	0.5918	0.6
0.6438	0.6849	0.7397	0.8575	0.9096	0.9644	1.0082	1.2822	1.3452	1.4
1.526	1.7205	1.989	2.2438	2.5068	2.6466	3.0384	3.1726	3.4411	4.4219
4.4356	4.5863	4.6904	4.7808	4.9863	5				

With the help of *Wolfram Mathematica*, we found that the best fitting for the above data is obtained by using $\alpha = 0.9, \beta = 1.5, a = 1, b = 0.5, \lambda = -0.35$ as indicated in the following figure.

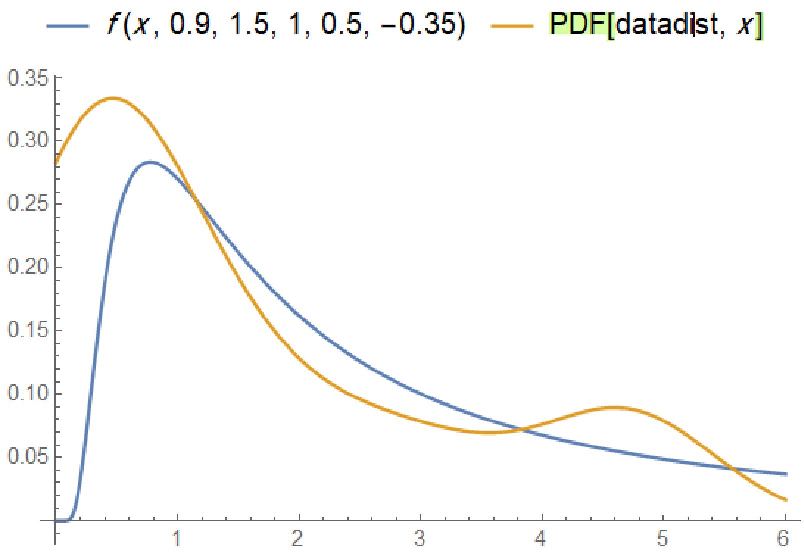

Fig. 1: Fitting data

11 Conclusion

We introduced in this paper a proposed 5-parameter flexible model that can be used to fit a variety of lifetime data sets. We studied the main mathematical properties of this model that we called a generalized transmuted Fréchet distribution. This model can be used to capture the complexity of lifetime data. Among the properties of this model, we studied its central moments, L-moments, order statistics, maximum-likelihood function, and moment-generating function. Then we applied the proposed model to a set of real lifetime data.

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