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Existence Results for a New Fractional Boundary Value Problem by Variational Methods

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Abstract: In this paper, using techniques from fractional variational calculus and some critical point theorems, we prove the existence of weak solution. Then, we deduce the existence of solution for the following fractional boundary value problem:

$$\begin{cases} {}_tD_T^{\alpha}({}_0D_t^{\alpha}k(t)) = f(t,\ {}_0D_t^{\alpha}k(t)), \ \ a.e.\ \ t\in[0,T], \\ k^{(j)}(0)=0, \ \ j=0,1,2,...,2(n-1), \\ k^{(l)}(T)=0, \ \ l=0,1,2,...,n-1, \end{cases}$$

where ${}_tD_T^{\alpha}$ and ${}_0D_t^{\alpha}$ are the right and left Riemann-Liouville fractional derivatives of order $n-1<\alpha<n$ which is a generalization of previous results and $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying some assumptions. We propose two examples to illustrate our results.

Keywords: Fractional boundary value problem, fractional variational calculus, critical point theorems, weak solution.

1 Introduction

A look at recent investigations shows that the mathematical modeling of many physical and engineering processes need to use fractional differential or integral operators, see [1]. Fractional calculus is an old branch of differential and integral calculus, with its basic idea attributed to "$L^{\cdot}Hopital$" and several relevant applications can be found in the study of neorons, electrochemistry, control, porous media, electromagnetic, ...etc. (see [2–9]). In recent years, many researches have addressed the existence of solution for fractional boundary value problems (see [10–18]).

Variational calculus is a generalization of differential calculus of functions like $f:\mathbb{R}^n\to\mathbb{R}^m$ to functionals on function spaces [19]. We know that a function space is a set of functions from X to Y, where commonly X and Y are equipped with different mathematical structures like topology, metric, norm, ...etc. Fractional variational calculus, was raised by "Riewe" [20,21], that presented a new method to get the nonconservative systems using specific functionals. Some researches were conducted the existence results of fractional differential equations using variational methods and critical point theory explained, as follows:

In [22], Jiao and Zhou presented a new method to study the existence of solutions of the following fractional boundary value problem:

$$\begin{cases} \frac{d}{dt}(\frac{1}{2}\,{}_0D_t^{-\rho}(v'(t))+\frac{1}{2}\,{}_tD_T^{-\rho}(v'(t))+\nabla F(t,v(t))=0, \\ v(0)=v(T)=0, \ \ a.e.\ t\in[0,T], \end{cases}$$

where ${}_0D_t^{-\rho}$ and ${}_tD_T^{-\rho}$ are the left and right Riemann-Liouville fractional integrals of order $0\le\rho<1$, F is a given function and $\nabla F(t,x)$ is the gradient of F at x.

In [23], Jiao and Zhou examined the fractional problem:

$$\begin{cases} {}_tD_T^{\tau}({}_0D_t^{\tau}(\delta(t)) = \nabla F(t,\delta(t)), \ \ a.e.\ t\in[0,T], \\ \delta(0)=\delta(T)=0, \ 0<\tau<1, \end{cases}$$

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they proved the existence of solutions for the above problem.

In this work, we consider the following boundary value problem:

$$\begin{cases} {}_tD_T^{\alpha}({}_0D_t^{\alpha}k(t)) = f(t,\ {}_0D_t^{\alpha}k(t)),\ \ a.e.\ t\in[0,T], \\ k^{(j)}(0)=0,\ \ j=0,1,2,...,2(n-1), \\ k^{(l)}(T)=0,\ \ l=0,1,2,...,n-1, \end{cases} \tag{1}$$

where $n-1<\alpha<n$, $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function and satisfies some assumptions. Using fractional variational calculus and some critical point theorems like Minimization principle and Mountain pass theorem, we show that under certain assumptions the critical points of defined variational functional on the suitable Hilbert space are the solutions of fractional BVP. We discuss weak solution and solution of the fractional BVP (1).

The rest of this paper is organized, as follows: we propose the definitions of the fractional calculus. We divide Sec.3 into two subsections, we first develope a fractional derivative space. Next, we will present some critical point theorems and prove the existence of weak solution for fractional BVP (1). In the second subsection, we will give some existence results of solution for fractional BVP (1), then two examples end this paper.

2 Preliminaries

In this section, definitions of fractional integral, fractional derivative and some properties of fractional calculus are presented.

Definition 1.([24]) *For $\rho>0$, the left (right) Riemann-Liouville fractional integral operator of order ρ of a function $g:[0,T]\to\mathbb{R}$ is given by*

$$ {}_0I_w^{\rho}g(w)=\frac{1}{\Gamma(\rho)}\int_0^w\frac{g(s)ds}{(w-s)^{1-\rho}},\quad w\in[0,T],$$

$$ {}_wI_T^{\rho}g(w)=\frac{1}{\Gamma(\rho)}\int_w^T\frac{g(s)ds}{(s-w)^{1-\rho}},\quad w\in[0,T]. \tag{2}$$

Here and in what follows, Γ denotes the Euler's Gamma function. If $g\in L^1$, then ${}_0I_w^{\rho}g$ and ${}_wI_T^{\rho}g$ are defined almost everywhere on $(0,T)$.

Definition 2.([24]) *For $m-1<\rho<m$ ($m\in\mathbb{N}$), the left (right) Riemann-Liouville fractional derivative operator of order ρ of a function $g:[0,T]\to\mathbb{R}$ is given by*

$$ {}_0D_w^{\rho}g(w)=\frac{d^m}{dw^m}\,{}_0I_w^{m-\rho}g(w)=\frac{1}{\Gamma(m-\rho)}\frac{d^m}{dw^m}\int_0^w\frac{g(s)ds}{(w-s)^{\rho-m+1}},w\in[0,T],$$

$$ {}_wD_T^{\rho}g(w)=(-1)^m\frac{d^m}{dw^m}\,{}_wI_T^{m-\rho}g(w)=(-1)^m\frac{1}{\Gamma(m-\rho)}\frac{d^m}{dw^m}\int_w^T\frac{g(s)ds}{(s-w)^{\rho-m+1}},w\in[0,T].$$

According to [24], the next result characterizes the conditions for the existence of the fractional derivatives ${}_0D_w^{\rho}$ and ${}_wD_T^{\rho}$ in the space $AC^k([0,T],\mathbb{R})$, ($k\in\mathbb{N}=1,2,3,...$) the space of real-valued functions $g(w)$ which have continuous derivatives up to order $k-1$ on $[0,T]$ such that $g^{k-1}\in AC([0,T],\mathbb{R})$ the space of functions $g(w)$ which are absolutely continuous on $[0,T]$. $AC^k([0,T],\mathbb{R})=\{g:[0,T]\to\mathbb{R}$ and $(D^{k-1}g)(w)\in AC([0,T],\mathbb{R})$ $(D=\frac{d}{dw})\}$. In particular, $AC^1[0,T]=AC[0,T]$.

Lemma 1.([24]) *Let $\rho>0$. If $g(w)\in AC^k([0,T],\mathbb{R})$, then the fractional derivatives ${}_0D_w^{\rho}$ and ${}_wD_T^{\rho}$ exist almost everywhere on $[0,T]$ and can be represented in the forms*

$$ {}_0D_w^{\rho}g(w)=\sum_{j=0}^{k-1}\frac{g^{(j)}(0)}{\Gamma(j-\rho+1)}w^{j-\rho}+\frac{1}{\Gamma(k-\rho)}\int_0^w\frac{g^{(k)}(s)ds}{(w-s)^{\rho-k+1}},\quad w\in[0,T], \tag{3}$$

$$ {}_wD_T^{\rho}g(w)=\sum_{j=0}^{k-1}\frac{(-1)^jg^{(j)}(T)}{\Gamma(j-\rho+1)}(T-w)^{j-\rho}+\frac{(-1)^k}{\Gamma(k-\rho)}\int_w^T\frac{g^{(k)}(s)ds}{(s-w)^{\rho-k+1}},\quad w\in[0,T]. \tag{4}$$

Lemma 2.([24]) *If $\rho,\tau>0$ and $y(t)\in L^p([0,T],\mathbb{R})$ $(1\le p\le\infty)$, then the following equations are satisfied*

$$_0I_t^{\rho}\ _0I_t^{\tau}\ y(t)=\ _0I_t^{\rho+\tau}\ y(t),\quad a.e\ on\ [0,T],$$

$$_tI_T^{\rho}\ _tI_T^{\tau}\ y(t)=\ _tI_T^{\rho+\tau}\ y(t),\quad a.e.\ on\ [0,T].$$

Lemma 3.([24]) *If $\tau>0$ and $y(t)\in L^p([0,T],\mathbb{R})$, $(1\le p\le\infty)$, then the following equalities hold*

$$_0D_t^{\tau}\ _0I_t^{\tau}\ y(t)=y(t),\quad a.e.\ on\ [0,T],$$

$$_tD_T^{\tau}\ _tI_T^{\tau}\ y(t)=y(t),\quad a.e.\ on\ [0,T].$$

According to ([24], Lemma 2.5, p.74), before presenting the next lemma, we must define the spaces of functions $_0I_t^{\tau}(L^p)$, $_tI_T^{\tau}(L^p)$ for $\tau>0$ and $(1\le p\le\infty)$.

$$_0I_t^{\tau}(L^p)=\{y:y=\ _0I_t^{\tau}\varphi,\varphi\in L^p(0,T)\},\tag{5}$$

$$_tI_T^{\tau}(L^p)=\{y:y=\ _tI_T^{\tau}\phi,\phi\in L^p(0,T)\}.\tag{6}$$

Lemma 4.([24]) *Let $\rho>0$, $n-1<\alpha<n$ and $y_{n-\rho}(t)=(_0I_t^{n-\rho}y)(t)$ be the fractional integral (2) of order $n-\rho$.*

(a)If $(1\le p\le\infty)$ and $y(t)\in\ _0I_t^{\rho}(L^p)$, then $(_0I_t^{\rho}\ _0D_t^{\rho}y)(t)=y(t)$.
(b)If $y(t)\in L^1(0,T)$ and $y_{n-\rho}(t)\in AC^n[0,T]$, then the following equality holds:

$$(_0I_t^{\rho}\ _0D_t^{\rho}y)(t)=y(t)-\sum_{j=1}^{n}\frac{y_{n-\rho}^{(n-j)}(0)}{\Gamma(\rho-j+1)}t^{\rho-j},\quad a.e.\ \ on\ [0,T].$$

Lemma 5.([24]) *Let $\rho>0$, $n-1<\alpha<n$ and $g_{n-\rho}(t)=(_tI_T^{n-\rho}g)(t)$ be the fractional integral (2) of order $n-\rho$.*

(a)If $(1\le p\le\infty)$ and $g(t)\in\ _tI_T^{\rho}(L^p)$, then $(_tI_T^{\rho}\ _tD_T^{\rho}g)(t)=g(t)$.
(b)If $g(t)\in L^1(0,T)$ and $g_{n-\rho}(t)\in AC^n[0,T]$, then the following equality holds:

$$(_tI_T^{\rho}\ _tD_T^{\rho}g)(t)=g(t)-\sum_{j=1}^{n}\frac{(-1)^{n-j}g_{n-\rho}^{(n-j)}(0)}{\Gamma(\rho-j+1)}(T-t)^{\rho-j},\quad a.e.\ \ on\ [0,T].$$

The next result yields the boundedness of the fractional integration operators $_0I_t^{\rho}y$ and $_tI_T^{\rho}y$ from the space $L^p[0,T]$ $(1\le p\le\infty)$ with the norm $\|\ y\ \|_{L^p}$.

Lemma 6.([24]) *Let $y\in L^p$. The fractional integration operators $_0I_t^{\rho}$ and $_tI_T^{\rho}$ with $\rho>0$ are bounded in $L^p(0,T)$ $(1\le p\le\infty)$:*

$$\|\ _0I_t^{\rho}y\ \|_{L^p}\le\frac{T^{\rho}}{\Gamma(\rho+1)}\|\ y\ \|_{L^p},$$

$$\|\ _tI_T^{\rho}y\ \|_{L^p}\le\frac{T^{\rho}}{\Gamma(\rho+1)}\|\ y\ \|_{L^p}\ .$$

3 Main result

In this section, we discuss the existence of weak solution and then solution of given fractional BVP in (1).

3.1 *Existence of weak solution*

According to [14], to organize a variational structure for fractional differential equations, we need the introduction of a suitable space of functions. We present the definition of the spaces $E_0^{\alpha,p}$, which depend on L^P-integrability of the Riemann-Liouville fractional derivative of a function and we present some of its properties in the following.

Denote by $C_0^{\infty}([0,T],\mathbb{R})$ the set of all functions $u\in C^{\infty}([0,T],\mathbb{R})$ with $u(0)=u(T)=0$. According to Lemma 1, for any $h\in C_0^{\infty}([0,T],\mathbb{R})$ and $1<p<\infty$, we have $h\in L^p$ and $_0D_t^{\alpha}h\in L^p$. Therefore, we can define the following space of functions.

Definition 3.*For $n-1<\alpha<n$ and $1<p<\infty$, the fractional derivative space $E_0^{\alpha,p}$ is defined as the closure of $C_0^\infty([0,T],\mathbb{R})$. It is clear that:*

$$E_0^{\alpha,p}=\{u\in L^p([0,T],\mathbb{R})\mid {}_0D_t^\alpha u\in L^p([0,T],\mathbb{R}) \text{ and } u(0)=u(T)=0\},$$

we endow $E_0^{\alpha,p}$ with the following norm:

$$\| u\|_{\alpha,p}=\left(\int_0^T |u(t)|^p\,dt+\int_0^T |{}_0D_t^\alpha u(t)|^p\,dt\right)^{\frac{1}{p}}. \tag{7}$$

In the next proposition, we show that fractional derivative and integral can commute in space $E_0^{\alpha,p}$, but before that we state the following property that will be useful further in proofs of propositions.

Property 1.([14], property A.7) Let $\alpha>0$ and $n-1<\alpha<n$, then the following statements are equivalent for $u\in C^\infty[0,T]$:

(a)${}_0I_t^\alpha\,{}_0D_t^\alpha u(t)=u(t),\quad t\in[0,T]$.
(b)$[{}_0I_t^{j-\alpha}u(t)]_{t=0}=0,\quad j=1,2,\ldots,n$.
(c)$u^{(j)}(0)=0,\quad j=0,1,\ldots,n-1$.

Also, this property holds for ${}_tI_T^\alpha u(t)$ and ${}_tD_T^\alpha u(t)$.

Proposition 1.*Let $n-1<\alpha<n$ and $1\le p\le\infty$. For all $u\in E_0^{\alpha,p}$ with $u^{(j)}(0)=0$ $(j=0,1,\ldots,n-1)$, we have ${}_0I_t^\alpha\,{}_0D_t^\alpha u(t)=u(t)$.*

Proof. According to property 1, the proof is obvious.

Proposition 2.*([24]) Let $\alpha>0$. If $h\in L^p([0,T],\mathbb{R})$ and $g\in L^q([0,T],\mathbb{R})$, $p\ge1,q\ge1$ and $\frac{1}{p}+\frac{1}{q}\le 1+\alpha$ ($p\ne1$ and $q\ne1$ in the case when $\frac{1}{p}+\frac{1}{q}=1+\alpha$), then*

$$\int_0^T h(t)\,({}_0I_t^\alpha g)(t)dt=\int_0^T g(t)({}_tI_T^\alpha h)(t)dt.$$

Proposition 3.*Let $n-1<\alpha<n$, $h\in E_0^{\alpha,p}$, $h^{(j)}(0)=0$ $(j=0,1,\ldots,n-1)$ and $g\in C_0^\infty([0,T],\mathbb{R})$, $g^{(j)}(0)=0$ $(j=0,1,\ldots,n-1)$ then*

$$\int_0^T ({}_0D_t^\alpha h)\;g\;dt=\int_0^T h({}_tD_T^\alpha\;g)\;dt,$$
$$\int_0^T ({}_tD_T^\alpha h)\;g\;dt=\int_0^T h({}_0D_t^\alpha\;g)\;dt.$$

Proof. Substituting $y^{(j)}(0)=0$ in (3), we have

$${}_0D_t^\alpha y(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{y^{(n)}(s)ds}{(t-s)^{\alpha-n+1}}={}_0I_t^{n-\alpha}y^{(n)}(t),$$

and using integration by parts, we have

$$\int_0^T({}_0D_t^\alpha h)gdt=\int_0^T gd({}_0I_t^{n-\alpha}h^{(n-1)})=g({}_0I_t^{n-\alpha}h^{(n-1)})\,|_0^T-\int_0^T({}_0I_t^{n-\alpha}h^{(n-1)})g'dt$$

Similarly, we can obtain the following integral

$$\int_0^T({}_0D_t^\alpha h)gdt=(-1)^n\int_0^T({}_0I_t^{n-\alpha}h)g^{(n)}dt,$$

also by Proposition 2

$$\int_0^T({}_0D_t^\alpha h)gdt=(-1)^n\int_0^T h({}_tI_T^{n-\alpha}g^{(n)})dt=(-1)^n\int_0^T h((-1)^n\,{}_tD_T^\alpha g)dt=\int_0^T h({}_tD_T^\alpha g)dt,$$
$$\Rightarrow\int_0^T({}_0D_t^\alpha h)gdt=\int_0^T h({}_tD_T^\alpha g)dt.$$

Proposition 4. *Let $n-1<\alpha<n$ and $1<p<\infty$. For all $u\in E_0^{\alpha,p}$ with $u^{(j)}(0)=0$ $(j=0,1,...,n-1)$, we have*

$$\| u \|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \| \, {}_0D_t^\alpha u \|_{L^p} \, .$$

If $\alpha>\frac{1}{p}$ and $\frac{1}{p}+\frac{1}{q}=1$, then

$$\| u \|_\infty \leq \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} \| \, {}_0D_t^\alpha u \|_{L^p} \, .$$

Proof. According to Proposition 1, we can replace u with ${}_0I_t^\alpha \, {}_0D_t^\alpha u$, so it suffices to prove

$$\| \, {}_0I_t^\alpha \, {}_0D_t^\alpha u \|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \| \, {}_0D_t^\alpha u \|_{L^p} \, .$$

We know that for $u\in E_0^{\alpha,p}$, we have ${}_0D_t^\alpha u\in L^p$. Thus, according to Lemma 6, substituting ${}_0D_t^\alpha u$ instead of y, we have

$$\| \, {}_0I_t^\alpha \, {}_0D_t^\alpha u \|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \| \, {}_0D_t^\alpha u \|_{L^p},$$

$$\Rightarrow \| u \|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \| \, {}_0D_t^\alpha u \|_{L^p} \, .$$

For the proof of second part, we also substitute u with ${}_0I_t^\alpha \, {}_0D_t^\alpha u$ and it is similar to ([23], Proposition 3.3).

Lemma 7. *The operator A from space $E_0^{\alpha,p}$ into $L_2^p([0,T],\mathbb{R})$*

$$A: E_0^{\alpha,p} \to A(E_0^{\alpha,p}) \subset L_2^p([0,T],\mathbb{R}) \tag{8}$$

$$u \mapsto A(u) = (u, \, {}_0D_t^\alpha u), \quad \forall u\in E_0^{\alpha,p}.$$

is an isometric isomorphic mapping.

Note that $L_2^p([0,T],\mathbb{R})$ is the cartesian product space $L^p([0,T],\mathbb{R})\times L^p([0,T],\mathbb{R})$, with the norm $\| v \|_{L_2^p} = (\sum_{i=1}^2 \| v_i \|_{L^p}^p)^{\frac{1}{p}}, (v_i\in E_0^{\alpha,p})$.

Proof. According to linearity property of Riemann-Liouville fractional derivative operator, it is obvious that A is a linear operator. We will show that A preserves the norm which means that

$$\forall u\in E_0^{\alpha,p}; \quad \| u \|_{\alpha,p} = \| Au \|_{L_2^p} \, .$$

from (7), we know that

$$\| u \|_{\alpha,p} \;=\! \left(\; \| u \|_{L^p}^p \; + \; \| \, {}_0D_t^\alpha u \|_{L^p}^p \right)^{\frac{1}{p}},$$

so by the definition, the operator A is an isometric isomorphic mapping.

Proposition 5. *Let $n-1<\alpha<n$ and $1<p<\infty$. The fractional derivative space $E_0^{\alpha,p}$ is a reflexive and separable Banach space.*

Proof. First, we will show that $E_0^{\alpha,p}$ is a Banach space. Let $(u_m)_{m\geq1}$ be a Cauchy sequence in $E_0^{\alpha,p}$. Then, $(u_m)_{m\geq1}$ and $({}_0D_t^\alpha u_m)_{m\geq1}$ are Cauchy sequences in $L^p([0,T],\mathbb{R})$. Since $L^p([0,T],\mathbb{R})$ is a Banach space, there exist functions v_1,v_2 in $L^p([0,T],\mathbb{R})$, such that $u_m\to v_1$, ${}_0D_t^\alpha u_m\to v_2$ in $L^p([0,T],\mathbb{R})$ as $m\to\infty$. Now, we are going to show that ${}_0D_t^\alpha v_l = v_2$. According to Proposition 3,

$$\int_0^T {}_0D_t^\alpha u_m(t)v(t)dt = \int_0^T u_m(t) \, {}_tD_T^\alpha v(t)dt, \quad \forall v\in E_0^{\alpha,p}([0,T],\mathbb{R}).$$

We obtain that

$$\int_0^T v_2(t)v(t)dt = \int_0^T v_1(t) {}_tD_T^\alpha v(t)dt = \int_0^T {}_0D_t^\alpha v_1(t)v(t)dt$$

$$\Rightarrow v_2(t) = {}_0D_t^{\alpha} v_1(t) \in L^p.$$

Therefore, $E_0^{\alpha,p}$ is a Banach space. Now, we want to show that $E_0^{\alpha,p}$ is a reflexive space. Since $L^p([0,T],\mathbb{R})$ is a reflexive Banach space, $L_2^p([0,T],\mathbb{R})$ is also a reflexive Banach space with respect to the $\| u \|_{L_2^p}$, where defined in Lemma 7. According to (8), we observed that the operator A is an isometric isomorphic mapping, then $A(E_0^{\alpha,p})$ is a closed subspace of $L_2^p([0,T],\mathbb{R})$, therefore $A(E_0^{\alpha,p})$ is reflexive. Consequently, $E_0^{\alpha,p}$ is also reflexive. We show that $E_0^{\alpha,p}$ is a separable space. Since $L^p([0,T],\mathbb{R})$ ia a separable Banach space, $L_2^p([0,T],\mathbb{R})$ is also a separable Banach space with respect to the $\| u \|_{L_2^p}$, then $A(E_0^{\alpha,p})$ is also separable. Consequently, $E_0^{\alpha,p}$ is a separable space.

*Remark.*According to Proposition 4, we can consider $E_0^{\alpha,p}$ with respect to the norm
$\| u \|_{\alpha,p} = \| {}_0D_t^{\alpha} u \|_{L^p}$.

Definition 4.*The sequence $\{x_n\}_{n=1}^{\infty} \subset X$ converges weakly to $x \in X$ i.e. $x_n \rightharpoonup x$ if for all $T \in X^*$, $T(x_n) \to T(x)$. (X^* is the dual space of X)*

Proposition 6.*([23]) Let $n-1 < \alpha < n$ and $1 < p < \infty$. Assume that $\alpha > \dfrac{1}{p}$ and the sequence $\{u_k\}$ converges weakly to u in $E_0^{\alpha,p}$. Then, $u_k \to u$ in $C([0,T],R)$, i.e. $\| u - u_k \|_{\infty} = 0$, as $k \to \infty$.*

Definition 5.*([25]) The functional φ is said to be Gateaux-differentiable in $u \in E_0^{\alpha,p}$ if the map:*

$$D\varphi(u) : E_0^{\alpha,p} \to \mathbb{R}$$

$$v \mapsto D\varphi(u)(v) := \lim_{h\to 0} \frac{\varphi(u+hv) - \varphi(u)}{h}$$

is well-defined for any $v \in E_0^{\alpha,p}$ and if it is linear and continuous. A critical point $u \in E_0^{\alpha,p}$ of φ is defined by $D\varphi(u) = 0$.

We present a proposition which will be useful to establish a variational structure for BVP (1) in $E_0^{\alpha,p}$.

Proposition 7.*Let $n-1 < \alpha < n$, $1 < p < \infty$ and $\alpha > \frac{1}{p}$,*

$$L : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

$$(t,x,y) \mapsto L(t,x,y),$$

be measurable in t for each $(x,y) \in \mathbb{R}\times\mathbb{R}$ and continuously differentiable in (x,y) for almost every $t \in [0,T]$. If there exist $a \in C(\mathbb{R}^+,\mathbb{R}^+)$, $b \in L^1([0,T],\mathbb{R}^+)$ and $c \in L^q([0,T],\mathbb{R}^+)$, $1 < q < \infty$, such that for a.e. $t \in [0,T]$ and every $(x,y) \in \mathbb{R}\times\mathbb{R}$, one has

$$| L(t,x,y) | \leq a(| x |)(b(t) + | y |^p),$$

$$| D_x L(t,x,y) | \leq a(| x |)(b(t) + | y |^p),$$

$$| D_y L(t,x,y) | \leq a(| x |)(c(t) + | y |^{p-1}),$$

where $\frac{1}{p} + \frac{1}{q} = 1$, then the functional φ defined by

$$\varphi(k) = \int_0^T L(t,k(t), {}_0D_t^{\alpha} k(t))dt,$$

is continuously differentiable on $E_0^{\alpha,p}$ and $\forall k,v \in E_0^{\alpha,p}$, we have

$$< \varphi'(k), v >= \int_0^T [(D_x L(t,k(t), {}_0D_t^{\alpha} k(t)).v(t))$$

$$+(D_y L(t,k(t), {}_0D_t^{\alpha} k(t)). {}_0D_t^{\alpha} v(t))]dt.$$

Proof. We refer the reader to ([26], Theorem 1.4).

We propose the definition of a weak solution and we continue the discussion of the existence of weak solution in Hilbert space $E_0^{\alpha,2}$ with respect to the norm $\| u \|_{\alpha,2}$ given by (7). If $f(.,x) \in L^1([0,T],\mathbb{R})$, then multiplying (1) by $v \in C_0^\infty([0,T],\mathbb{R})$, by integration and Proposition 3, we have

$$\int_0^T [({}_tD_T^\alpha({}_0D_t^\alpha k(t))v(t)) - (f(t,\ {}_0D_t^\alpha k(t))v(t))]dt$$

$$= \int_0^T [({}_0D_t^\alpha k(t){}_0D_t^\alpha v(t)) - f(t,\ {}_0D_t^\alpha k(t))v(t)]dt = 0. \tag{9}$$

Definition 6. *A function $k \in E_0^{\alpha,2}$ such that $f(.,x) \in L^1([0,T],\mathbb{R})$ is said to be a weak solution of fractional BVP (1) if k satisfies (9) for all $v \in C_0^\infty([0,T],\mathbb{R})$.*

We consider the following conditions:

(H_1)Let $\dfrac{dF}{dx} = f(t,x)$ such that $F(t,x)$ is measurable in t for each $x \in \mathbb{R}$, continuously differentiable in x for almost every $t \in [0,T]$ and there exist $a \in C(\mathbb{R}^+,\mathbb{R}^+)$ and $b \in L^1([0,T],\mathbb{R}^+)$ such that

$$| f(t,x) | \ \le a(| x |)b(t),$$

$$| F(t,x) | \ \le a(| x |)b(t),$$

for all $x \in \mathbb{R}$ and a.e. $t \in [0,T]$.

$(H_2)$$F \in C([0,T]\times\mathbb{R},\mathbb{R})$ and there exists $\mu \in [0,\frac{1}{2})$ and $M > 0$ such that $0 < F(t,x) \le \mu f(t,x){}_0I_t^\alpha x$ for all $x \in \mathbb{R}$ with $| x | \ \ge M$ and $t \in [0,T]$.

(H_3)There exist constants $\theta \in (0,2), a \in [0,\frac{\Gamma^2(\alpha+1)}{2T^{2\alpha}})$ and the functions $b(t) \in L^{\frac{2}{\theta}}([0,T],\mathbb{R}), c(t) \in L^1([0,T],\mathbb{R})$ such that

$$| F(t,x) | \ \le a\,| x |^2 + b(t)\,| x |^{2-\theta} + c(t) \ \ a.e\ \ t \in [0,T],\ \ x \in \mathbb{R}.$$

$(H_4)$$\limsup_{|x|\to 0} \dfrac{F(t,x)}{| x |^2} < \dfrac{1}{2}$ uniformly for $t \in [0,T]$ and $x \in \mathbb{R}$.

Corollary 1. *([26], Corollary 1.1) Let $L : [0,T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be defined by*

$$L(t,x,y) = \frac{1}{2} | y |^2 - F(t,x),$$

where

$$F : [0,T]\times\mathbb{R} \to \mathbb{R}$$

$$(t,x) \to F(t,x)$$

satisfies (H_1). If $n-1 < \alpha < n$, $\alpha > \frac{1}{2}$ and $k \in E_0^{\alpha,2}$ is a solution of Euler equation $\varphi'(k) = 0$, where φ is defined as

$$\varphi(k) = \int_0^T \frac{1}{2} |\ {}_0D_t^\alpha k(t) |^2 dt - \int_0^T F(t,\ {}_0D_t^\alpha k(t))dt, \tag{10}$$

then k is a weak solution of BVP (1). (In fact the critical point of the functional φ is a weak solution of BVP (1)).

Proof. By Proposition 7 and (9),

$$0 = \ <\varphi'(k),v>= \int_0^T ({}_0D_t^\alpha k(t){}_0D_t^\alpha v(t))dt - \int_0^T f(t,\ {}_0D_t^\alpha k(t))v(t)dt = 0,$$

$$\Leftrightarrow \int_0^T {}_tD_T^\alpha({}_0D_t^\alpha k(t))v(t)dt - \int_0^T f(t,\ {}_0D_t^\alpha k(t))v(t)dt = 0$$

$$\Leftrightarrow \int_0^T [{}_tD_T^\alpha({}_0D_t^\alpha k(t)) - f(t,\ {}_0D_t^\alpha k(t))]v(t)dt = 0,$$

for all $v \in E_0^{\alpha,2}$ with $v^{(j)}(0) = 0$ $(j = 0,1,...,n-1)$ and so for all $v \in C_0^\infty([0,T],\mathbb{R})$. Thus, k is a weak solution of BVP (1).

According to the above corollary, we know that to find a weak solution of BVP (1), it is enough to obtain the critical point of functional φ that is defined in (10). Thus, we use some critical point theorems, like Minimization principle and Mountain pass theorem to find the critical points. We present some definitions and propositions without their proofs which are used further in this paper.

Definition 7.([23]) *Let X be a real Banach space and $\varphi \in C^1(X,\mathbb{R})$ (the set of functionals that are Gateaux differentiable and their Gateaux derivatives are continuous on X). If any sequence $\{u_k\} \subset X$ for which $\{\varphi(u_k)\}$ is bounded and $\varphi'(u_k) \to 0$ as $k \to \infty$ possesses a convergent subsequence, then we say φ satisfies Palais-Smale condition (denoted by P.S. condition for short).*

Definition 8.([26]) *A function $\varphi : X \to \mathbb{R}$ is lower semi-continuous (resp. weakly lower semi-continuous) if for every sequence $(u_k)_{k\in\mathbb{N}}$*

$$u_k \to u \ \ (u \in X) \Longrightarrow liminf_{k\to\infty}\varphi(u_k) \geq \varphi(u)$$

$$(resp. u_k \rightharpoonup u \ \ (u \in X) \Longrightarrow liminf_{k\to\infty}\varphi(u_k) \geq \varphi(u)).$$

Proposition 8.([26]) *If X is a normed space and $\varphi : X \to \mathbb{R}$ is lower semi-continuous and convex, then φ is weakly lower semi-continuous.*

Proposition 9.([26]) *(Minimization principle) Let X be a real reflexive Banach space. If the functional $\varphi : X \to \mathbb{R}$ is weakly lower semi-continuous and coercive, then there exists $u_0 \in X$ such that $\varphi(u_0) = \inf_{u\in X}\varphi(u)$. Moreover, if φ is also Gateaux differentiable on X, then $\varphi'(u_0) = 0$.*

Hence, we state and prove the coercivity and weakly lower semi-continuity conditions of φ in the next two theorems.

Theorem 1. Let $n-1 < \alpha < n$, $\alpha > \frac{1}{2}$ *and F satisfies (H_1) and (H_3). Then, φ is coercive (i.e. $lim_{\|u\|\to+\infty}\varphi(u) = +\infty$).*

Proof. Using (H_3) and previous Remark

$$\varphi(k) = \int_0^T \frac{1}{2} \mid {}_0D_t^\alpha k(t) \mid^2 dt - \int_0^T F(t,\ {}_0D_t^\alpha k(t))dt$$

$$\geq \int_0^T \frac{1}{2} \mid {}_0D_t^\alpha k(t) \mid^2 dt - a\int_0^T \mid {}_0D_t^\alpha k(t) \mid^2 dt - \int_0^T b(t) \mid {}_0D_t^\alpha k(t) \mid^{2-\theta} dt - \int_0^T c(t)dt$$

$$= \frac{1}{2} \| {}_0D_t^\alpha k(t) \|_{L^2}^2 - a \| {}_0D_t^\alpha k(t) \|_{L^2}^2 - \int_0^T b(t) \mid {}_0D_t^\alpha k(t) \mid^{2-\theta} dt - \overline{C}$$

$$\geq (\frac{1}{2} - a) \| {}_0D_t^\alpha k(t) \|_{L^2}^2 - (\int_0^T \mid b(t) \mid^{\frac{2}{\theta}} dt)^{\frac{\theta}{2}} (\int_0^T \mid {}_0D_t^\alpha k(t) \mid^2 dt)^{1-\frac{\theta}{2}} - \overline{C}$$

$$= (\frac{1}{2} - a) \| {}_0D_t^\alpha k(t) \|_{L^2}^2 - \overline{b} \| {}_0D_t^\alpha k(t) \|_{L^2}^{2-\theta} - \overline{C}$$

$$= (\frac{1}{2} - a) \| k(t) \|_{\alpha,2}^2 - \overline{b} \| k(t) \|_{\alpha,2}^{2-\theta} - \overline{C},$$

where

$$\overline{b} = (\int_0^T \mid b(t) \mid^{\frac{2}{\theta}} dt)^{\frac{\theta}{2}},\ \overline{C} = \int_0^T c(t)dt,$$

so, when $\| k \|_{\alpha,2} \to +\infty$, we have $\varphi(k) = +\infty$, therefore φ is coercive.

Now, we prove that φ is weakly lower semi-continuous. We seperate the functional $\varphi(k)$ into the following form

$$\varphi(k) = \int_0^T \frac{1}{2} \mid {}_0D_t^\alpha k(t) \mid^2 dt - \int_0^T F(t,\ {}_0D_t^\alpha k(t))dt,$$

let

$$H(k):\int_0^T \frac{1}{2}\mid {}_0D_t^{\alpha}k(t)\mid^2 dt,\quad G(k):\int_0^T F(t,\,{}_0D_t^{\alpha}k(t))dt.$$

We show that H and G are weakly lower semi-continuous. We know H is continuous, so according to Proposition 8, we must show that H is convex, We suppose that k, v $\in E_0^{\alpha,p}$ and $\lambda\in(0,1)$, and we must show:

$$H((1-\lambda)k+\lambda v)\leq(1-\lambda)H(k)+\lambda H(v).$$

$$H((1-\lambda)k+\lambda v)=\frac{1}{2}\int_0^T\mid {}_0D_t^{\alpha}((1-\lambda)k+\lambda v)\mid^2 dt.$$

We know that the Riemann-Liouville fractional derivative operator ${}_0D_t^{\alpha}$ is linear, so

$${}_0D_t^{\alpha}((1-\lambda)k+\lambda v)=(1-\lambda)_0D_t^{\alpha}k(t)+\lambda_0D_t^{\alpha}v(t).$$

$$\rightarrow\frac{1}{2}\int_0^T\mid(1-\lambda)_0D_t^{\alpha}k(t)+\lambda_0D_t^{\alpha}v(t)\mid^2 dt$$

$$\leq\frac{1}{2}\int_0^T[(1-\lambda)\mid {}_0D_t^{\alpha}k(t)\mid^2+\lambda\mid {}_0D_t^{\alpha}v(t)\mid^2]dt,$$

$$\leq\frac{1}{2}\int_0^T(1-\lambda)\mid {}_0D_t^{\alpha}k(t)\mid^2 dt+\frac{1}{2}\int_0^T\lambda\mid {}_0D_t^{\alpha}v(t)\mid^2 dt,$$

$$\leq\frac{1}{2}(1-\lambda)\int_0^T\mid {}_0D_t^{\alpha}k(t)\mid^2 dt+\frac{1}{2}\lambda\int_0^T\mid {}_0D_t^{\alpha}v(t)\mid^2 dt,$$

$$=(1-\lambda)H(k)+\lambda H(v).$$

H is lower semi-continuous and convex, so H is weakly lower semi-continuous.

Now, as we observed in Proposition 6, with weakly convergence assumption of u_k to u in $E_0^{\alpha,p}$, u_k will be convergent to u in $C([0,T],\mathbb{R})$, which means F is continuous with respect to the second variable for all $t\in[0,T]$. Hence, we can write $F(t,\,{}_0D_t^{\alpha}k_n(t))\rightarrow F(t,\,{}_0D_t^{\alpha}k(t))$ for all $t\in[0,T]$. On the other hand, since the sequence $({}_0D_t^{\alpha}k_n)$ is convergent in $C([0,T],\mathbb{R})$, it is bounded in this space. Hence, according to the lebesgue dominated convergence theorem, we have

$$\int_0^T F(t,\,{}_0D_t^{\alpha}k_n(t))dt\rightarrow\int_0^T F(t,\,{}_0D_t^{\alpha}k(t))dt.$$

That, this convergence shows the weakly continuity and then the weakly lower semi-continuity of G. Therefore, the functional $\varphi(k)$ is weakly lower semi-continuous.

Hence, as we observed if φ is weakly lower semi-continuous and coercive, then according to Proposition 9, φ has a minimum. According to Theorem 1, we saw that when F satisfies $(H1)$ and $(H3)$, φ will be coercive, so all the terms in Proposition 9, satisfy. We have already shown that, it suffices to find the critical points of the functional and we saw that the critical points can be a weak solution, so we are looking for the conditions that under which we show the corresponding functional has critical points. To achieve this, we use the critical points theorems like Proposition 9 and now to find a nonzero critical point of functional φ, we use Mountain pass theorem.

Proposition 10.([23]) *(Mountain pass theorem) Let X be a real Banach space and $\varphi\in C^1(X,\mathbb{R})$ satisfying P.S. condition. Suppose that*

1.$\varphi(0)=0$;
2.there exist $\rho>0$ and $\sigma>0$ such that $\varphi(z)\geq\sigma$ for all $z\in X$ with $\|z\|=\rho$;
3.there exists z_1 in X with $\|z_1\|>\rho$ such that $\varphi(z_1)\leq\sigma$.

Then, φ possesses a critical value $c\geq\sigma$. Moreover, c can be characterized as

$$c=\inf_{g\in\Lambda}\max_{z\in g([0,1])}\varphi(z),$$

where $\Lambda=\{g\in C([0,1],X):g(0)=0,g(1)=z_1\}$.

We show in the following theorem with some assumptions the Mountain pass theorem conditions satisfy, so the functional φ has critical point and the BVP (1) has a weak solution.

Theorem 2. *Suppose that F satisfies the conditions $(H_1),(H_2)$ and (H_4), then BVP (1) has at least a nonzero weak solution on $E_0^{\alpha,2}$.*

Proof. We prove that φ satisfies all the Mountain pass theorem conditions. First, we prove that φ satisfies P.S. condition. We suppose that $\{k_n\}$ is a sequence in $E_0^{\alpha,2}$ such that $\lim_{n\to\infty}\varphi'(k_n)=0$ and $\varphi(k_n)$ is bounded, i.e.

$$\exists L>0;\ \mid \varphi(k_n)\mid\ \leq L\ \ (n=1,2,...).$$

We know f is continuous and from (H_1), F is continuous. Hence, there exist $\mu\in[0,\frac{1}{2})$ and $M>0$ such that for $\mid x\mid\ \leq M$ and $0\leq t\leq T$ the function $F(t,x)-\mu f(t,x)_0I_t^{\alpha}x$ is continuous, so there exists $c\in\mathbb{R}^+$ such that $F(t,x)\leq \mu f(t,x)_0I_t^{\alpha}x+c$, by (H_2) condition:

$$F(t,x)\leq \mu f(t,x)_0I_t^{\alpha}x+c,\ \ t\in[0,T],\ x\in\mathbb{R}. \tag{11}$$

By Proposition 7 and (9), we have

$$< \varphi'(k_n),k_n >=\int_0^T[({}_0D_t^{\alpha}k_n\,{}_0D_t^{\alpha}k_n)-f(t,\,{}_0D_t^{\alpha}k_n)k_n]dt \tag{12}$$

$$=\int_0^T[\mid {}_0D_t^{\alpha}k_n(t)\mid^2-f(t,\,{}_0D_t^{\alpha}k_n)k_n]dt$$

$$=\parallel {}_0D_t^{\alpha}k_n\parallel_{L^2}^2-\int_0^T f(t,\,{}_0D_t^{\alpha}k_n)k_ndt$$

$$=\parallel k_n\parallel_{\alpha,2}^2-\int_0^T f(t,\,{}_0D_t^{\alpha}k_n)k_ndt.$$

From (11) and (12), for $n=1,2,...$, we conclude that

$$L\geq\varphi(k_n)=\int_0^T\frac{1}{2}\mid {}_0D_t^{\alpha}k_n\mid^2dt-\int_0^T F(t,\,{}_0D_t^{\alpha}k_n)dt$$

$$=\frac{1}{2}\parallel {}_0D_t^{\alpha}k_n\parallel_{L^2}^2-\int_0^T F(t,\,{}_0D_t^{\alpha}k_n)dt$$

$$=\frac{1}{2}\parallel k_n\parallel_{\alpha,2}^2-\int_0^T F(t,\,{}_0D_t^{\alpha}k_n)dt$$

$$\geq\frac{1}{2}\parallel k_n\parallel_{\alpha,2}^2-\mu\int_0^T(f(t,\,{}_0D_t^{\alpha}k_n)_0I_t^{\alpha}\,{}_0D_t^{\alpha}k_n)dt-CT$$

$$=\frac{1}{2}\parallel k_n\parallel_{\alpha,2}^2-\mu\int_0^T(f(t,\,{}_0D_t^{\alpha}k_n)k_ndt-CT$$

$$=\frac{1}{2}\parallel k_n\parallel_{\alpha,2}^2-\mu\,(\parallel k_n\parallel_{\alpha,2}^2-<\varphi'(k_n),k_n>)-CT$$

$$=\frac{1}{2}\parallel k_n\parallel_{\alpha,2}^2-\mu\parallel k_n\parallel_{\alpha,2}^2+\mu<\varphi'(k_n),k_n>-CT$$

$$=(\frac{1}{2}-\mu)\parallel k_n\parallel_{\alpha,2}^2+\mu<\varphi'(k_n),k_n>-CT$$

$$\geq(\frac{1}{2}-\mu)\parallel k_n\parallel_{\alpha,2}^2-\mu\parallel\varphi'(k_n)\parallel_{\alpha,2}\parallel k_n\parallel_{\alpha,2}-CT,$$

since $\varphi'(k_n)\to 0$, so there exists $N_0\in\mathbb{N}$ such that $\parallel\varphi'(k_n)\parallel_{\alpha,2}\leq 1$,

$$L\geq(\frac{1}{2}-\mu)\parallel k_n\parallel_{\alpha,2}^2-\mu\parallel k_n\parallel_{\alpha,2}-CT,\ \ n>N_0.$$

This shows that $\{k_n\}\subset E_0^{\alpha,2}$ is bounded. Since $E_0^{\alpha,2}$ is a reflexive Banach space, so we can suppose that there exixts $\{k_{n_k}\}\subset\{k_n\}$, such that $k_{n_k}\rightharpoonup k$ in $E_0^{\alpha,2}$, so we have

$$<\varphi'(k_{n_k})-\varphi'(k),k_{n_k}-k>$$

$$=<\varphi'(k_{n_k}),k_{n_k}-k>-<\varphi'(k),k_{n_k}-k>$$

$$\leq\|\varphi'(k_{n_k})\|_{\alpha,2}\|k_{n_k}-k\|_{\alpha,2}-<\varphi'(k),k_{n_k}-k>$$

$$\leq(\|\varphi'(k_{n_k})\|_{\alpha,2}-\|\varphi'(k)\|_{\alpha,2})\|k_{n_k}-k\|_{\alpha,2}$$

$$=\|\varphi'(k_{n_k})-\varphi'(k)\|_{\alpha,2}\|k_{n_k}-k\|_{\alpha,2}\quad\to 0\quad as\;\; n_k\to\infty. \tag{13}$$

In addition, by Proposition 6, $\| k_{n_k}-k\|_\infty=0\;\; as\;\; n_k\to\infty$ and $\| k_{n_k}\|_\infty\leq M$, we have also $k_{n_k}\to k$ in $C([0,T],\mathbb{R})$ implies that ${}_0D_t^\alpha k_{n_k}\to {}_0D_t^\alpha k$, so $f(t,\,{}_0D_t^\alpha k_{n_k}(t))\to f(t,\,{}_0D_t^\alpha k(t))$. If $t\in[0,T]$ and $|f(t,\,{}_0D_t^\alpha k_{n_k}(t))|\;\;\leq sup_{x\in[-M,M]}|f(t,x)|$, then by the lebesgue dominated convergence theorem, we have

$$\int_0^T f(t,\,{}_0D_t^\alpha k_{n_k}(t))dt\to\int_0^T f(t,\,{}_0D_t^\alpha k(t))dt,\quad as\;\; n_k\to\infty. \tag{14}$$

So, for a large n_k, we have

$$<\varphi'(k_{n_k})-\varphi'(k),k_{n_k}-k>$$
$$=\int_0^T({}_0D_t^\alpha k_{n_k}-\,{}_0D_t^\alpha k)^2dt-\int_0^T(f(t,\,{}_0D_t^\alpha k_{n_k})-f(t,\,{}_0D_t^\alpha k))(k_{n_k}(t)-k(t))dt$$
$$\geq\|k_{n_k}-k\|_{\alpha,2}^2-\left|\int_0^T(f(t,\,{}_0D_t^\alpha k_{n_k}(t))-f(t,\,{}_0D_t^\alpha k)dt\right|\|k_{n_k}-k\|_\infty .$$

Combining (13) and (14), we have

$$\|k_{n_k}-k\|_{\alpha,2}^2\to 0,\quad as\;\; n_k\to\infty,$$

therefore $k_{n_k}\to k$ in $E_0^{\alpha,2}$, so the intended convergence is proved. Thus, φ satisfies the P.S. condition. Now, we consider the second condition in Mountain pass theorem. Since condition (H_4) is satisfied, we conclude

$$\exists\varepsilon\in(0,1),\delta>0;\;\; F(t,x)\leq(\frac{1}{2})(1-\varepsilon)\,|x|^2,\; for\;\; t\in[0,T],\;\; |x|\leq\delta.$$

By proposition 4, let $\delta=\dfrac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2(\alpha-1)+1)^{\frac{1}{2}}}\|k\|_{\alpha,2},\;\; \rho=\dfrac{\Gamma(\alpha)(2(\alpha-1)+1)^{\frac{1}{2}}}{T^{\alpha-\frac{1}{2}}}\delta$ and $\sigma=\dfrac{1}{2}\varepsilon\rho^2>0$, so

$$\varphi(k)=\int_0^T[\frac{1}{2}|\,{}_0D_t^\alpha k(t)|^2-F(t,\,{}_0D_t^\alpha k(t))]dt$$
$$=\frac{1}{2}\|\,{}_0D_t^\alpha k(t)\|_{L^2}^2-\int_0^T F(t,\,{}_0D_t^\alpha k(t))dt$$
$$\geq\frac{1}{2}\|k(t)\|_{\alpha,2}^2-\frac{(1-\varepsilon)}{2}\int_0^T|\,{}_0D_t^\alpha k(t)|^2\,dt$$
$$=\frac{1}{2}\|k(t)\|_{\alpha,2}^2-\frac{1}{2}(1-\varepsilon)\|k(t)\|_{\alpha,2}^2$$
$$=\frac{1}{2}\varepsilon\|k(t)\|_{\alpha,2}^2=\sigma,$$

for all $k\in E_0^{\alpha,2}$ with $\|k\|_{\alpha,2}=\rho$. This concludes the second condition in Mountain pass theorem.
Now, we show that φ satisfies the third condition in Mountain pass theorem. For all $x\in\mathbb{R}$, there exist $c_1,c_2>0$ and $\mu\in[0,\frac{1}{2})$ such that $F(t,x)\geq c_1\,|x|^{\frac{1}{\mu}}-c_2$, we suppose that $\eta>0$ and $0\neq k\in E_0^{\alpha,2}$, we have

$$\varphi(\eta k)=\frac{1}{2}\|\eta\,{}_0D_t^\alpha k\|_{L^2}^2-\int_0^T F(t,\,{}_0D_t^\alpha\eta k)dt$$
$$=\frac{1}{2}\|\eta\, k(t)\|_{\alpha,2}^2-\int_0^T F(t,\eta\,{}_0D_t^\alpha k)dt$$

$$\leq \frac{\eta^2}{2} \parallel k(t) \parallel_{\alpha,2}^2 - c_1 \int_0^T \mid \eta \ {}_0D_t^\alpha k(t) \mid^{\frac{1}{\mu}} dt + c_2 T$$

$$= \frac{\eta^2}{2} \parallel k(t) \parallel_{\alpha,2}^2 - c_1 \eta^{\frac{1}{\mu}} \parallel \ {}_0D_t^\alpha k(t) \parallel_{L^{\frac{1}{\mu}}}^{\frac{1}{\mu}} + c_2 T \to -\infty, \quad as\ \eta \to +\infty,$$

so there exists a big enough η_0 that $\varphi(\eta_0 \ k) \leq 0$. Consequently, the third condition in Mountain pass theorem satisfies. We know that $\varphi(0) = 0$, but for the critical point k, we have $\varphi(k) \geq \sigma > 0$. Therefore, k is a nontrivial solution of BVP (1).

3.2 *Existence of solution*

In this subsection, we present the definition of solution for fractional BVP (1) and a lemma which is useful for our further theorem. Afterward, we prove that if n is even, then every weak solution of BVP (1) is also a solution of BVP (1).

Definition 9. *A function $k : [0,T] \to \mathbb{R}$ is called a solution of BVP (1) if*

(i)${}_tI_T^{n-\alpha} \ {}_0D_t^\alpha k(t)$ is n-times derivable,
(ii)k satisfies BVP (1).

Lemma 8. *Let $n-1 < \alpha < n$, if $k \in E_0^{\alpha,2}$ is a weak solution of BVP (1), then*

$$ {}_0D_t^\alpha k(t) = \ {}_tI_T^\alpha f(t, \ {}_0D_t^\alpha k(t)) + \ {}_tD_T^{n-\alpha} P(t), \ \ a.e. \ \ t \in [0,T]. \tag{15}$$

where

$$P(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \ \ c_i \in \mathbb{R}.$$

Proof. By assumption, $k \in E_0^{\alpha,2}$ is a weak solution of BVP (1). Thus, by the definition of weak solution, we have $\forall h \in C_0^\infty([0,T],\mathbb{R})$;

$$\int_0^T [{}_0D_t^\alpha k(t) {}_0D_t^\alpha h(t) - f(t, \ {}_0D_t^\alpha k(t))h(t)]dt = 0. \tag{16}$$

Since $f \in L^1([0,T],\mathbb{R})$ and according to Lemma 6, we get that ${}_0I_t^\alpha f, \ {}_tI_T^\alpha f, \in L^1([0,T],\mathbb{R})$. We define $w(t)$ by

$$w(t) = \ {}_tI_T^\alpha f(t, \ {}_0D_t^\alpha k(t)), \ \ t \in [0,T],$$

so $w \in L^1([0,T],\mathbb{R})$ and

$$\int_0^T w(t) {}_0D_t^\alpha h(t) dt = \int_0^T \ {}_tD_T^\alpha w(t) h(t) dt$$
$$= \int_0^T \ {}_tD_T^\alpha \ {}_tI_T^\alpha f(t, \ {}_0D_t^\alpha k(t)) h(t) dt$$
$$= \int_0^T f(t, \ {}_0D_t^\alpha k(t)) h(t) dt.$$

By (16), we have for every $h \in C_0^\infty([0,T],\mathbb{R})$,

$$\int_0^T [{}_0D_t^\alpha k(t) {}_0D_t^\alpha h(t) - f(t, \ {}_0D_t^\alpha k(t))h(t)]dt = 0,$$

$$\Rightarrow \int_0^T [{}_0D_t^\alpha k(t) {}_0D_t^\alpha h(t) - w(t) {}_0D_t^\alpha h(t)]dt = 0,$$

$$\Rightarrow \int_0^T ({}_0D_t^\alpha k(t) - w(t)) {}_0D_t^\alpha h(t) dt = 0.$$

According to (3)

$$\int_0^T ({}_0D_t^\alpha k(t) - w(t))({}_0I_t^{n-\alpha} h^{(n)}(t)) dt = 0,$$

$$\int_0^T \ {}_tI_T^{n-\alpha} ({}_0D_t^\alpha k(t) - w(t)) h^{(n)}(t) dt = 0.$$

In view of ${}_tI_T^{n-\alpha}({}_0D_t^{\alpha}k(t)-w(t)) \in L^1([0,T],\mathbb{R})$, Lemma 2 and Lemma 3 ([19]), implies that

$$ {}_tI_T^{n-\alpha}({}_0D_t^{\alpha}k(t)-w(t)) = P(t) \quad a.e. \;\; on\ [0,T], \tag{17}$$

where

$$P(t) = c_0 + c_1t + c_2t^2 + ... + c_{n-1}t^{n-1}, \;\; c_i \in \mathbb{R},$$

$$\begin{aligned}
&\to {}_tD_T^{n-\alpha}\, {}_tI_T^{n-\alpha}({}_0D_t^{\alpha}k(t)-w(t)) = {}_tD_T^{n-\alpha}P(t),\\
&\to {}_0D_t^{\alpha}k(t)-w(t) = {}_tD_T^{n-\alpha}P(t),\\
&\to {}_0D_t^{\alpha}k(t) = w(t) + {}_tD_T^{n-\alpha}P(t) \quad a.e. \;\; t\in[0,T],
\end{aligned}$$

and with substituting, we have

$$ {}_0D_t^{\alpha}k(t) = {}_tI_T^{\alpha}f(t,\,{}_0D_t^{\alpha}k(t)) + {}_tD_T^{n-\alpha}P(t),$$

and this completes the proof.

*Remark.*According to (17) and Lemma 2, we have

$$\begin{aligned}
{}_tI_T^{n-\alpha}\, {}_0D_t^{\alpha}k(t) &= {}_tI_T^{n-\alpha}w(t) + P(t),\\
{}_tI_T^{n-\alpha}\, {}_0D_t^{\alpha}k(t) &= {}_tI_T^{n-\alpha}\, {}_tI_T^{\alpha}f(t,\,{}_0D_t^{\alpha}k(t)) + P(t),\\
&= {}_tI_T^{n}f(t,\,{}_0D_t^{\alpha}k(t)) + P(t) \quad a.e. \;\; on\ [0,T].
\end{aligned}$$

We will show the main theorem of this subsection that states if n is even, then every weak solution of BVP (1) is also a solution of BVP (1).

Theorem 3.*Let $n-1<\alpha<n$ and $k\in E_0^{\alpha,2}$. If k is a weak solution of BVP (1) and n is even, then k is also a solution of BVP (1).*

Proof. According to the above Remark, we have

$$\begin{aligned}
&{}_tI_T^{n-\alpha}\, {}_0D_t^{\alpha}k(t) = {}_tI_T^{n}f(t,\,{}_0D_t^{\alpha}k(t)) + P(t),\\
\Rightarrow\ &{}_tD_T^{n}\, {}_tI_T^{n-\alpha}\, {}_0D_t^{\alpha}k(t) = {}_tD_T^{n}\, {}_tI_T^{n}f(t,\,{}_0D_t^{\alpha}k(t)) + {}_tD_T^{n}P(t),\\
&\Rightarrow (-1)^n\, {}_tD_T^{\alpha}\, {}_0D_t^{\alpha}k(t) = f(t,\,{}_0D_t^{\alpha}k(t)).
\end{aligned}$$

So, we verified that if n is even, then k satisfies BVP (1). Moreover, $k\in E_0^{\alpha,2}$ implies that $k(0)=k(T)=0$. In a similar way, $k^{(j)}(0)=0,\ (j=0,1,2,...,2(n-1))$ and $k^{(l)}(T)=0,\ (l=0,1,2,...,n-1)$.

We study the existence results by the following examples:

*Example 1.*Consider the following problem

$$\begin{cases}
{}_tD_1^{\frac{5}{2}}\, {}_0D_t^{\frac{5}{2}}k(t) = 3\mid {}_0D_t^{\frac{5}{2}}k(t)\mid {}_0D_t^{\frac{5}{2}}k(t), \;\; a.e. \;\; t\in[0,1]\\
k^{(j)}(0)=0, \;\; j=0,1,2,3,4\\
k^{(l)}(1)=0, \;\; l=0,1,2
\end{cases} \tag{18}$$

where $2<\alpha=\frac{5}{2}<3$, n=3, T = 1, $f(t,{}_0D_t^{\frac{5}{2}}k(t)) = 3\mid {}_0D_t^{\frac{5}{2}}k(t)\mid {}_0D_t^{\frac{5}{2}}k(t)$ and

$$F(t,x) = \int_0^x f(t,s)ds = \mid {}_0D_t^{\frac{5}{2}}k(t)\mid^3 .$$

Hence, conditions $(H_1),(H_2)$ and (H_4) hold with $\mu=\frac{1}{3}\in[0,\frac{1}{2})$. Thus, according to Theorem 2, the problem (18) has at least a nonzero weak solution on $E_0^{\alpha,2}$.

*Example 2.*Let us consider problem (18) again with $\alpha=\frac{7}{2}$. So, in this case, $n=4$ is even. Similar to the previous example, conditions $(H_1),(H_2)$ and (H_4) hold. Hence, according to Theorem 2, the problem has at least a nonzero weak solution. Consequently, according to Theorem 3, this weak solution is also a solution of the problem.

Conflict of Interest

The authors declare that they have no conflict of interest.

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