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Numerical solution of fractional bioheat transfer model under moving heat source using wavelet method

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Abstract: In this article, an efficient wavelet collocation method is proposed for solving the fractional bioheat transfer model within the skin tissue under intense moving heat source. Unlike the existing operational methods based on orthogonal functions, we formulate the Haar wavelet operational matrices of fractional order integration without using the block pulse functions. The thermal damages in the skin tissues are investigated by the extent of the denatured protein employing with the Arrhenius equation. The temperature distribution over the skin tissue has been investigated for different values of the fractional parameter α and moving heat source velocity v. The numerical outcomes suggest that the desired temperature decrease with an decrease in the order of the fractional derivative and moving heat source velocity. The numerical outcomes of thermal injuries and temperatures distributions are introduced graphically.

Keywords: Bioheat model, Thermal Damage, Haar wavelet, Collocation method, Operational matrices, Fractional derivative.

1 Introduction

In recent times, the field of medical science has witnessed a tremendous growth especially in the development of bio-mathematics, which in essence is the intertwining of biological science and mathematics [1,2,3,4]. The mathematical study of heat transfer in living tissues is one of core areas of research and different mathematical approaches are being regularly employed to understand the mechanism of heat transfer in biological tissues. The use of heat to destroy unwanted tissue has found immense applications in therapeutic purposes, for instance, laser, microwave and magnetic fluid hyperthermia [1]. Between these therapies, the moving heat sources applications to living tissue considering exudation rate is observed in certain plastic surgery operation such as the removal of moles or laser tattoos or the thermal action of the cornea to correct hyperopia. Therefore, appropriate modelling and analysis of the underpinning therapy play a vital role in optimizing the temperature distribution in the region of treatment. As a result, several bioheat transfer models have been proposed and successfully applied for modelling of heat inside the living biological tissues. Among several available mathematical models in the open literature, the Pennes bioheat transfer model [2] has gained a respectable status mainly due to its simplicity and lucid nature. Mathematically, the Pennes bioheat transfer model is described by a second order partial differential equation

$$\rho c\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + \rho_b\omega_b c_b(T_b - T) + Q_m + Q_{ext}, \tag{1.1}$$

where the parameters $\rho, c, k, \rho_b, \omega_b, c_b, T_b$, and T involved in the equation (1.1) denote density of the tissue, specific heat of the tissue, thermal conductivity of the tissue, blood mass density, blood perfusion rate, specific heat of the blood, temperature of arterial blood, and tissue temperature, respectively. The terms Q_m and Q_{ext} involved in (1.1) represent the heat generated by the metabolic process and moving line heat source. Since model (1.1) is based on the Fourier's law of heat conduction that depicts infinite speed of thermal wave, which is highly unrealistic due to the non-homogeneous nature of the biological tissues. To overcome this infeasibility, Cattaneo [5] and Vernotte [6] independently proposed a modified version of the Fourier's law of heat conduction by introducing phase-lag term τ_r in the time variable and combined with

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the Pennes model (1.1), a general form of bioheat transfer model in skin tissue is established by [7]

$$k\frac{\partial^2 T}{\partial x^2} = \left(1+\tau_r\frac{\partial}{\partial t}\right)\left(\rho c\frac{\partial T}{\partial t}+\rho_b\omega_b c_b(T-T_b)-Q_m-Q_{ext}\right), \tag{1.2}$$

On the other hand, fractional calculus has fascinated the scientific community due to its widespread applications in mathematics, engineering and biological sciences. Indeed, it has been recognized as one of the valuable tools to describe many physical phenomena such as heat conduction in biological and physical processes [8]. Motivated and inspired by the fact that the fractional order models are more exact and accurate than the conventional integer-order models, we consider a fractional form of bioheat transfer model (1.2) by replacing the classical time derivative with the fractional derivative in Caputo sense so that the governing equation takes the form [9]

$$k\frac{\partial^2 T}{\partial x^2} = \left(1+\frac{\tau_r^\alpha}{\Gamma(\alpha+1)}\frac{\partial^\alpha}{\partial t^\alpha}\right)\left(\rho c\frac{\partial T}{\partial t}+\rho_b\omega_b c_b(T-T_b)-Q_m-Q_{ext}\right), \tag{1.3}$$

where τ_r is the thermal lagging time. Till date, a series of numerical and analytical techniques have been successfully applied for solving the model (1.3). For instance, finite-decomposition method [10], homotopy perturbation method [11], Galerkin method [12], finite difference method [13], Laplace transform method [14,15].

Wavelets are the recent addition to the class of orthogonal functions with magnificent and attractive features: orthogonality, compact support, arbitrary regularity, and good localization [16,17]. As an outcome, they are broadly used in seeking numerical solutions of various types of differential equations emerging in various disciplines of physical, chemical and biological sciences. During the last decade or so, the operational matrices of integration for the Haar wavelets, Legendre wavelets, Chebyshev wavelets, CAS wavelets, Bernoulli wavelets and the spline wavelets have been developed in order to solve a wide variety of differential, integral and integro-differential equations. In particular, considerable interest has been stimulated for the numerical solution of the Pennes bioheat transfer model using different wavelet methods. For instance, Kumar et al.[18] have obtained a numerical solution of the hyperbolic space-fractional bioheat transfer model using wavelet-Galerkin and finite element methods, whereas the same strategy has been followed by Pandey et al.[19] to investigate the one-dimensional bioheat transfer model (1.1) in cylindrical living tissues. Recently, Awana and Shah [20] have successfully employed a wavelet collocation method based on Haar wavelets for the numerical solution of the classical bioheat transfer model (1.1).

The main goal of this article is to present and analyze a new stable algorithm for the numerical solution of modified form of fractional bioheat transfer model (1.3) based upon the constructed Haar wavelet operational matrix of integration without invoking the block pulse functions. The motivation and philosophy behind this approach is that it converts the underlying problem to a set of algebraic equations by expanding the term, which has maximum derivative, given in the equation as Haar functions with unknown coefficients and thus, simplifying the solution process of the problem to a significant extent.

The remainder of the article is organized as follows. Mathematical formulation of the fractional bioheat transfer model (1.3) is presented in Section 2. Section 3 is devoted to the derivation of Haar wavelet operational matrix of fractional order integration without using the block pulse functions. The method of solution is illustrated in Section 4. In Section 5, we discuss the obtained numerical results. Finally, a conclusion is drawn in Section 6.

2 Formulation of the Model

In this Section, we formulate the thermal waves model of fractional bioheat equation using the machinery of Caputo derivative and the classical Fourier's law of heat conduction. The traditional Fourier law of heat conduction is given by

$$q(x,t) = -k\nabla T(x,t), \tag{2.1}$$

where x,t,k and T denote the space variable, time variable, thermal conductivity and temperature, respectively. From equation (2.1), it assumes that the heat flux vector $q(x,t)$ and temperature gradient ∇T implies unphysical infinite propagation speed of thermal disturbance. In order to overcome this unphysical behavior, a modified constitutive relation was proposed by Cattane[5] and Vernotte[6] independently, by introducing a thermal lag time in Fourier's law of heat conduction (2.1):

$$q(x,t+\tau_r) = -k\nabla T(x,t). \tag{2.2}$$

where τ_r is the thermal lagging time. Now by applying the theory of partial time derivative of the fractional order α $(0<\alpha\leq 1)$ proposed by Ezzat et.al.[9], the Fourier law (2.2) takes the form

$$-k\nabla T(x,t)=\left(1+\frac{\tau_r^{\alpha}}{\Gamma(\alpha+1)}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)q(x,t+\tau_r),\quad 0\leq\alpha\leq 1. \tag{2.3}$$

where

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}f(t)=D^{\alpha}f(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t\frac{f^m(\tau)}{(t-\tau)^{\alpha-m+1}}\,d\tau,\quad m-1<\alpha\leq m,m\in\mathbb{N}. \tag{2.4}$$

and the notation D^{α} denotes the Caputo fractional derivative of a function $f(t)$. Some of the basic properties of Caputo fractional derivative are as follows [21]:

(i) $D^{\alpha}\left(\gamma f(t)+\delta g(t)\right)=\gamma D^{\alpha}f(t)+\delta D^{\alpha}g(t)$, where γ,δ are constants,

(ii) $D^{\alpha}t^{\beta}=\dfrac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}t^{\beta-\alpha},\quad 0<\alpha<\beta+1,\beta>-1;$

(iii) $J^{\alpha}D^{\alpha}f(t)=f(t)-\sum\limits_{k=0}^{m-1}f^k(0^+)\dfrac{t^k}{k!},\quad m-1<\alpha\leq m,m\in\mathbb{N};$

(iv) $D^{\alpha}C=0,\quad C$ is a constant.

Based on the model (2.3), proposed by the Ezzat et.al. and the Pennes model, the modified form of the fractional bioheat transfer model is obtained as

$$k\frac{\partial^2 T}{\partial x^2}=\left(1+\frac{\tau_r^{\alpha}}{\Gamma(\alpha+1)}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\left(\rho c\frac{\partial T}{\partial t}+\rho_b\omega_b c_b(T-T_b)-Q_m-Q_{ext}\right), \tag{2.5}$$

with initial and boundary conditions given by

$$T(x,0)=T_b,\quad -k\frac{\partial T(x,t)}{\partial x}\bigg|_{x=0}=0,\quad -kT(x,t)\big|_{x=R}=0. \tag{2.6}$$

Consider the dimensionless variables

$$\left.\begin{aligned}&\theta=\frac{T-T_b}{T_b},\ \ \tau_o=\frac{\omega_b\rho_b c_b}{\rho c}\tau_r,\ \ \zeta=\frac{\omega_b\rho_b c_b}{\rho c}t,\ \ \eta=\sqrt{\frac{\omega_b\rho_b c_b}{k}}x,\\&Q_m=\frac{Q_m}{\omega_b\rho_b c_b}T_b,Q_{ext}=\frac{Q_{ext}}{\omega_b\rho_b c_b}T_b\end{aligned}\right\}. \tag{2.7}$$

Subsequently, the system of equations (2.5)–(2.6), reduces to the following dimensionless form

$$\left.\begin{aligned}&\frac{\partial^2\theta}{\partial\eta^2}=\left(1+\frac{\tau_o^{\alpha}}{\Gamma(\alpha+1)}\frac{\partial^{\alpha}}{\partial\zeta^{\alpha}}\right)\left(\frac{\partial\theta}{\partial\zeta}+\theta-Q_m-Q_{ext}\right),\\&\theta(\eta,0)=0,\quad \frac{\partial\theta(\eta,\zeta)}{\partial\eta}\bigg|_{\eta=0}=0,\quad \theta(\eta,\zeta)\big|_{\eta=1}=0\end{aligned}\right\}. \tag{2.8}$$

The dimensionless form of the external heat source $Q_{ext}(\eta,\zeta)$ is a movable thermal resource which can be expressed as [22]

$$Q_{ext}(\eta,\zeta)=Q_o\delta(\eta-v\zeta), \tag{2.9}$$

where Q_o,v and δ denote the constant, velocity and delta function, respectively.

2.1 Evaluation of thermal damage

The main concern in the bio-engineering sciences is to accurately assess the optimal temperature during heat therapy to prevent thermal damage to the skin tissue. The most reliable method for quantifying the thermal damage is the one developed by Henriques and Moritz [23,24]. The expression governing the assessment of thermal damage reads as:

$$\Omega = \int_0^{\zeta} B \exp\left(-\frac{E_a}{R\theta}\right), \tag{2.10}$$

where Ω, $B = 3.1 \times 10^{98}\ s^{-1}$, $E_a = 6.28 \times 10^5$ J/mol, and $R = 8.313$ J/mol are the thermal damage index, frequency factor, activation energy and universal gas constant, respectively.

3 Haar Wavelets and Operational Matrices of Fractional Order

Haar wavelet is the first known wavelet proposed by Hungarian mathematician Alfred Haar in 1910. The Haar wavelet is the simplest and oldest orthonormal wavelet having compact support. The Haar wavelet family for $\eta \in [0,1]$ is defined as follows:

$$h_i(\eta) = h_i(2^j\eta - k) = \begin{cases} 1, & \frac{k}{2^j} \le \eta < \frac{k+0.5}{2^j} \\ -1, & \frac{k+0.5}{2^j} \le \eta < \frac{k+1}{2^j} \\ 0, & \text{elsewhere} \end{cases} \tag{3.1}$$

where $i = 0,1,2,\ldots,m-1, m = 2^M$ and M is a positive integer which is called the maximum level of resolution. Here, j and k represent the integer decomposition of the index i; that is, $i = k + 2^j - 1, 0 \le j < i$ and $1 \le k < 2^j + 1$. For more about Haar wavelets and their applications, we refer to the monographs [16,17].

By invoking the Haar basis functions, any square integrable function $\theta \in L^2[0,1]$ can be expressed as

$$\theta(\eta) = d_0 h_0(\eta) + d_1 h_1(\eta) + d_2 h_2(\eta) + \cdots = \sum_{i=0}^{\infty} d_i h_i(\eta), \tag{3.2}$$

where the Haar coefficients d_i, $i = 0,1,2,\ldots$, are given by

$$d_i = \langle \theta, h_i \rangle = \int_0^1 \theta(\eta) h_i(\eta)\, d\eta. \tag{3.3}$$

Although the series expansion (3.2) is an infinite sum, we can reasonably approximate $\theta(\eta)$ by using finitely many terms, provided $\theta(\eta)$ is a piecewise constant or it may be approximated as a piecewise constant function over each sub-interval; that is,

$$\theta(\eta) \backsimeq \theta_m(\eta) = \sum_{i=0}^{m-1} d_i h_i(\eta). \tag{3.4}$$

An analogous expression for (3.4) in the matrix form is given by

$$\psi = \mathbf{D}_m^T \mathbf{H}_m, \tag{3.5}$$

where ψ represents the discrete form of the continuous function $\theta(\eta)$ and $\mathbf{D}_m^T = [d_0, d_1, \ldots, d_{m-1}]$ is the m-dimensional row vector. Moreover, $\mathbf{H}_m$ denotes the Haar wavelet matrix of order $m = 2^M$ and is given by $\mathbf{H}_m = [\mathbf{h}_0, \mathbf{h}_1, \ldots, \mathbf{h}_{m-1}]^T$; that is,

$$H_m = \begin{pmatrix} h_0 \\ h_1 \\ \vdots \\ h_{m-1} \end{pmatrix} = \begin{pmatrix} h_{0,0} & h_{0,1} & \ldots & h_{0,m-1} \\ h_{1,0} & h_{1,1} & \ldots & h_{1,m-1} \\ \vdots & \vdots & \vdots & \vdots \\ h_{m-1,0} & h_{m-1,1} & \ldots & h_{m-1,m-1} \end{pmatrix}. \tag{3.6}$$

For obtaining the Haar wavelet approximations, we consider the following collocation points:

$$\eta_\ell = \zeta_\ell = \frac{\ell - 0.5}{m}, \quad \ell = 1,2,\ldots,m. \tag{3.7}$$

3.1 Operational Matrices of Fractional Order

We now intend to construct operational matrices of fractional order integration without using the block pulse functions. The integration of the vector $H_m(\eta) = [h_0(\eta), h_1(\eta), \ldots, h_{m-1}(\eta)]^T$ can be approximated by Haar series as [25,26]:

$$\int_0^\eta H_m(\eta)d\eta \cong U_m H_m(\eta), \tag{3.8}$$

where U_m is called the Haar wavelet operational matrix of integration of order m. Using the equation (2.4), we can construct the Haar operational matrix of fractional order integration U^α as

$$\begin{aligned} U_m^\alpha H_m(\eta) = J^\alpha H_m(\eta) &= \left[J^\alpha h_0(\eta), J^\alpha h_1(\eta), \ldots, J^\alpha h_{m-1}(\eta)\right]^T, \\ &= \left[U_m h_0(\eta), U_m h_1(\eta), \ldots, U_m h_{m-1}(\eta)\right]^T, \end{aligned} \tag{3.9}$$

where

$$U_m h_0(\eta) = \frac{1}{\sqrt{m}}\,\frac{\eta^\alpha}{\Gamma(1+\alpha)}, \quad 0 \le \eta \le 1, \tag{3.10}$$

$$U_m h_i(\eta) = \frac{1}{\sqrt{m}} \begin{cases} 0, & 0 \le \eta < \dfrac{k-1}{2^j} \\ 2^{j/2}\phi_1(\eta), & \dfrac{k-1}{2^j} \le \eta < \dfrac{k-0.5}{2^j} \\ 2^{j/2}\phi_2(\eta), & \dfrac{k-0.5}{2^j} \le \eta < \dfrac{k}{2^j} \\ 2^{j/2}\phi_3(\eta), & \dfrac{k}{2^j} \le \eta < 1 \end{cases} \tag{3.11}$$

$$\begin{aligned} \phi_1(\eta) &= \frac{1}{\Gamma(\alpha+1)}\left(\eta - \frac{k-1}{2^j}\right)^\alpha, \\ \phi_2(\eta) &= \frac{1}{\Gamma(\alpha+1)}\left(\eta - \frac{k-1}{2^j}\right)^\alpha - \frac{2}{\Gamma(\alpha+1)}\left(\eta - \frac{k-0.5}{2^j}\right)^\alpha, \\ \phi_3(\eta) &= \frac{1}{\Gamma(\alpha+1)}\left(\eta - \frac{k-1}{2^j}\right)^\alpha - \frac{2}{\Gamma(\alpha+1)}\left(\eta - \frac{k-0.5}{2^j}\right)^\alpha + \frac{1}{\Gamma(\alpha+1)}\left(\eta - \frac{k}{2^j}\right)^\alpha. \end{aligned}$$

4 Method of Solution

The present section is devoted to solving the fractional bioheat transfer model (2.8) in skin tissue by employing the Haar wavelet operational matrices of fractional order integration constructed in Section 3. To facilitate the method of solution, we recall the dimensionless form of the model (2.8)

$$\frac{\partial^2\theta}{\partial\eta^2} = \left(1 + \frac{\tau_o^\alpha}{\Gamma(\alpha+1)}\frac{\partial^\alpha}{\partial\zeta^\alpha}\right)\left(\frac{\partial\theta}{\partial\zeta} + \theta - Q_m - Q_{ext}\right), \tag{4.1}$$

subjected to the conditions

$$\theta(\eta,0) = 0, \quad \left.\frac{\partial\theta(\eta,\zeta)}{\partial\eta}\right|_{\eta=0} = 0, \quad \left.\theta(\eta,\zeta)\right|_{\eta=1} = 0. \tag{4.2}$$

Next, the Haar wavelet expansion for approximating the highest order derivative $\dfrac{\partial^3\theta}{\partial\eta^2\partial\zeta}$ is given by

$$\frac{\partial^3\theta}{\partial\eta^2\partial\zeta} \approx H_m^T(\eta) D H_m(\zeta). \tag{4.3}$$

Integrating (4.3), with respect to ζ from 0 to ζ and using conditions (4.2), we obtain

$$\begin{aligned}\frac{\partial^2\theta}{\partial\eta^2} &= \int_0^\zeta \frac{\partial^3\theta}{\partial\eta^2\partial\zeta}d\zeta + \frac{\partial^2\theta}{\partial\eta^2}\Big|_{\zeta=0} \\ &\approx \int_0^\zeta \left[H_m^T(\eta)DH_m(\zeta)\right]d\zeta + \frac{\partial^2\theta}{\partial\eta^2}\Big|_{\zeta=0} \\ &= H_m^T(\eta)DU^1_{m\times m}H_m(\zeta). \end{aligned} \tag{4.4}$$

Again integrating (4.3), twice with respect to η from 0 to η, then we get

$$\begin{aligned}\frac{\partial\theta}{\partial\zeta} &= \int_0^\eta\int_0^\eta \frac{\partial^3\theta}{\partial\eta^2\partial\zeta}d\eta + \eta\frac{\partial\theta}{\partial\zeta}\Big|_{\eta=0} + \frac{\partial\theta}{\partial\zeta}\Big|_{\eta=0} \\ &\approx \int_0^\eta\int_0^\eta [H_m^T(\eta)DH_m(\zeta)]d\eta + \eta\frac{\partial\theta}{\partial\zeta}\Big|_{\eta=0} + \frac{\partial\theta}{\partial\zeta}\Big|_{\eta=0} \\ &= H_m^T(\eta)[U^2_{m\times m}]^T DH_m(\zeta). \end{aligned} \tag{4.5}$$

Now integrating (4.5), with respect to ζ from 0 to ζ, then we obtain

$$\begin{aligned}\theta(\eta,\zeta) &\approx H_m^T(\eta)[U^2_{m\times m}]^T DU^1_{m\times m}H_m(\zeta) + \theta(\eta,0) \\ &= H_m^T(\eta)[U^2_{m\times m}]^T DU^1_{m\times m}H_m(\zeta). \end{aligned} \tag{4.6}$$

To approximate the fractional order derivative appearing in (4.4), we shall make use of the properties of Caputo derivative as

$$\begin{aligned}\frac{\partial^\alpha\theta(\eta,\zeta)}{\partial\zeta^\alpha} &= J^{1-\alpha}\left(\frac{\partial\theta(\eta,\zeta)}{\partial\zeta}\right) \\ &= J^{1-\alpha}\left[H_m^T(\eta)[U^2_{m\times m}]^T DH_m(\zeta)\right] \\ &= H_m^T(\eta)[U^{2-\alpha}_{m\times m}]^T DH_m(\zeta). \end{aligned} \tag{4.7}$$

Substituting (4.4)–(4.7) in (4.1), we obtain the following system of algebraic equations

$$\begin{aligned}&H^T(\eta)DU^1H(\zeta) - \frac{\tau_o^\alpha}{\Gamma(\alpha+1)}H^T(\eta)[U^2]^T DU^{2-\alpha}H(\zeta) - H^T(\eta)[U^2]^T DH(\zeta) \\ &- \frac{\tau^\alpha}{\Gamma(\alpha+1)}H^T(\eta)[U^2]^T DU^{1-\alpha}H(\zeta) - H^T(\eta)[U^2]^T DU^1H(\zeta) = -Q_m - Q_{ext}. \end{aligned} \tag{4.8}$$

We can solve the above system of algebraic equations (4.8) to obtain the unknown coefficient vector D. Upon substituting the obtained value of D in (4.6), we shall obtain the desired approximation solution $\theta(\eta,\zeta)$.

5 Numerical Results and Discussion

The aim of this section is to discuss the numerical outcomes of the proposed method for solving fractional bioheat transfer model (2.8) in skin tissue. To facilitate the numerical computations, we use some typical thermophysical properties of skin tissue. Some of the properties that are employed in this paper are [27]: $Q_0 = 2\times10^3 Wm^{-2}, Q_m = 1.19\times10^3 Wm^{-3}, \rho = 1000 kgm^{-3}, \rho_b = 1060 kgm^3, c_b = 3860 Jkg^{-1}, \omega_b = 1.87\times10^{-3}s^{-1}, T_b = 37^\circ C, c = 4187 Jkg^{-1}k^{-1}, \tau_0 = 16s, R = 0.05m$. All the computations work was carried out by using MATLAB (R2019a) software.

In order to see the impact of the fractional model during thermal therapy, we first present a graphical comparison of the fractional bioheat model (1.3) with the classical bioheat model (1.1) in fig.1 for fixed time $\zeta = 3$ min. Fig 1 clearly demonstrates that the fractional bioheat model attains higher temperature than the classical bioheat model, moreover, it is quite evident that the temperature predicted in fractional case will always be greater than that of classical case. To examine the influence of the fractional parameter α on the temperature distribution over the skin tissue, we plot the dimensionless temperature profile for $\alpha = 0.1, 0.3, 1$ for fixed time $\zeta = 3$ min in Fig. 2. It is observed from Fig. 2, that the temperature profile in the skin tissue decreases to the normal temperature as the value of α increases, which in turn

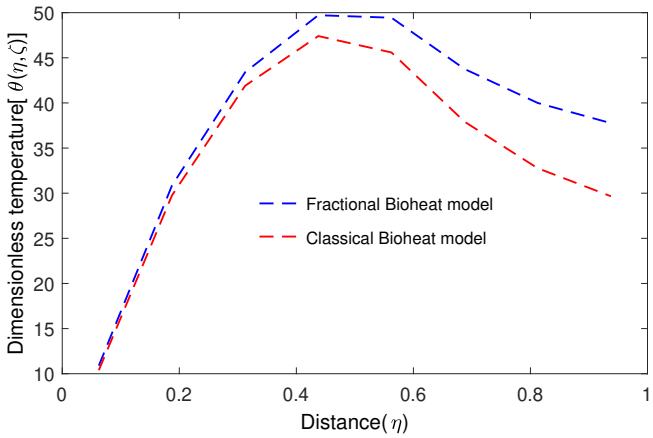

Fig. 1: Comparison between fractional and classical bioheat model for $\zeta = 3$ min.

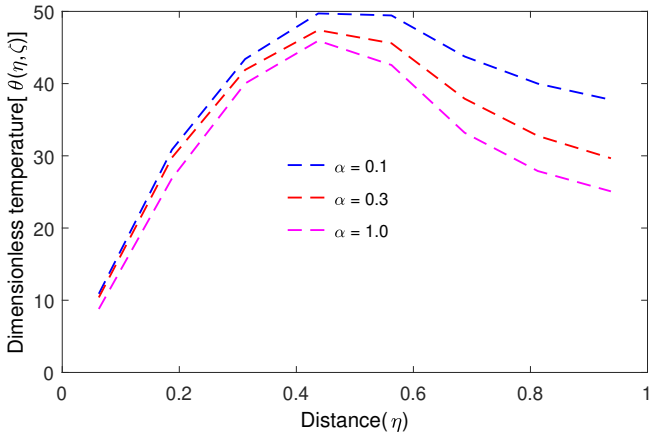

Fig. 2: Dimensionless temperature(θ) vs η for different values of α and $\zeta = 3$ min.

exhibits the appreciable effect of fractional derivative in the temperature distribution during thermal therapy process. Fig. 3 demonstrates the comparison of temperature distribution for different heat source velocities $v = 0.2, 0.3, 0.4$, with respect to distance η. It is worth noticing that the moving heat source provides temperature distribution on the skin surface which depends on its velocity. Moreover, the temperature distribution gradually increases and attains the peak value at a particular location when the value of moving heat source decreases. In continuation, Fig. 4, shows the the influence of the fractional parameter on the thermal damage of skin tissue. It is quite evident from Fig. 4, that the thermal damage decreases with the increase in fractional parameter α. As expected, the fractional order parameter has a great effect on the distribution of field quantities.

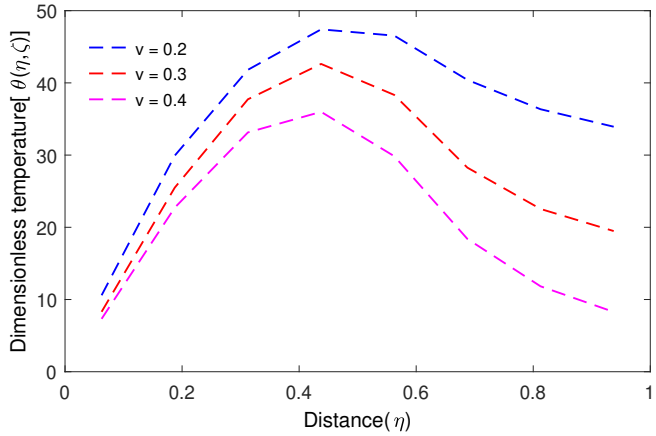

Fig. 3: Dimensionless temperature(θ) vs η for different values of heat source velocity(v).

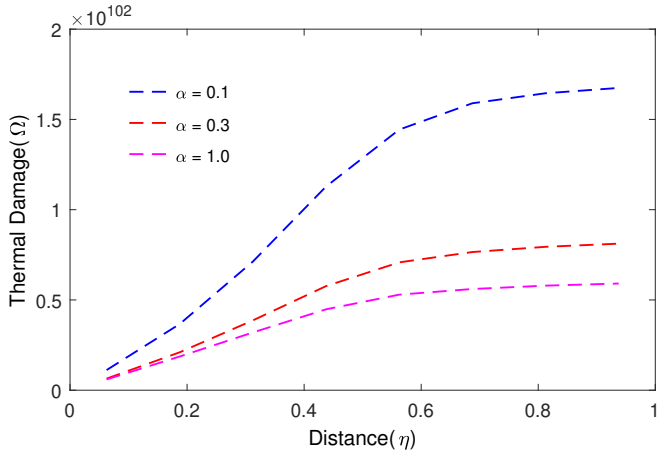

Fig. 4: Thermal Damage(Ω) vs η for different values of fractional parameter(α).

6 Conclusion

In this work, an efficient wavelet collocation method of fractional order integration based on Haar wavelets is derived and used to solve the fractional bioheat transfer model (2.8) in skin tissue under concentric moving heat source. The temperature distribution in skin tissue is investigated for different values of the fractional parameter α and it has been observed that the temperature in skin tissue during the thermal therapy is significantly dependent upon the time fractional order derivative α and varies inversely with α. It is hoped that the proposed method can be employed for solving a wide range of physical and biological problems.

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