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Construction of Caputo-Fabrizio Fractional Differential Mask for Image Enhancement

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Abstract: The present paper aims to introduce an algorithm based on the Caputo-Fabrizio fractional differential mask for image contrast enhancement. Experiments show that the method can control the degree of contrast enhancement by varying the fractional differential order. The contrast performance is measured using Peak Signal to Noise Ratio (PSNR).The final numerical procedure is given for contrast enhancement. The experimental results asserted the effectiveness of the algorithm (higher PSNR values) compared with other proposed fractional differential mask.

Keywords: Caputo-Fabrizio fractional differential operator, Caputo-Fabrizio fractional differential mask, contrast image enhancement.

1 Introduction

The concept of fractional derivative is a generalization of the integer-order differentiation. In recent years, the fractional derivative has become popular due to its applications in numerous fields of science and engineering [1–4] . It also helps solve differential and integral equations as well as other problems such as contrast image enhancement, image denoising and image restoration [5–19]. Several known forms of the fractional derivatives have been used to obtain fractional differential masks. In [7], the authors proposed $1 \sim 2$ order fractional differential masks based on Riemann-Liouville definition. In [5], Chen Qing-li, Huang Guo and Zhang Xiu-qiong proposed $0 \sim 1$ order fractional differential masks based on Riemann-Liouville definition. In the same line, in [8, 9] , we can find some fractional differential masks based on the Grümwald-Letnikov fractional derivative.
Recently, Caputo and Fabrizio have introduced a new fractional derivative [20] . The interest in this new approach is due to the necessity of using a model describing structures with different scales [20] as well as facilitating detection of edges and regions in an image. In literature, no results have been obtained on image contrast enhancement using Caputo-Fabrizio fractional derivative. Indeed, motivated by [5–9], our interest in this work is to develop the Caputo-Fabrizio fractional differential mask and demonstrate its capability for image enhancement by varying the fractional differential order. The paper is organized as follows: in Section 2, we briefly review the basic definitions concerning the Caputo-Fabrizio fractional operators. Caputo-Fabrizio fractional differential mask is constructed in Section 3. Section 4 presents the experimental performance of the proposed method. Conclusion is presented in Section 5.

2 Basic definitions

Here, we present some definitions concerning the Caputo-Fabrizio fractional operators used in our subsequent discussion.

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Definition 1. *Suppose $a,\alpha \in \mathbb{R}$ such that $\alpha \in (0,1)$. The Caputo-Fabrizio fractional integral of order α is defined by*

$$I_{ax}^{\alpha}u(x) = (1-\alpha)u(x) + \alpha \int_a^x u(s)ds. \tag{1}$$

Definition 2. *Suppose $a,\alpha \in \mathbb{R}$ such that $\alpha \in (0,1)$. The Caputo-Fabrizio fractional derivative of order α is defined by*

$$D_{ax}^{\alpha}u(x) = \frac{1}{1-\alpha}\int_a^x e^{-\frac{\alpha}{1-\alpha}(x-\xi)}u'(\xi)d\xi. \tag{2}$$

For more details, see [12, 14, 21-29]. In particular, when $a=0$, (2) can be approximated as

$$\begin{aligned} D_{0x}^{\alpha}u(x) &= \frac{1}{1-\alpha}\int_0^x e^{-\frac{\alpha}{1-\alpha}(x-\xi)}u'(\xi)d\xi, \\ &\approx \frac{1}{1-\alpha}\sum_{k=0}^{N-1}\int_{k\cdot\frac{x}{N}}^{(k+1)\cdot\frac{x}{N}} e^{-\frac{\alpha}{1-\alpha}(x-\xi)}u'(\xi_k)d\xi. \end{aligned} \tag{3}$$

3 Construction of Caputo-Fabrizio fractional differential mask

Let's take a partition of $N+1$ nodes $x_k = k\Delta x, k = 0,1,...,N$ of the interval $[0,x]$, with step $\Delta x = x/N$. Then, the $N+1$ pixels can be given by

$$\begin{cases} u_0 = u(0), \\ u_1 = u(x/N), \\ \vdots \\ u_k = u(kx/N), \\ \vdots \\ u_N = u(x). \end{cases} \tag{4}$$

By approximating, we obtain

$$\begin{aligned} &\int_{k\cdot\frac{x}{N}}^{(k+1)\cdot\frac{x}{N}} e^{-\frac{\alpha}{1-\alpha}(x-\xi)}u'(\xi_k)d\xi, \\ &\approx \frac{u(\frac{kx+x}{N}) - u(\frac{kx}{N})}{\Delta x}\cdot\int_{kx/N}^{(kx+x)/N} e^{-\frac{\alpha}{1-\alpha}(x-\xi)}d\xi, \\ &= \frac{1-\alpha}{\alpha}\cdot\frac{u(\frac{kx+x}{N}) - u(\frac{kx}{N})}{\Delta x}\cdot\left[e^{-\frac{\alpha}{1-\alpha}(N-k-1)\Delta x} - e^{-\frac{\alpha}{1-\alpha}(N-k)\Delta x}\right]. \end{aligned} \tag{5}$$

Then, taking (5) into (3), we have

$$\begin{aligned} D_{0x}^{\alpha}u(x) &\approx \frac{1}{\alpha}\cdot\sum_{k=0}^{N-1}\left\{\left[\frac{u((k+1)\cdot\frac{x}{N}) - u(k\cdot\frac{x}{N})}{x/N}\right]\cdot\left[e^{-\frac{\alpha}{1-\alpha}[N-(k+1)]\frac{x}{N}} - e^{-\frac{\alpha}{1-\alpha}[N-k]\frac{x}{N}}\right]\right\} \\ &= \frac{1}{\alpha\cdot\Delta x}\left\{\begin{array}{c} \left(1-e^{-\frac{\alpha}{1-\alpha}\Delta x}\right)u_N + \left(2e^{-\frac{\alpha\cdot\Delta x}{1-\alpha}} - e^{-2\frac{\alpha\cdot\Delta x}{1-\alpha}} - 1\right)u_{N-1} + \\ \left(2e^{-\frac{2\alpha\cdot\Delta x}{1-\alpha}} - e^{-3\frac{\alpha\cdot\Delta x}{1-\alpha}} - e^{-\frac{\alpha\cdot\Delta x}{1-\alpha}}\right)u_{N-2} + \cdots + \\ \left(2e^{-\frac{\alpha\cdot(N-j)\cdot\Delta x}{1-\alpha}} - e^{-\frac{\alpha\cdot(N-j-1)\cdot\Delta x}{1-\alpha}} - e^{-\frac{\alpha\cdot(N-j+1)\cdot\Delta x}{1-\alpha}}\right)u_j \\ + \cdots + \left(2e^{-\frac{\alpha\cdot(N-1)\cdot\Delta x}{1-\alpha}} - e^{-\frac{\alpha\cdot(N-2)\cdot\Delta x}{1-\alpha}} - e^{-\frac{\alpha\cdot N\cdot\Delta x}{1-\alpha}}\right)u_1 \\ + \left(2e^{-\frac{\alpha\cdot N\cdot\Delta x}{1-\alpha}} - e^{-\frac{\alpha\cdot(N-1)\cdot\Delta x}{1-\alpha}} - e^{-\frac{\alpha\cdot(N+1)\cdot\Delta x}{1-\alpha}}\right)u_0 \end{array}\right\} \end{aligned} \tag{6}$$

From (6), we obtain $N+1$ nonzero coefficients $c_i(i=0,\ldots,N)$ which are a function of fractional order α. The nonzero coefficients are

$$\begin{cases} c_0 = \dfrac{1}{\alpha\cdot\Delta x}\left(1-e^{-\frac{\alpha}{1-\alpha}\Delta x}\right), \\ c_1 = \dfrac{1}{\alpha\cdot\Delta x}\left(2e^{-\frac{\alpha}{1-\alpha}\Delta x}-e^{-\frac{2\alpha}{1-\alpha}\Delta x}-1\right), \\ c_2 = \dfrac{1}{\alpha\cdot\Delta x}\left(2e^{-2\frac{\alpha}{1-\alpha}\Delta x}-e^{-3\frac{\alpha}{1-\alpha}\Delta x}-e^{-\frac{\alpha}{1-\alpha}\Delta x}\right), \\ \vdots \\ c_j = \dfrac{1}{\alpha\cdot\Delta x}\left(2e^{-\frac{\alpha}{1-\alpha}(N-j)\Delta x}-e^{-\frac{\alpha}{1-\alpha}(N-j-1)\Delta x}-e^{-\frac{\alpha}{1-\alpha}(N-j+1)\Delta x}\right), \\ \vdots \\ c_{N-1} = \dfrac{1}{\alpha\cdot\Delta x}\left(2e^{-\frac{\alpha}{1-\alpha}(N-1)\Delta x}-e^{-\frac{\alpha}{1-\alpha}(N-2)\Delta x}-e^{-\frac{\alpha}{1-\alpha}\cdot N\cdot\Delta x}\right), \\ c_N = \dfrac{1}{\alpha\cdot\Delta x}\left(2e^{-\frac{\alpha}{1-\alpha}\cdot N\cdot\Delta x}-e^{-\frac{\alpha}{1-\alpha}(N-1)\Delta x}-e^{-\frac{\alpha}{1-\alpha}(N+1)\Delta x}\right). \end{cases} \tag{7}$$

Taking $\Delta x = 1$, then the approximate differences of fractional partial differentiation on x-and y-coordinate are defined by

$$\frac{\partial^\alpha u(x,y)}{\partial x^\alpha} := \frac{1}{\alpha}\cdot\left\{\begin{array}{c} \left(1-e^{-\frac{\alpha}{1-\alpha}}\right)u(x,y)+\left(2e^{-\frac{\alpha}{1-\alpha}}-e^{-2\frac{\alpha}{1-\alpha}}-1\right)u(x-1,y) \\ +\cdots+\left(2e^{-\frac{\alpha}{1-\alpha}(N-j)}-e^{-\frac{\alpha}{1-\alpha}(N-j-1)}\right. \\ \left.-e^{-\frac{\alpha}{1-\alpha}(N-j+1)}\right)u(x-k,y)+\cdots \\ +\left(2e^{-\frac{\alpha}{1-\alpha}N}-e^{-\frac{\alpha}{1-\alpha}(N-1)}-e^{-\frac{\alpha}{1-\alpha}(N+1)}\right)u(x-n,y) \end{array}\right\}$$

and

$$\frac{\partial^\alpha u(x,y)}{\partial y^\alpha} := \frac{1}{\alpha}\cdot\left\{\begin{array}{c} \left(1-e^{-\frac{\alpha}{1-\alpha}}\right)u(x,y)+\left(2e^{-\frac{\alpha}{1-\alpha}}-e^{-2\frac{\alpha}{1-\alpha}}-1\right)u(x,y-1) \\ +\cdots+\left(2e^{-\frac{\alpha}{1-\alpha}(N-j)}-e^{-\frac{\alpha}{1-\alpha}(N-j-1)}\right. \\ \left.-e^{-\frac{\alpha}{1-\alpha}(N-j+1)}\right)u(x,y-k)+\cdots \\ +\left(2e^{-\frac{\alpha}{1-\alpha}N}-e^{-\frac{\alpha}{1-\alpha}(N-1)}-e^{-\frac{\alpha}{1-\alpha}(N+1)}\right)u(x,y-n) \end{array}\right\}$$

respectively.

4 Experimental performance

This section aims to demonstrate the performance of the proposed Caputo-Fabrizio fractional diffrerntial mask method. For this purpose, three test images (i.e. dark living-room image, panther image and Goldhill) are used, as shown in Fig. 1

Table 1: Values for PSNR of the proposed mask.

alpha	dark living-room	phanter	Goldhill
10^{-3}	54.0831	36.7739	39.6978
10^{-4}	74.0857	56.7765	59.7004
10^{-5}	94.0860	76.7768	79.7007
10^{-6}	114.0862	96.7770	99.7009
10^{-7}	134.0619	116.7528	119.6769
10^{-8}	153.1599	135.8507	138.7748
10^{-9}	136.4216	119.1024	122.0041
10^{-10}	162.7767	145.5665	148.5020
10^{-11}	162.7767	145.5665	148.5020
10^{-12}	76.6730	59.3534	62.2519

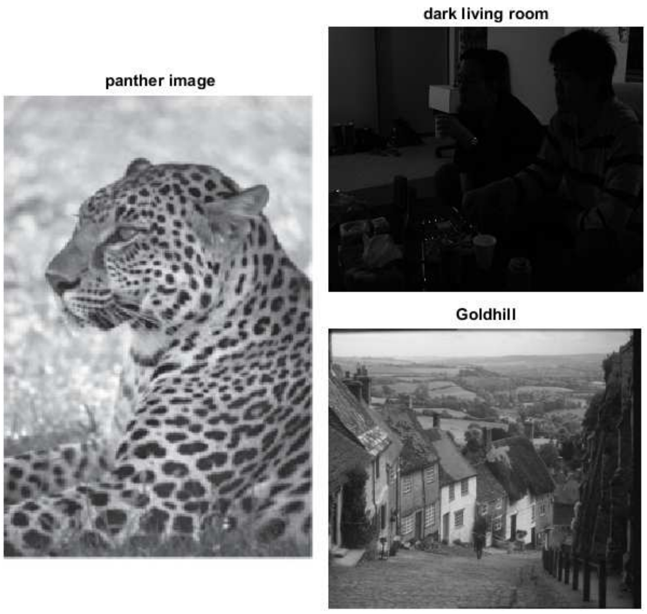

Fig. 1: Original images: dark living-room, panther and Goldhill

Table 2: Values of PSNR considering the mask given in [30].

$alpha$	dark living-room	phanter	Goldhill
0.01	21.4133	4.1688	6.9463
0.02	21.6614	4.4217	7.1844
0.09	23.5280	6.3397	8.9538
0.1	23.8140	6.6367	9.2209
0.2	26.9128	9.9494	12.0130
0.3	29.8277	13.4326	14.3663
0.399	30.2471	14.2526	14.5970
0.4	30.2340	14.2382	14.5864
0.5	28.0883	11.7231	12.8364
0.6	25.7333	9.0613	10.7878

Table 3: Values of PSNR considering the mask given in [16].

$alpha$	dark living-room	phanter	Goldhill
0.03	25.9788	8.7088	11.6943
0.04	28.3511	11.0815	14.0401
0.05	31.4253	14.2607	17.0251
0.06	35.3429	18.9058	20.6147
0.0699	37.6893	24.2845	22.3918
0.07	37.6822	24.3034	22.3834
0.0701	37.6745	24.3205	22.3744
0.08	34.5865	20.1482	19.6139
0.09	30.8791	15.1783	16.2085
0.1	28.0412	11.8444	13.5055

For ease of calculation, only the first three coefficients are considered in this paper,

$$c_0 = \frac{1}{\alpha}\left(1 - e^{-\frac{\alpha}{1-\alpha}}\right), \tag{8}$$

$$c_1 = \frac{1}{\alpha}\left(2e^{-\frac{\alpha}{1-\alpha}} - e^{-\frac{2\alpha}{1-\alpha}} - 1\right), \tag{9}$$

$$c_2 = \frac{1}{\alpha}\left(2e^{-2\frac{\alpha}{1-\alpha}} - e^{-3\frac{\alpha}{1-\alpha}} - e^{-\frac{\alpha}{1-\alpha}}\right). \tag{10}$$

From (8)-(10), we construct the following fractional differential mask (see Table 4)

Table 4: Proposed mask.

C_2	C_2	C_2	C_2	C_2
C_2	C_1	C_1	C_1	C_2
C_2	C_1	C_0	C_1	C_2
C_2	C_1	C_1	C_1	C_2
C_2	C_2	C_2	C_2	C_2

which will be used in this paper. For the comparison purpose, we used the PSNR (Peak Signal to Noise Ratio). The maximum value of PSNR in Tables 1, 2 and 3 are $\alpha = 10^{-10}$, $\alpha = 0.399$ and 0.0699, respectively. These tables indicate that the proposed mask has higher Peak Signal to Noise Ratio (PSNR) values in comparison with the results obtained using other methods.

Figures 2 to 7 show the visual results of the proposed method applied on dark living-room, goldhill and panther images with different values of fractional order $\alpha \in (0,1)$. Based on these results, the proposed mask can control the degree of contrast enhancement by varying the fractional order α. In Figure 8, we compared the visual result of the different masks and observed that the proposed mask has demonstrated significant advantages over other known methods.

Fig. 2: Results of applying the proposed method on a dark image(taken from [5]) with different values of fractional parameter α.

Fig. 3: Results of applying the proposed method on a dark image with different values of fractional parameter α.

Fig. 4: Results of applying the proposed method on a dark image with different values of fractional parameter α.

Fig. 5: Results of applying the proposed method on a dark image with different values of fractional parameter α.

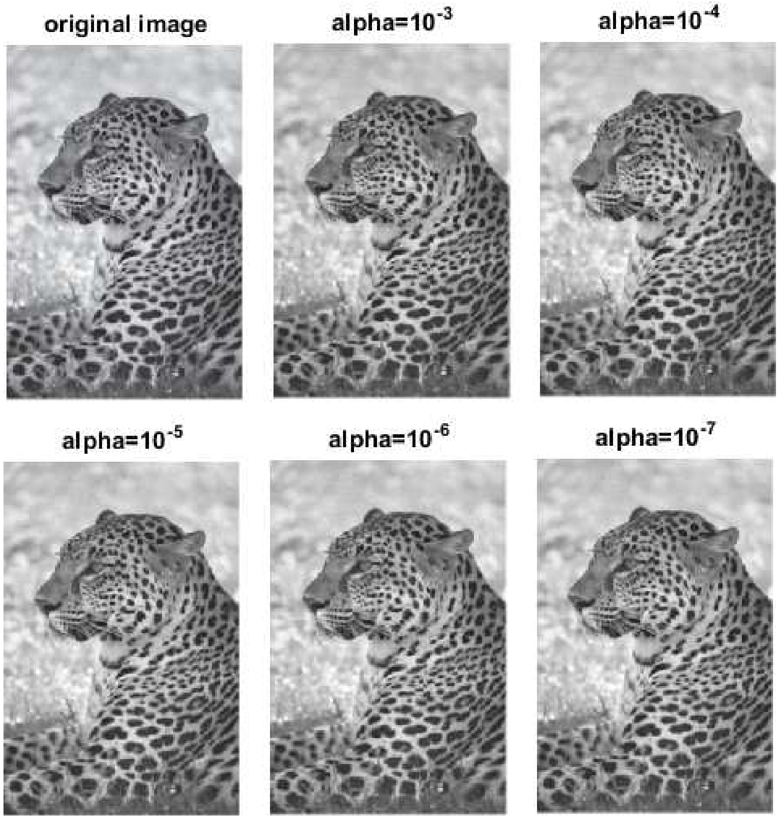

Fig. 6: Results of applying the proposed method on a dark image with different values of fractional parameter α.

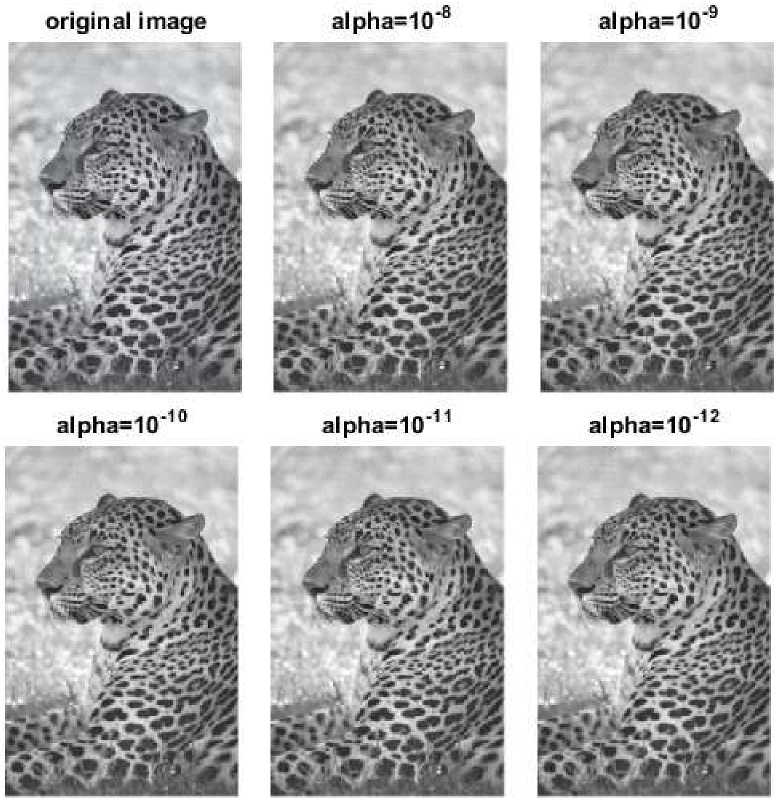

Fig. 7: Results of applying the proposed method on a dark image with different values of fractional parameter α.

Fig. 8: Results obtained for the best value of each model.

5 Conclusion

In this paper, we presented the Caputo- Fabrizio fractional differential mask, which can enhance both contrast and texture of a real image. The method can control the degree of contrast enhancement by varying the fractional differential order $\alpha \in (0,1)$. Experiments showed that the proposed mask did not only enhance edges and texture of the image but also improved some traditional algorithms. After employing the proposed method with different fractional orders, we observed that the value with the best contrast was $\alpha = 10^{-10}$. As a future work, we plan to improve the proposed Caputo-Fabrizio fractional differential mask for contrast image enhancement.

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