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Bayesian Statistics Application on Reliability Prediction and Analysis

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Abstract: Reliability predictions focus on developing the appropriate reliability model suitable for existing data. A reliability assessment comes not only from testing the product itself but it is affected by information which is available prior to the start of the test. Bayesian methods are considered efficient in the reliability modeling field when the use of "fault trees" and "reliability diagrams" are not possible. Bayes augment likelihood methods with prior information. Bayesian methods are capable of using a variety of information sources: statistical data, expert opinions, historical information, etc. to reach a probability distribution that is used to describe the prior beliefs about the parameter or set of parameters under study. This paper introduces a comprehensive review of using "Bayesian network approach" for modeling reliability and different methods and statistical distributions used in systems reliability studies.

Keywords: Reliability prediction, Bayesian theory, Statistical distribution, Prior information.

1 Introduction

Although Bayes theorem that was introduced in the 1770s is still relatively inexplicable, "Bayesian Statistics" is continually attracting attention and is being applied in different fields of researches. "Bayesian statistics" is a mathematical method that applies probabilities to statistical problems [1]. Bayesian methods make use of well known, statistically accurate, and logically sensible techniques to combine different types of data.

Bayesian approach and analysis are useful for a wide range of applications. For example; Singh et al presented an algorithm for reliability prediction that permits the system engineer to analyze system reliability prior it is built while considering the estimates of component reliability and their expected use [2]. Incorporated with the Unified Modeling Language (UML), this method enables the identification of crucial components and investigates the influence of their replacement by the other reliable ones. "Reliability assessment algorithm", employs these estimates as "prior probabilities". In the "Bayesian estimation structure", "posterior probability" of failure is calculated on basis of the priors and testing data. A non-parametric Bayesian model for unverified semantic parsing was proposed [3]. In order to demonstrate statistical dependences between meaning representations of predicates and those of their arguments, the authors applied Pitman-Yor "hierarchical processes". The authors had also evaluated the proposed method experimentally. Amrin et al introduced a procedure that systematizes the BN generation process [4]. The approach uses an engineering design demonstration method to create a BN for analyzing system reliability and evaluating it automatically. Bayesian designs that are optimized through applying "binary longitudinal responses"; examined with "mixed logistic regression" to describe linear time effects were investigated [5]. Based on "penalized quasi-likelihood" (PQL1) of the first order, the authors optimized a scalar function of the estimated information matrix to reach the optimum number of time points, cost limitations, and covariance structures of the random influences. The results displayed that the optimum number of time-points is dependent on the cost ratio of "subject-to-measurement". Moreover, the maxim in Bayesian D-optimal designs is extremely efficient in cases of prior changes.

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1.1 Bayesian Model Application Fields

The demand curve data is usually analyzed using individual-specific non-linear "least squares regression approach" to minimalize the residual sum of squares for each item of the dataset. Yi Ho et al reviewed different approaches to analyzing the demand curve data [6]. They also reviewed both; non-linear least square and the mixed effects regression, proposing a new Bayesian hierarchical model. Simulation analyses were conducted for the sake of comparing the three approaches' performance.

It had been seen that ML and Bayesian method are both frequently used to fit mixed logit models to special data. The type and the number of "quasi-random draws" employed for the simulation of the "likelihood" and for choosing the priors in "Bayesian estimation" have a great influence on the estimates. Akinc and Vandebroek [7] compared the different approaches computing the relative" root mean square errors" of the resulted estimates. The authors performed a simulation study focusing on the prior for the "covariance matrix" in Bayesian estimation.

As an extension of established Bayesian approaches, Dommert et al. presented a method that offers reliable estimations of the uncertainties of "neutron spectra" by proposing a new "parameterized model" that defines the main features of the expected "neutron spectra". In addition to measured information, the parameterization was based on exhaustive Monte Carlo simulations [8]. Bayesian networks are used in fault diagnosis in robot behavior and sensor fault [9]. In the same context, Bacha et al presented a Bayesian network-based system for diagnosing faults in DC motor using Bayesian parameters and structure learning algorithms [10]. Cognitive science field has grown to deal with progressively more complex problems [11]. A review article presented by Jacobs and Kruschke [12], focused on the main mechanisms of Bayesian information processing. The authors delivered several examples that demonstrate the role of "Bayesian approaches" in studying human awareness.

In order to investigate "dendritic morphology" and its effect on brain function, Lopez et al proposed a simulation algorithm to find new "brain dendrites", using a sample delivered from "Bayesian networks" [13]. To validate their results, the authors used a number of "univariate statistical tests" and a new multivariate test to check the similarity of virtual dendrites and real ones.

Amin et al. presented an availability assessment technique that is based on Dynamic Bayesian network (DBN). The technique offers a flexible demonstration of a number of failure scenarios and their cause's interdependency [14]. The authors also performed a sensitivity analysis to designate failure causes that are most influential and applied the proposed methodology on two fire alarm systems and risk consequences in a steam generation system. As BNs are originally used in the open-loop systems, some precise areas of closed-loop systems were not carefully investigated. Thus, an approach based on Cyclic Bayesian Networks (CBNs) [15] was presented to enhance GO method's capability to model and analyze complex closed-loop systems. Li et al [16] presented a Cyclic Bayesian Networks-based approach to enhance the capability of "GO methodology" for modeling systems with feedback signals. The approach is based on introducing an operator to simulate the feedback signal component. Comparing the decision capability of the GO methodology and BN' approach, in dynamic structure and uncertainty managing proved that enhancing GO-methodology with loop structured system modeling, offered a more practical methodology for addressing compound engineering systems with feedback loops.

In their research, Zhang et al focused on the "Bayesian duality" of a diversion in statistical results of the exponential distribution of censored data from life tests. A new simulation generating function was used to construct the manifold based on obtained data [17]. The "Bregman divergence" between two parameter points was examined attaining dual coordinate system and its function. Discussing the proposed structure, the results demonstrated that ML appraisal can be attained to reach a minimum value of the "Bregman divergence" prompted from the dual function.

In a recent study, a "Classical Latent Bayesian network model" was developed and compared with "the quantum-like Bayesian network" by means of "Prisoner's Dilemma game experiment" [18]. The study concluded that it is possible to simulate the violated probability theory axiom using the "Classical Latent Variable model" which caused an increased exponential complexity. Moreover, observed and unobserved conditions are not predictable. Targeting to further investigate of the case of possible calculation of the posterior distribution analytically was presented by Kyriazis [19]. In his article, the author analyzed the influence of two rectangular prior distributions assuming non-informative priors. This analysis was carried out for "signal-to-noise ratios" (SNR). The author used a Bayesian approach with approximations based on the "posterior mode", discussing the advantages and disadvantages of taking the mean of the posterior as estimations of the decay time value.

To demonstrate the developed "Bayesian finite element (FE) model", Das & Debnath applied the proposed approach on a truss' structure; of two dimensions, to demonstrate its efficiency considering multiple damage cases [20]. The results showed satisfactory performances in updating the model and estimation of probability. Another advantage of the proposed approach is having better computational efficiency compared with two well-known "Markov Chain Monte Carlo" (MCMC) techniques.

A broad survey on "Bayesian forecasting" techniques was implemented on flood forecasting from 1999 to present [21]. At first, an outline of the basics of advancements in BFS was presented. The second step was the method applied to forecast the real-time flood. This was followed by a critical analysis to evaluate the benefits and restrictions of Bayesian forecasting methods and other prognostic uncertainty flood forecasting assessment methods. Results demonstrated that the Bayesian flood forecasting method is considered an estimation approach as it takes into account all sources of uncertainties giving efficient, precise, and reliable estimates. Some evolving Bayesian forecasting methods (of multi-model combination) offer a reliable way to overcome a single model or fixed model weight limitations and effectively reduce uncertainty prediction.

Over the last few years, one of the serious engineering infrastructures, railway turnouts (RTs) was studied and analyzed to develop a new BN-based model with exceptional algorithm [21]. This unique investigation showed the causal relations between primary causes and the subsystem failures that result in the derailment in case of extreme weather-related conditions.

1.2 Bayesian Model Applications in Health Sector

The likelihood-tree techniques give advanced knowledge of "cognitive models" applied to behavioral studies. These techniques made it possible to connect the theoretical models of cognizance and "cognitive neuroscience"[11]. In the field of modeling "cognitive neuroscience", the parameters of the models are related to statistical models of neural activation to demonstrate brain function by mechanistic interpretations [18], [22]. Some of the models used are "ideal observer analysis" and a relatively new discipline called Quantum Cognition that is based on quantum probability [23].

Psychophysics is offering an evidence that humanly related computations are "Bayes' optimal", indicating the "Bayesian coding hypothesis". This signifies that the brain represents sensory information probabilistically [24]. Although Neuro-physiological data on the hypothesis are rare, a number of computational arrangements have lately been suggested so this would be realized in "populations of neurons". Experimental testing of these concepts is still a major challenge for neuroscientists.

A review considering the application of Bayesian models of learning in pain that essentially accommodate uncertainty and action was proposed by Tabor and Burr [25]. The suggested Bayesian models are essential in understanding learning in both acute and persistent cases of pain.

A narrative review presented by Pfuhl [26] discussed the basics of using Bayesian decision theory approach to delusion. The crucial importance of this review is its ability to deliver a measure for belief perseverance. The author presented experimental tasks for measuring parameters and adjustment of two standard reasoning tasks. It is proposed to test the parameters using "Bayesian decision theory".

To calculate the human efficiency, the ratio of human to ideal performance using ideal observer analysis was applied by Houpt & Bittener [27]. This method could be directly applied to control the human differences in performance that could be a result of motivation and/or task specific requirements. The authors; whom had previously addressed this problem using "ideal observer analysis" to test the effects of variation in experimental conditions on the efficiency using "ANOVAs" and pair-wise associations, presented another model combining "Bayesian estimates" of "psychometric functions" with "hierarchical logistic regression" for inference concerning human performance measures and efficiencies. In the last era of the twentieth century, a number of models capable of predicting or explaining human decisions that conflict with the conventional probability theory logic had been proposed. These models are part of a more or less new discipline called "Quantum Cognition" that is based on quantum probability [28]. Several "quantum-like" models have been suggested to contain violations of probability laws in the literature. Recent works had recommended "quantum-like Bayesian networks" as an alternative model to predict scenarios with high uncertainty.

2 **Bayesian** ***Statistics***

2.1 Bayes' Theorem [29]

One of the axioms of probability theory states that the probability of occurrence of the two events, A and B, is equal to the multiplication of the occurrence of one probability "labeled P(A)", and the conditional probability of the other, given that the first occurred, "labeled P(B|A)".

$$P\ (A \text{ and } B) = P\ (A)\ P\ (B|(1)\ A) = P(A\cap B) \qquad (1)$$

The symbol (∩) is called an “intersection.” One notes that it makes no difference if one interchanges A with B since the calculation is the probability they both occur. There need not be time ordering of the events. Thus, the following expression must also be true.

P (B and A) = P (B) P (A|B) =P (B∩A) (2)

Setting equations (1) and (2) equal to one another, it follows directly that

P (B) P (A|B) = P (A) P (B|A) (3)

Solving these two terms for P (A|B), one arrives at "Baye’s Theorem".

P (A|B) = [P (B|A) / P (B)] P (A) (Bayes’ Theorem) (4)

Posterior = [Relative Likelihood] × Prior (5)

The terms in the square brackets are termed the relative "likelihood". The probability P (A) is called the "prior probability" and P (A|B) is called the "posterior probability", it is the "conditional probability" of the event (A) occurring given information about the event (B).

$$f_{prior}(R) \propto R^{N_m\pi}(1-R)^{N_m(1-\pi)}, \qquad prior\ distribution \qquad (6)$$

$$L(n,s|R) \propto R^{s}(1-R)^{n-s}, \qquad likelihood\ function \qquad (7)$$

$$f_{posterior}(R) \propto R^{N_m\pi+s}(1-R)^{N_m(1-\pi)+n-s}, \qquad \text{posterior distribution} \qquad (8)$$

N (_m) and (/Pi) are parameters chosen by the subject matter experts/test engineers based on prior analysis e.g. simulation results, or experience from similar systems. The variables s and n are the numbers of successes in n tests performed on the component (or system) of interest. Figure 1 shows a representation of prior and posterior beliefs.

2.2 Background in Bayesian Statistics [30]

2.2.1 Prior Distributions

A "prior distribution" of a parameter is the "probability distribution" representing the uncertainty about the parameter (or the belief in the parameter) before the recent data are studied. To get the "posterior distribution" of the parameter, the prior distribution is multiplied by the "likelihood function". The posterior distribution is used to perform all inferences. Then, the prior distribution is essential to any Bayesian inference or development of system models.

2.2.2 Non-Informative Priors

In general, a "prior distribution" is non-informative if it is “flat” relative to the "likelihood function". Thus, a prior π(θ) is non-informative if it has negligible influence on the "posterior distribution" of (θ).Flat prior is commonly chosen for a non-informative prior, i.e. ”a prior distribution" that assigns equal "likelihood" on all possible values of the parameter. However, this may not be true in all cases. For more explanation, assume there is a "binomial" experiment with n "Bernoulli trials" where y's are perceived. To draw “inferences" about the unidentified success probability p, a uniform "prior" on p, may seem to be non-informative. But, assuming the "uniform prior" is actually as if two observations are added to the data, (1, and 0). In case of small n and y, the two added observations can have a great influence on the parameter estimate of p.

$$\pi\ (p)\ \alpha\ 1 \qquad (9)$$

Notice that the likelihood is this:

$$p^{y}(1-p)^{n-y} \qquad (10)$$

The "maximum likelihood estimator" (MLE) of p is (y/n). The uniformly distributed prior can be fitted to a "beta distribution" with shape and scale parameters (α and β) being:

$$\pi(p)\alpha p^{\alpha-1}(1-p)^{\beta-1} \tag{11}$$

The "posterior distribution" of p is proportional to:

$$p^{\alpha+y-1}(1-p)^{\beta+n-y-1} \tag{12}$$

Which is beta (α+y, β+ n – y) Hence, the "posterior" means is this:

$$\frac{\alpha+y}{\alpha+\beta+n} = \frac{1+y}{2+n} \tag{13}$$

This may be pretty different from the "MLE" if both n and y are small [31].

2.2.3 Informative Priors

A prior distribution that powerfully affects the likelihood is clearly an informative prior and it would have an impact on the posterior distribution. Using prior distributions properly demonstrates the Bayesian method's power of information from a study, where there is no enough data.

2.2.4 Conjugate Priors

If both "prior" and "posterior" distributions are from the same family, the prior is considered a conjugate prior, which means that the "posterior" has the same form of distribution as the "prior". This means that in case the "likelihood" is "binomial, y ~ Bin (n,θ)", thus the "conjugate prior" ofθ is the "beta" distributed; this leads to a posterior distribution of θ that also follows a beta distribution. Conjugacy in posterior sampling is not used in Bayesian procedures.

2.2.5 Jeffreys' Prior

Jeffreys' prior is very helpful [32] as it assures that a prior does not considerably change over the significant region of "likelihood" and does not presume significant values outside that range. Jeffreys' prior is based on the Fisher information matrix and is defined as:

$$\mathrm{I}\,(\theta) = \mathrm{E}\left[\frac{\partial^2 \log p(y|\theta)}{\partial\theta^2}\right] \tag{14}$$

As Jeffreys' prior is locally uniform it is non-informative.

2.3 Bayesian Inference

As it is known in statistics, "inference "is the process of drawing conclusions about a parameter that is needed to be measured or estimated. Choosing the best measure of many measurements is a decision that is often faced in scientific study.

"Bayesian inference" about θ is mainly associated with the distribution of the "posterior" of (θ) which may be summarized in different ways, such as reporting the findings by point estimates. The posterior distribution could also be used to construct hypothesis tests or probability statements.

Bayesian estimation is considered one of the principal approaches of statistical inference as it incorporates logical expectations; or prior judgments, as well as new observations. Likelihood approach is another method that eschews "prior probabilities" in order to calculate the parameter value that would be most "likely" to produce the observed distribution of experimental results.

2.3.1 Point Estimation and Estimation Error

Although classical methodologies usually depend on the "maximum likelihood estimator" (MLE) or the "method of moment's estimator" (MOME) of a parameter, Bayesian methods generally utilize the posterior mean.The definition of the posterior mean is given by:

$$E((\theta|y) = \int \theta p(\theta|y)\, d\theta \tag{15}$$

$$\theta P p(\theta \geq median|y) = p(median|\theta/y) = \frac{1}{2} \tag{16}$$

The posterior mode is defined as the value of θ that maximizes:

$$\text{Ho} = \text{is } \theta \in \Theta 0 \quad (17)$$

The uncertainty in the parameter is described by the posterior variance which is normally used in a Bayesian analysis to distinguish the "dispersion of the parameter". In "multidimensional" models, "covariance" or correlation matrices are employed.

If the posterior density of interest has a known distributional form, the exact posterior point estimates can be defined. If models are too difficult to analyze analytically, simulation algorithms, such as the "Markov chain Monte Carlo" (MCMC) method can be used to attain "posterior" estimates [33]. All posteriors that are estimated; using Bayesian procedures, rely on MCMC. If the used samples are of a finite number, simulations would elevate the level of uncertainty of the accuracy of the estimate. "Monte Carlo standard error" (MCSE); which is the "posterior" mean-estimate standard error, measures the simulation accuracy.

MCSE and posterior standard deviation are entirely different concepts as MCSE only defines the doubt "uncertainty" in the parameter estimate while the posterior standard deviation depicts the uncertainty in the parameter. The "posterior standard deviation" is a function of the sample size, whereas the "MCSE" is a function of the number of simulation runs. Figure (1) illustrates probability density functions (pdf) of the prior and posterior beliefs.

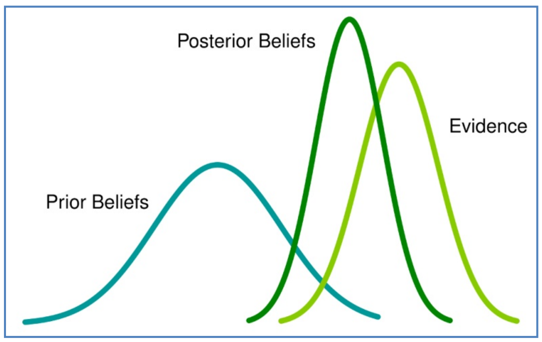

Fig.1: Probability density functions of the prior and posterior beliefs.

2.3.2 Maximum Likelihood Method

Valuation of the "likelihood" function L_{ji} is considered the core of the calculations related to the Bayesian techniques **[**34**]**. Dealing with large data, the p-value can be relatively large, hence the product Π resulting in figures that are small for the computer to handle. Thus, it is essential to implement a scaling procedure. This is realized by applying a logarithmical transformation of (L_{ji}) in the form of:

$$ln = [L_{ji}] = \ln\,[\prod_{k=1}^{p} f_j(\alpha_k|\theta_{ji}) or \ln[L_{ji}] = \sum_{k=1}^{p} ln\,[f_j(\alpha_k|\theta_{ji})] \quad (18)$$

Which reduces the product of too small figures in the summation of large negative values. The "scaling factor SF" is chosen such that {max ln [L_{ji}]} + SF = 0 . Therefore SF =- maximum value.
The Bayesian method for data combination of "pile capacity" was joined with full-scale tests results using a Bayesian approach to determine the "probability density functions" (pdf) of the model uncertainty.

2.4 Bayesian Networks

Presently, the complexity of the systems arising from applying smart tools and decision techniques in the analysis of dependability, reliability, and maintenance shows an increasing trend. The Bayesian Network (BN) methods proved to be highly effective for these applications.
Bayesian networks (BN) offer an appropriate basis to express; graphically, probabilistic relationships between multiple variables. Hence, it is considered a "directed acyclic graph (DAG)"illustration of a "multivariate distribution", decomposing it into a grouping of marginal and conditional probabilities [35]. Figure (2) presents an example of a DAG model. Each node in the BN exemplifies a random variable while the arcs connecting the nodes express the conditional probabilities. In case an arc exists, the upstream node is nominated the "parent node “while the downstream one is nominated the "child node". A mathematical description of nodes dependency could be expressed by "conditional probability distribution"; based on the "directed Markov condition".

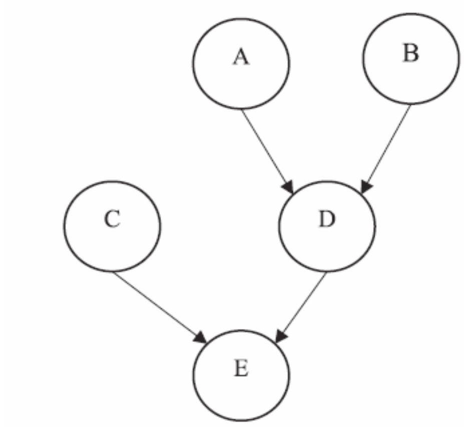

Fig.2: Directed, acyclic graph (DAG) model.

3 Bayesian Methods Applied to Reliability Analysis and Prediction

"Bayesian network" (BN) is generally used in a probabilistic study of uncertainty in the industry, nevertheless large, complex systems modeling is required especially in cases of fault diagnosis and reliability evaluation**.** To reduce the whole complexities of BN fault diagnosis, and immediate fault recording, Cai et al [36, 37] presented a "real-time fault diagnosis" approach of repetitive structures based on "object-oriented Bayesian networks" (OOBNs).

To estimate and predict system "functional reliability", Rebello et al [38] used both "Hidden Markov Model (HMM)" and "Dynamic Bayesian Network (DBN)". The authors proposed model used system "functional indicators" and data concerning the elements condition in a continuous time domain. The suggested model establishes a logical causal connection between the degradation states of components and system functional state. The methodology introduced by Amrin et al [4] to accelerate BN generation of process reliability was validated through an application on an automotive power train system.

To evaluate warships reliability; that are typical dynamic systems, Liang et al proposed method that develops the current a methods applied on static systems to be applicable for warships reliability evaluation based on numerical simulation[5][39]. This method is also suitable for other complex dynamic systems.

Wang & Roy constructed a two-block "Gibbs sampler" for "Bayesian logistic linear mixed models" with normal priors on "regression parameters" and "truncated Gamma priors" on "precision parameters" [40]. The developed model was based on data augmentation technique regularized support vector machines (SVM's); that was presented by Polson et al [41].

Bayesian approach is also considered a powerful method for handling missing data. Ma and Chen [42] reviewed the latest advances and applications of "Bayesian methods" for managing "ignorable" and "non-ignorable" missing data. The authors introduced missing data mechanisms and "Bayesian" framework for handling those including missing data models and other related topics. After that, significant topics of "Bayesian inference", comprising “prior" construction, "posterior" computation, model comparison and sensitivity analysis, were discussed.

4 Bayesian Statistics for Reliability Distribution Models

As computational power is the steadily rising, extremely complicated models can be predicted using modern computers. Thus, modeling progressed from restricted mean models to probabilistic distributional models where "covariate" effects can have flexible forms [43]. In addition, the development of "Monte Carlo Markov Chain" methods, lead to huge improvements in computational capabilities which in turn lead to increased use of "Bayesian methods" in reliability applications [44]. The main concern when using Bayesian methods in Reliability is from where the needed "prior" distributions should emerge from.

"Hierarchical learning models"; e.g. mixture models and "Bayesian networks", are extensively applied for unverified tasks, such as "clustering analysis". As these models comprise observable and latent variables; representing input data and generation procedure, "conventional statistical analysis" is not appropriate. Recently, analyzing the precision of forecasting observable variables employing a technique based on algebraic geometry; using Bayesian estimation became possible [45]. The study suggested an applicable methodology if the range of the latent variable is surplus compared with the model generating data. Error functions of the Bayesian estimation of "multidimensional latent" variables were formulated and their asymptotic forms were derived.

As mentioned above (sec 2.1), Bayesian analyses result in a set of distribution functions called posterior distributions that represent the predicted ranges of possible reliabilities and associated probabilities of the components, subsystems and the full system attaining those reliabilities.

Different types of distributions are used to represent the prior distribution. The following section illustrates a number of frequently used methods and distributions.

4.1 Rayleigh Distribution

The "Bayes estimators" for the parameter "reliability function of the generalized Rayleigh distribution" are attained based on "square error (SE)" and "LINEX (linear-exponential) loss functions". Risk comparison of the parameters estimated from the two methods (SE & LINEC) loss functions was reported. Numerical and simulation examples were also included [46, 47, and 48].

Bayes' estimators for cases of "symmetric and asymmetric loss functions" were found for Weibull Rayleigh distribution unknown parameters [49]. If the three parameters are unknown, the "closed-form" expressions of the "Bayes estimators" are not obtainable. The authors used "Lindley's approximation" to compute the Bayes' estimates. Bayes' estimators of the reliability characteristic were also obtained in the same way and its performance was compared conducting a "Monte Carlo" simulation study. To illustrate the results a numerical study was provided.

The "generalized Rayleigh probability density function" (pdf) is:

$$\boldsymbol{f(x|\sigma)} = \frac{\boldsymbol{2x\theta}}{\boldsymbol{\sigma^2}}\boldsymbol{e}^{\frac{-\boldsymbol{X^2}}{\boldsymbol{2\sigma^2}}}\left[\boldsymbol{1-e}^{-(\frac{x}{\sigma})^2}\right]^{\boldsymbol{\theta-1}}; \qquad \boldsymbol{x \geq 0, \theta \geq 0, \sigma} > 0 \tag{19}$$

The reliability function at mission time t is given by:

$$\boldsymbol{F(x|\sigma)} = \left[\boldsymbol{1-e}^{-\left(\frac{x}{\sigma}\right)^2}\right]^{\boldsymbol{\theta}}; \boldsymbol{x \geq 0}, \qquad \boldsymbol{\theta \geq 0, \sigma} > 0 \tag{20}$$

Estimating a parameter θ by$(\widehat{\theta})$, the "loss function" is given by:

$$\boldsymbol{L}(\Delta) = \boldsymbol{e}^{\boldsymbol{a}\Delta} - \boldsymbol{a}\Delta - \boldsymbol{1}, \text{ and } \Delta = \frac{\boldsymbol{\theta}}{\boldsymbol{\theta}} \tag{21}$$

Bayes Estimate of θ:

To estimate the "shape parameter θ" of the "generalized Rayleigh" model, suppose n ordered values $X_{(1)}$, $X_{(2)}$, $X_{(3)}$ are observed from the "generalized Rayleigh distribution" pdf given by the equation: The "likelihood function" (LF) is given by:

$$f(\underline{X}\theta) = \prod_{i=1}^{n} f(X_i) \tag{22}$$

$$f(\underline{X}|\theta) = \left[\frac{2\theta}{\sigma^2}\right]^n \prod_{i=1}^{n}\left[1-e^{\left(\frac{X_i}{\sigma}\right)^2}\right]^{\theta-1}; \qquad \theta > 0 \tag{23}$$

Bayes estimators of the reliability function:

If R = F (x) is the probability a system survives a definite task time t. Then, the reliability function of "Burr type X" distribution is obtained as:

$$R(x) = 1 - \left(1-e^{-x^2}\right)^{\theta} \tag{24}$$

The "maximum likelihood" estimate of R is given by:

$$\mathrm{R(x)} = 1 - (1-\mathrm{e}^{-\mathrm{x}^2})^{\theta_{\mathrm{M1}}} \tag{25}$$

Where $\hat{\theta}_{Ml}$ is given by$\hat{\theta}_{Ml}$

$$\hat{\boldsymbol{\theta}}_{\boldsymbol{Ml}} = \frac{\boldsymbol{n}}{\boldsymbol{T}}, \tag{26}$$

where

$$T = -\sum \ln\left(1 - e^{-x^2}\right) \tag{27}$$

$$\hat{\theta}_{SS} = 1 - \sum_{j=1}^{K} p_j E\left[\prod_i^N \left(1 - \theta_j\right)^{b p_{ij}}\right| \tag{28}$$

n_j tests of thej^{th}scenario

resulted in r_j failure

4.2 Beta Distribution

The Bayesian structure offers a thorough approach for incorporating "Unified Modeling Language" (UML) interpretations into the reliability prediction model. Singh et al [50], assumed that the "failure probability" of a component Ci is (θi) which is a random variable having "beta distributions" that are considered very flexible and capable of designating a varied range of probability distributions. The probability density function of the failure probability (θi of the component Ci; i =1, 2,……, K) is represented by the equation:

$$g_i(\theta_i) = \frac{1}{B(a_i, b_i)} \theta_i^{a_i - 1} (1 - \theta_i)^{b_i - 1}, \qquad 0 \le \theta_i \le 1 \tag{29}$$

Assuming that "failure probabilities “of the component are independent, the system "failure probability"θs is given by:

$$\theta_S = 1 - \sum_{j=1}^{K} p_j \prod_{i=1}^{N} \left(1 - \theta_j\right)^{bp_{ij}} \tag{30}$$

Thus, (1- θs) is a linear combination of the product of "beta random variables". The explicit expression of the" probability density function" of θs is, mostly, difficult to obtain.

4.3 Bernoulli Distribution

"Bayesian Inference" considering a "Bernoulli process" was applied by Gunawan & Papalambros [51]. The method assumes a "Bayesian binomial inference" method to estimate reliability. This estimate was used to maximize the confidence in the design as the targeted reliability will be met or exceeded. Applying Bernoulli process, "probabilities of success and failure” are given by "p and (1−p)", respectively. Given N independent trials, the probability of (r) successes out of these trials follows a "Binomial distribution: r~Bin (N, p)".

An inference of this process search is calculating p on basis of the outcomes of the trial. The "probability distribution" of p can be calculated using "Bayes’ theorem" as in the following equation:

$$f(p|r) = \frac{f(p) \times f(r|p)}{\int_0^1 f(p) \times f(r|p) dp} \tag{31}$$

Where: f (p)≡"prior distribution" of p

f (p | r)≡ the "posterior distribution" of p

f (r| p) is the "likelihood" of "r given p"

To reach a proper probability distribution, a normalizing factor is used (the integral in the denominator). The "posterior distribution" is the sought distribution. It is the estimate of p based on the outcome of the trials. Using a "uniform prior" and a "Binomial likelihood" function in Eq. (32) resulted in a "Beta posterior" distribution, Eq. (33)

$$L\,((p|n,y) = \binom{n}{y} p^{y} \tag{32}$$

$$f(p|r) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)-\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1} \tag{33}$$

Where: $\alpha = r + 1$ and $\beta = (N-r) + 1$.

This means that p is distributed according to a "Beta distribution", and the parameters depend on the outcome of the trials: "p~Beta (r+1, (N−r) +1)".

The authors performed a reliability-based optimization and applied the model on a "piston-ring/cylinder-liner assembly" that accommodates the combustion process inside an engine.

4.4 Weibull Distribution

Bayesian reliability models were also used for systems of failure following Weibull distribution. To formulate the reliability model, "Bayesian estimation of Weibull parameters" and "goodness-of-fit" test were as suggested. To predict the reliability of the non-repairable product, the authors suggested a Bayesian methodology to be applied to a Weibull distribution [52].
Ion and Sander [53] suggested a "Bayesian model" for a manufactured item that has the "time to failure" following "Weibull distribution", while "scale and shape parameters" are normally distributed. Assuming a known shape parameter of "Weibull distribution" Fard et al [54, estimated "prior distribution" for "scale parameter" through the "maximum likelihood method" that was previously investigated by Ion and Sander [53]. They also investigated four different "prior distributions" for scale parameter such as "Gamma, Exponential, Inverted Gamma", and "truncated Normal".
A study by Roy [55] showed that optimal "Accelerated Life Tests" (ALT) plans using "Weibull" model are considerably changed from the resulted plans of applying "exponential model". The paper also presented a "comparative study" between two commonly used "Bayesian design criteria". As design criteria are created based on the particular objective. The optimal resulting "ALT plans" from two dissimilar criteria can be quite different in real applications.
"Weibull mixture" is a highly significant model to detect non-homogeneous collected data. Regrettably, the performance of "classical estimation methods" (ML, Bayes approach) is risked as a result of the high number of parameters and the heavy censoring [56]. Hence, Ducros and Pamphile proposed a Bayesian "bootstrap" method, called "Bayesian Restoration Maximization" (BRM) to compensate for heavy censoring for "Weibull mixture parameters estimation". Using simulation, they proved that BRM outperforms the "EM and S-EM algorithms" in terms of estimates accuracy.
Ahmed and Ibrahim [57] compared the performance of the Bayesian using "Jeffrey prior" with "maximum likelihood "process for estimating the "parameters of Weibull distribution" with censored data applying a simulation study. Assessments of these estimators performance were made with respect to the "Mean Square Error (MSE)" and "Mean Percentage Error (MPE)". "MSE and MPE" both decreased as sample size increased for all investigated cases.

The Weibull distribution of failure, with its two "parameters shape and scale", permits the modeling of different regions of the "bathtub" curves in a vast number of components' lifecycle [58].

The Weibull pdf is defined by

$$f(t|a,b,\tau) = \left(\frac{a}{b}\right)\left(\frac{t-\tau}{b}\right)^{a-1} exp\left[-\left(\frac{t-\tau}{b}\right)^{a}\right], \tag{34 }$$

Where "a is the shape parameter", "b is scale the parameter", and τ is the parameter of location (delay). It could be noticed that, if $\theta = 1$, the distribution becomes "exponential".

The "hazard rate" is:

$$h(t) = \left(\frac{a}{b}\right)\left(\frac{t-\tau}{b}\right)^{a-1}, \tag{35}$$

In practice and for physical reasons, the shape parameter θ is bounded. The reliability is determined by the relation:

$$R(t) = \int_t^{\infty} f(u)du = \exp\left[-\left(\frac{t-\tau}{b}\right)^{a}\right]. \tag{36}$$

Estimation of "mixed Weibull distribution" by other methods such as "maximum likelihood" is usually complicated as a result of unsteady estimates due to limited data. This could be overcome using "Bayesian techniques" which stabilize the estimates through the "priors" [60]. The author introduced a Bayesian-based method that reduced the number of "numeric integrations"; required for estimation from "mixed Weibull situations", from five to two.

The choice of prior's distribution is a crucial step in Bayesian estimation. The informative/non-informative distribution that reflects the prior information is distinguished [61]. Bayesian estimate Eq. (37) reduces the ML estimate Eq. (38) close to the prior distribution mean, depending on the sample size of the prior distribution. Hence, a common Bayesian estimator is the mean of the posterior distribution:

$$\hat{\theta}^{Bayes} = E[\theta|x, c, z)] \quad (37)$$

For Weibull mixture, in the case where z is known, the ML estimator is [61]:

$$\hat{\theta}^{MLE} = \arg\max_1^n \Upsilon\, lnL(x, c, z|\theta) \quad (38)$$

Where the likelihood function is

$$L(x, c, z|\theta) = \left[\prod_{i=1,c_1=0}^{n} \prod_{k=1}^{2} (\alpha_k . f_w(x_i|\beta_k, \eta_k)^{1(z_1=k)}\right] = \left[\prod_{i=1,c_1=0}^{n} \prod_{k=1}^{2} (\alpha_k . R_w(x_i|\beta_k, \eta_k)^{1(z_1=k)}\right] \quad (39)$$

4.6 Exponential Distribution

Although "maximum likelihood estimation (MLE)" is the main parameter estimation method used, "Markov Chain Monte Carlo (MCMC)" was used to estimate the parameters of Bayes estimators of "generalized exponential distribution" [62].

A "Bayesian study" of the model showed how "posterior simulations" based on "Markov chain Monte Carlo" algorithms can be direct and repetitive in R-codes. Likewise, Mobolaji et al. and presented the "exponential power distribution" as a reliability model software and carried out the "Bayesian analysis" in Open BUGS using "informative gamma priors" for the parameters [63].

The distribution function of "Generalized Exponential distribution" (GE) is considered as a simple structure; hence it could be effectively used to analyze lifetime data, especially for a non-negative random variable where a skewed distribution is needed [64]. The estimators of exponential distributions (one and two-parameters) were derived, using the squared error loss function [65]. In some cases, the Bayes estimators are derived in closed form. “Mean squared errors" (MSE) was used to investigate the performance and was compared with the maximum likelihood estimators.

The "exponential power (EP) model" with "shape parameter" $\gamma > 0$ and "scale parameter" $\alpha > 0$ is defined by the following "probability density function" (pdf) [66, 67].

$$f(t) = \frac{\gamma}{\alpha^\gamma} t^{\gamma-1} exp\left(\frac{t}{\alpha}\right)^\gamma exp\left[1 - exp\left(\frac{t}{\alpha}\right)^\gamma\right] \quad (40)$$

The corresponding "reliability and failure rate" functions are:

$$R(t) = exp\left[1 - exp\left(\frac{t}{\alpha}\right)^\gamma\right] \quad (41)$$

$$h(t) = \frac{\gamma}{\alpha^\gamma} t^{\gamma-1} exp\left(\frac{t}{\alpha}\right)^\gamma \quad (42)$$

4.7 Geometric Distribution

E-Bayesian estimation is a new method that computes estimates of the parameter and reliability function of the geometric distribution [68]. "One-parameter geometric (Geo (θ)) distribution” “probability mass function" is:

$$p(X;\theta) = \theta(1-\theta)^{x-1},\ \ x = 1,2,3,\dots\dots,\qquad 0<\theta<1 \tag{43}$$

The estimates are derived based on scale parameter conjugate prior. The balanced loss function is used for the computations. The balanced "loss function" includes both "symmetric and asymmetric". Examples using both actual and simulated record values were utilized to demonstrate the results. These results are interesting especially in a situation when only record values exist.

"Bayesian and E-Bayesian methods “were used to find the parameter estimators, "reliability", and "hazard functions" based on recorded statistics from a "geometric distribution". Assuming a "conjugate prior" for the parameter, Bayes estimators were derived on basis of a "balanced squared error loss" (BSEL) function. E-Bayesian properties are obtained ensuring the rule that the prior parameters α and βought to be selected to assure that f (θ|α,β) is a declining function of (θ) [69].

The derivative of f (θ|α,β) *w.r.t.* θ is:

$$\frac{df(\theta|\alpha,\beta)}{d\theta} = \frac{1}{B(\alpha,\beta)}\theta^{x-2}(1-\theta)^{\beta-2}[(\alpha-1)(1-\theta)-(\beta-1)\theta]. \tag{44}$$

"Monte Carlo simulation" was used for comparing the estimated target parameters from new methodology with that estimated from Bayesian and ML. The results showed that for real-life data the new estimation method "E-Bayesian" is efficient and performs justly.

4.8 Gamma Distribution

In a study for estimating residual life of devices in a mechanical system, an exponential degradation model was introduced to designate its characteristics [70]. The model was based on a "gamma-prior Bayesian updating" methodology and an "acceptance-rejection algorithm". Either historical or real-time observed data could be used.
In the case of a generalized Gamma mixture model (G ΓMM), Bayesian inference approach that emerges from a variational expectation-maximization procedure was proposed by Liu et al [71].Both shape and inverse scale parameters, in addition to GΓMM mixing coefficients, are considered as random variables. In this case, power parameters are just regarded as parameters. The author's proposed approach simplifies assigning the conjugate prior distributions. Applying the whole suggested process, the computational demand was reduced and the effective number of components could be resolved automatically. The method's effectiveness was verified experimentally.

Dehbozorgi & Nematollahi suggested a correction on the Das & Dey- Lemma derivation employing a logical transformation, instead of the "ad-hoc transformation" presenting the proper derivation using the two approaches [72].

The gamma function is defined by [67]:

$$\Gamma(x) = \int_o^{\infty} e^{-t}t^{x-1}dt for x > 0 \tag{45}$$

The "gamma function" satisfies the relation that "**Γ(x + 1) = xΓ(x)**"

Gamma distribution probability density function is given as:

$$g(\alpha) = \frac{1}{\gamma^k\Gamma(k)}\alpha^{k-1}e^{-\frac{\alpha}{\gamma}} \tag{46}$$

4.9 Normal Distribution

"Normal distribution" is the most generally used in statistics. "A random variable" T is assumed to be "normally distributed; with mean ν and variance τ^2, T~N (ν, τ^2)" when the probability density of T is:

$$f(t) = \frac{1}{\sqrt{2\pi}.\tau}e^{-\frac{(1-\nu)^2}{2\tau^2}} for \qquad -\infty < t < \infty \tag{47}$$

Gibbs sampler [74] is Using the projected normal distribution "Bayesian analysis"; of a regression model for circular data, the inferences also was using samples from the posterior densities that were acquired from the missing data problem was also addressed employing a "predictive criterion" for model selection. The authors presented an illustration of the procedure using two simulated datasets that were previously analyzed [74].

4.10 Generalized Lognormal Distribution

"Lognormal distributions" (LN) are usually adapted to model lifetime data that are employed in a reliability analysis. LN variable proposes appropriate fit for different types of data derived by experiments or observations due to inherited particular properties, Martín [74]. The generalized "lognormal" distribution (Log GN) is attained from the "exponential transformation" of a variable that is GN distributed. Analyzing the distribution from a Bayesian perspective, Gibbs sampling was applied to perform inferences. Generally, if a random variable X has a log GN distribution, the random variable Y = log X is normally distributed.

$$f(x) = \frac{s}{2x\sigma\Gamma(\frac{1}{s})} exp\left(-\left|\frac{logx-u}{\sigma}\right|^{s}\right) \quad (48)$$

With $x > 0$, $-\infty < \mu < +\infty$, $\sigma > 0$ and $s \geq 1$. $\boldsymbol{\Gamma}$ denotes the gamma function.

LN distribution is widely utilized to express positive random variables distribution. This case is applicable to health and biological data. Zou et al. [75], suggested procedures regarding ‘‘method of variance estimates recovery’’ (MOVER), while Krishnamoorthy and Mathew [76] discussed another approach "generalized confidence interval" which is based on simulation. Harvey& van der Merwe conducted a simulation study to assess the performance and the coverage accuracy of each of the two methods, using "Bayesian methodology “with a variety of "prior distributions" [77].

X. G. Hua et al. investigated "Bayesian finite element (FE) model" using "modal measurements" in "maximizing the posterior probability" rather than using any sampling method [78]. Most of the previously utilized Bayesian parameters updating employed normal distribution, although some statistical issues may prevail if non-negative parameters are used.

Various studies are related to city sizes modeling. Recently, Kwong & Nadarajah[78] proposed that city sizes are better modeled by "finite mixtures of exponential-type distributions “rather than by "Pareto" form of distributions. "Finite mixtures of LN distributions" are adopted when the dataset is relatively small. Considered as a generalized form of "LN distributions", the authors also investigated “finite mixtures of log-exponential distributions" (LEPs). Both distributions are eligible as an exponential-type distribution with closed-form density and distribution functions.

The probability density function f (x); taken as a mixture of K distributions with probability density functions $f_k(x)$, k = 1, 2. . . K. is modeled by:

$$f(x) = \sum_{k=1}^{k} \omega_k f_k(x), f_k(x) = \frac{1}{2x\sigma_k^2 \alpha_k^{\frac{1}{\alpha_k}} \Gamma(1+\frac{1}{\alpha_k})} exp\left(-\frac{1}{\alpha_k}\left|\frac{lnx-\mu_k}{\sigma_k^2}\right|^{\alpha_k}\right) \quad (49)$$

Where ω_k are the weights $\in$ (0, 1) and $\sum_{k=1}^{k} \omega_k = 1$, f(x) is considered a mixture of LN distributions when $\alpha_k = 2$ for all k = 1, 2… k

Based on the two investigated datasets, the authors concluded that it is right to suggest that mixtures of "LN distributions" are much more applicable than "Pareto type distributions" for modeling city sizes. However, other finite mixture distributions could offer adequate fit to these datasets as well.

5 Conclusions

"Bayesian method" is sensitive to the choice of "prior". Nevertheless, it generally performs well, when compared to "MLE", unless "priors" are poorly chosen. For example, the "prior distribution" should not be selected to result in "high values of the characteristic life and mixture ratio" simultaneously. Thus it is better a "prior" is chosen so that y is not overestimated.

The review shows that recently there has been a great interest in Bayesian models as most of the surveyed studies are very recent, in fact, a great number of them are published in 2018. Also, it could be seen that Bayesian models outperform a number of other methods. The following conclusions are deducted:

1. "Bayesian method" can very much decrease the errors in "Weibull parameter" estimates using "maximum likelihood estimation MLE" method as long as the "prior" is rightly selected.

2. The "lognormal distribution" is extensively applied for describing the distribution of positive random variables.

3. The "generalized form of the lognormal distribution", presented from a "Bayesian" point of view, offers the possibility of taking expert opinions into account.

4. "Combined normal and lognormal distributions" based on "Bayesian updating approach" demonstrated good efficiency in "FE model" updating. The methodology based on such "combined normal-lognormal distributions" demonstrates a capability in the identification of damages with good efficiency.

5. The "mixtures of LN distributions" are more suitable models than "Pareto type distributions" for estimating city sizes models.
6. "Bayesian confidence intervals" have to sustain "coverage probabilities" and sometimes perform better than "MOVER" and "generalized confidence interval".
7. "Bayesian inference procedures “may be extended to the difference between "two lognormal means".
8. Using "Quantum-like Bayesian network model" is advantageous in case of "classical model with latent variables" as it simulates both "observed and unobserved phenomena" in a single network.
9. "Maxim in Bayesian D-optimal designs" are highly efficient for cases of changing “priors".
10. Furthermore, the closed form studies obtained here opens the scope of further studies in mathematics, science, and engineering disciplines especially in some recent interesting practical works (see for examples, [40-47]).

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