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Numerical Solution of Partial Differential Equation by using Legendre Wavelets

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Abstract:A numerical method for solving partial differential equation is presented. The proposed method is based on Legendre wavelet in which Legendre polynomial is used. First we use the 2-point Euler backward differentiation formula, and then we use collocation points that convert the differential equation into a system of algebraic equations. Two examples are included to show the efficiency and accuracy of the present technique.
Keywords:partial differential equation; 2-point Euler backward differentiation formula; Legendre wavelets collocation method.

1 Introduction

Mathematical modeling of real-life problems usually founds in several equations, e.g. integral and integro-differential equation, partial differential equations, stochastic equations and others. More mathematical formulations of physical problem contain partial differential equations. These equations arise in biological models, fluid dynamics, and chemical kinetics [1-2]. Partial differential equations (PDEs) are classified according to linearity of differential operator, highest derivative, degree of highest derivative, type of coefficients, and the values of these coefficients.

We consider the problem of numerically computing an approximation to $u(x,t)$, the solution of the one dimensional partial differential equation (PDE)

$$u_t(x,y) - u_{xx}(x,t) = f(x,t,u), \qquad \text{in} \quad Q \equiv \Omega \times I, \tag{1.1}$$

associated with the boundary conditions

$$u(a,t) = g_1(t), \quad u(b,t) = g_2(t) \quad on \quad \partial\Omega \times I \tag{1.2}$$

and the initial condition

$$u(0,t) = u_0(x), \qquad x \in \Omega, \tag{1.3}$$

Where$I$denote a finite time interval $(0,T)$, $T>0$; and Ω is a bounded domain$[a,b]$, time-independent coefficients andg_1, g_2 are continuous functions in their variables.

Equations which contain partial have been numerically solved by using a variety of techniques. In [3, 4], in fact, a hybrid algorithm, based on Monte Carlo and classical domain decomposition methods, has been proposed for solving linear partial differential equation. All the three main types of equations: elliptic, parabolic and hyperbolic equations are covered by the finite element method [5]. Some meshless schemes to solve partial differential equations Galerkin method [6], the finite point method [7], the partition of unity method [8], the reproducing kernel particle method [9], local Petrov-Galerkin method [10], the finite difference method [11], and radial basis functions used in [12].

The organization of the rest of the paper is as follows. In section 2, we describe the basic formulation of wavelets and Legendre wavelets required for our subsequent development. Section 3 is devoted to the solution of PDE using Legendre wavelets. In Section 4, by including

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numerical examples, we report our numerical finding and demonstrate the accuracy of the proposed scheme. Conclusion is shown in Section 5.

2 Legendre Polynomials

2.1 Properties of Legendre polynomial

Wavelets are very successfully used in signal analysis for waveform representations and segmentations, time-frequency analysis and fast algorithms for easy implementation [13-19]. Wavelets constitute a family of signal functions constructed from dilation and translation of a signal function called the mother wavelet. The following family of continuous wavelets is found when the dilation parameter a and the translation parameter b vary continuously [20-22].

$$(t) = |a|^{-\frac{1}{2}}\varphi\left(\frac{t-b}{a}\right), \qquad a,b \in R, \quad a \neq 0. \tag{2.1}$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}$, $b = nb_0a_0^{-k}$, $a_0 > 1, b_0 > 0$ and n, k are positive integers, we have the following family of discrete wavelets:

$$(2.2)\varphi_{k,n}(t) = |a_0|^{-\frac{k}{2}}\varphi(a_0^k x - nb_0),$$

where $\varphi_{k,n}(x)$ forms a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$, then $\varphi_{k,n}(x)$ form an orthonormal basis [20-24]. Legendre wavelets $\varphi_{nm}(t) = \varphi(k,n,m,t)$ have four arguments; translation argument $n = 1,2,3,\dots,2^{k-1}$,dilation argument k can assume any positive integer, m is the order for Legendre polynomials [23-24]. They are defined on the interval $[0,1)$ as

$$\varphi_{n,m}(x)\begin{cases}\sqrt{m+\frac{1}{2}}\,2^{\frac{k}{2}}L_m(2^kx-2n+1), & \frac{n-1}{2^{k-1}} \leq x \leq \frac{n}{2^{k-1}}\\ 0, & \text{otherwise}\end{cases} \tag{2.3}$$

where $n = 1,2,3,\dots,2^{k-1}$. $m = 0,1,2,\dots,M-1$,the coefficient $\sqrt{m+\frac{1}{2}}$ is for orthonormality, and k, M are positive integers. $L_m(x)$are the Legendre polynomials of order m which are defined on the interval $[-1,1]$ and is given by the following recurrence relations

$$L_0(x) = 1 \tag{2.4}$$

$$L_1(x) = x \tag{2.5}$$

$$L_{m+1}(x) = \left(\frac{2m+1}{m+1}\right) x\, L_m(x) - \left(\frac{m}{m+1}\right) L_m(x), \ m = 1,2,3,\dots \tag{2.6}$$

The set of Legendre wavelets are an orthonormal set [23-28].

2.2 Function Approximation

A function $u(t)$ defined over $[0,1)$ can be expanded as

$$u(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\varphi_{nm}(t), \tag{2.7}$$

where

$$c_{nm} = \langle u(t), \varphi_{nm}(t)\rangle, \tag{2.8}$$

in which $\langle\cdot,\cdot\rangle$ denotes the inner product. Equation (2.7) can be written as

$$u(t) \cong \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\varphi_{nm}(t) = C^T\emptyset(t) \tag{2.9}$$

where C and $\emptyset$ are $2^{k-1}M \times 1$ matrices given by

$$C = [c_{10}, c_{11}, \dots, c_{1M-1}, c_{20}, \dots, c_{2M-1}, \dots, c_{2^{k-1}0}, \dots, c_{2^{k-1}M-1}]^T \tag{2.10}$$

$$\emptyset = [\varphi_{10}, \varphi_{11}, \dots, \varphi_{1M-1}, \varphi_{20}, \dots, \varphi_{2M-1}, \dots, \varphi_{2^{k-1}0}, \dots, \varphi_{2^{k-1}M-1}]^T \tag{2.11}$$

3 Solution of Partial Differential Equation

In this section Legendre wavelet collocation method is used to obtain a numerical solution to the second order partial differential equation (1.1)with the boundary conditions (1.2)and the initial condition (1.3). To construct a numerical solution, we first consider the nodal points t_i, defined in $[0,T]$, where

$$0 = t_0 < t_1 < \dots < t_i < \dots < T, \quad t_{i+1} - t_i = \tau, \tau = {}^{T}/_{l}, \tag{3.1}$$

$$t_i = i\tau, \quad \text{for} \quad i = 1,2,\dots,l.$$

The initial condition in equation (1.3) is approximated as follows:

$$u(x,0) = u_0 = u(x,t_0), \qquad \forall x \in \Omega \tag{3.2}$$

Next, the 2-point Euler backward differentiation formula is manipulated to approximate $u_t(x,t)$, given in equation (1.1), at the time-level t_i for $i = 1,2,\dots,l$. Therefore, we have

$$\frac{u^i(x)-u^{i-1}(x)}{\tau} - (u'')^i(x) = f^i(x), i = 1,2,\dots,l \tag{3.3}$$

Where $f^i(x) = f(x,t_i)$and$u^i(x) = u(x,t_i)$.Equivalently, we can rewrite equation (3.3) as

$$u^i(x) - \tau(u'')^i(x) = F^i(x), i = 1,2,\dots,l \tag{3.4}$$

where

$F^i(x) = \tau f^i(x) + u^{i-1}(x), i = 1,2,\dots,l.$ (3.5)

Second we approximate the unknown function $u^i(x)$ with help of Legendre wavelet as

$$u^i(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c^i_{n,m}\varphi_{n,m}(x) = C^{i^T}\emptyset(x), i = 1,2,\dots,l$$

(3.6)

$(u'')^i(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c^i_{n,m}\varphi''_{n,m}(x) = C^{i^T}\emptyset''(x), i = 1,2,\dots,l.$ (3.7)

Where C^i and $\emptyset$ are $2^{k-1}M \times 1$ matrices given by equation

$C^i = \left[c^i_{1,0},\dots,c^i_{1,M-1},c^i_{2,0},\dots,c^i_{2,M-1},\dots,c^i_{2^{k-1},M-1}\right]^T, i = 1,2,\dots,l$ (3.8)

$$\varphi = [\varphi_{1,0},\dots,\varphi_{1,M-1},\varphi_{2,0},\dots,\varphi_{2,M-1},\dots,\varphi_{2^{k-1},M-1}],$$

(3.9)

then we can rewrite equation (3.4) as

$$(\mathrm{C^i})^{\mathrm{T}}\emptyset(x) - \tau(\mathrm{C^i})^{\mathrm{T}}\emptyset''(x) = \mathrm{F^i}(x), i = 1,2,\dots,l$$

(3.10)

subject to the boundary conditions

$(C^i)^T\emptyset(a) = g_1(t_i), \quad (C^i)^T\emptyset(b) = g_2(t_i), i = 1,2,\dots,l$

(3.11)

Now collocating equation (3.10) at zeros of Chebyshev polynomial which given by

$$\mathrm{x_j} = \cos\left[\frac{(2\mathrm{j}-1)\pi}{2^{\mathrm{k}}\mathrm{M}}\right], \ j = 1,2,\dots,2^{k-1}M-2$$

(3.12)

Using the collocation points in equation (3.12), we get

$(\mathrm{C^i})^{\mathrm{T}}\emptyset(x_j) - \tau(\mathrm{C^i})^{\mathrm{T}}\emptyset''(x_j) = \mathrm{F^i}(x_j), \ j = 1,2,\dots,2^{k-1}M-2$ (3.13)

we can rewrite equation (3.13) as

$$(\mathrm{C^i})^{\mathrm{T}}[\emptyset(x_j) - \tau\emptyset''(x_j)] = \mathrm{F^i}(x_j), \quad j = 1,2,\dots,2^{k-1}M-2$$

(3.14)

From equation (3.6) there are $2^{k-1}M$ unknown constants. To find out the values of these unknown, we need $2^{k-1}M$equation out of which 2 equation are obtained from boundary conditions (3.11) which the remaining $2^{k-1}M-2$ equations are obtained from (3.14). By solve this system of equations given the unknown constants. By equation (3.6) we can give the approximate solution.

Theorem: Convergence theorem

The series solution (3.6) of problem (3.4) using Legendre wavelet method converges towards $u^i(x), i = 1,2,\dots,l$.

Proof: Let $L^2(R)$ be the Hilbert space and $\varphi_{k,n}(t) = |a|^{-\frac{1}{2}}\varphi(a_0^k t - nb_0)$ where $\varphi_{k,n}(t)$ form a basis of $L^2(R)$. As $a_0 = 2, b_0 = 1$, then $\varphi_{k,n}(t)$ forms an orthonormal basis.

Let

$u^i(x) = \sum_{j=1}^{M-1} C^i_{1j}\varphi_{1j}(x), \qquad i = 1,2,\dots,l$ (3.15)

where

$$C^i_{1j} = \langle u^i(x), \varphi_{1j}(x)\rangle, \ i = 1,2,\dots,l \quad, k = 1$$

(3.16)

and $\langle\cdot,\cdot\rangle$ represent an inner product.
$u^i(x) = \sum_{j=1}^{n}\langle u^i(x), \varphi_{1j}(x)\rangle\varphi_{1j}(x).$ (3.17)
Let us denote $\varphi_{1j}(x)$ as $\varphi(x)$.

Let $\alpha^i_j = \langle u^i(x), \varphi(x)\rangle, \quad i = 1,2,\dots,l.$

And the sequence of partial sums $\{S^i_n\}$ of $\{\alpha^i_j\varphi(x_j)\}$. Let S^i_n and S^i_m be arbitrary partial sums with $n \geq m$. We are going to prove that $\{S^i_n\}$ is a Cauchy sequence in Hilbert space.

Let $S^i_n = \sum_{j=1}^{n}\alpha^i_j\varphi(x_j), \quad i = 1,2,\dots,l$

$$\langle u^i(x), S^i_n\rangle = \langle u^i(x), \sum_{j=1}^{n}\alpha^i_j\varphi(x_j)\rangle, \ i = 1,2,\dots,l$$

(3.18)

$$\langle u^i(x), S^i_n\rangle = \sum_{j=1}^{n}\bar{\alpha}^i_j\langle u^i(x), \varphi(x_j)\rangle, \ i = 1,2,\dots,l$$

(3.19)

$$\langle u^i(x), S^i_n\rangle = \sum_{j=1}^{n}\bar{\alpha}^i_j\,\alpha^i_j = \sum_{j=1}^{n}|\alpha^i_j|^2, \ i = 1,2,\dots,l$$

(3.20)

And $\|S^i_n - S^i_m\|^2 = \sum_{j=m+1}^{n}|\alpha^i_j|^2$ for $n > m$. (3.21)

Then

$$\left\|\sum_{j=m+1}^{n}\alpha_j^i\varphi(x_j)\right\|^2 = \left\langle\sum_{k=m+1}^{n}\alpha_k^i\varphi(x_k), \sum_{j=m+1}^{n}\alpha_j^i\varphi(x_j)\right\rangle \tag{3.22}$$

$$\left\|\sum_{j=m+1}^{n}\alpha_j^i\varphi(x_j)\right\|^2 = \sum_{k=m+1}^{n}\sum_{j=m+1}^{n}\alpha_k^i\bar{\alpha}_j^i\left\langle\varphi(x_k), \varphi(x_j)\right\rangle \tag{3.23}$$

$$\left\|\sum_{j=m+1}^{n}\alpha_j^i\varphi(x_j)\right\|^2 = \sum_{j=m+1}^{n}\alpha_j^i\bar{\alpha}_j^i = \sum_{j=m+1}^{n}\left|\alpha_j^i\right|^2 \tag{3.24}$$

So $\left\|S_n^i - S_m^i\right\|^2 = \sum_{j=m+1}^{n}\left|\alpha_j^i\right|^2$ for $n > m$. (3.25)

From Bessel's inequality, then $\left\|S_n^i - S_m^i\right\|^2 \to 0\ as\ m, n \to \infty$. So $\{S_n^i\}$ is a Cauchy sequnce and it converges to S^i.

We need to show that$u^i(x) = s^i$

As $\langle S^i - u^i(x), \varphi(x_j)\rangle = \langle S^i, \varphi(x_j)\rangle - \langle u^i(x) - \varphi(x_j)\rangle$ (3.26)

$$\langle S^i - u^i(x), \varphi(x_j)\rangle = \lim_{n\to\infty}\langle S_n^i, \varphi(x_j)\rangle - \langle u^i(x) - \varphi(x_j)\rangle \tag{3.27}$$

$$\langle S^i - u^i(x), \varphi(x_j)\rangle = \alpha_j^i - \alpha_j^i = 0 \tag{3.28}$$

Hence $u^i(x) = s^i$ and $\sum_{j=1}^{n}\alpha_j^i\varphi(x_j)$ converges to $u^i(x)$.

4 Illustrative Examples

In this section, we present Legendre wavelet (LW) collocation method for the numerical solution of partial differential equation in comparison with existing method to demonstrate the capability of the proposed method and error analysis are shown in tables and figures. Error function is presented to verify the accuracy and efficiency of the following numerical results. We use MATLAB program to obtain the solution.

Example 1:

Consider the following partial differential equation

$$u_t(x,t) - u_{xx}(x,t) = 2x^2t - 2t^2,\ in\quad Q \equiv (0,1)\times(0,T) \tag{4.1}$$

with the boundary conditions

$$u(0,t) = 2, \qquad u(1,t) = t^2 + 3 \tag{4.2}$$

the exact solution is $u(x,t) = x^2t^2 + x + 2$ (4.3)

Table 4.1. Comparison between exact and numerical solution.

x	$t = 1$		
	Exact solution	Wavelets $M = 4, k = 1$	Wavelets $M = 8, k = 1$
0.0	2.000000	2.000000	2.000000
0.1	2.110000	2.110017	2.110015
0.2	2.240000	2.240012	2.240006
0.3	2.390000	2.390000	2.389997
0.4	2.560000	2.560001	2.559990
0.5	2.750000	2.749996	2.749988
0.6	2.960000	2.959993	2.959993
0.7	3.190000	3.189994	3.190004
0.8	3.440000	3.439998	3.440017
0.9	3.710000	3.710006	3.710026
1.0	4.000000	4.000000	4.000000

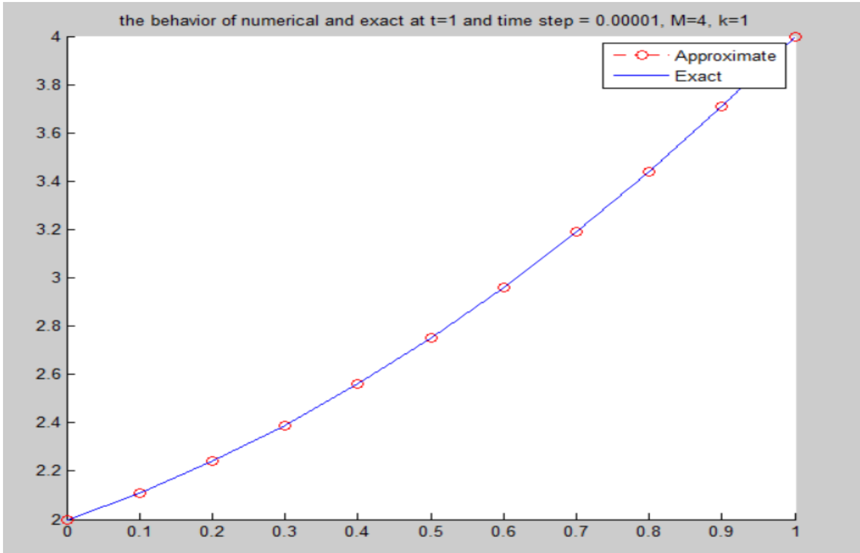

Fig. (4.1): Comparison of numerical solutions and exact solution of example 1.

Example 2:

Consider the following partial differential equation

$u_t(x,t) - u_{xx}(x,t) = e^x - t\,e^x, \ in \ \ Q \equiv (0,1) \times (0,T)$ (4.4)

$u(0,t) = \mathrm{t} + 2, \qquad u(1,t) = te^1 + 2$ (4.5)

the exact solution of equation (4.4) is as follows

$u(x,t) = t\,e^x + 2$ (4.6)

Table 4.2. Comparison between exact and numerical solution.

x	$t = 0.1$		
	Exact solution	Wavelets $M = 4, k = 1$	Wavelets $M = 8, k = 1$
0.0	2.100000	2.100000	2.100000
0.1	2.110517	2.110461	2.110523
0.2	2.122140	2.122054	2.122140
0.3	2.134986	2.134923	2.134980
0.4	2.149182	2.149201	2.149176
0.5	2.164872	2.165019	2.164869
0.6	2.182212	2.182510	2.182214
0.7	2.201375	2.201806	2.201380
0.8	2.222554	2.223040	2.222556
0.9	2.245960	2.246344	2.245960
1.0	2.271828	2.271828	2.271828

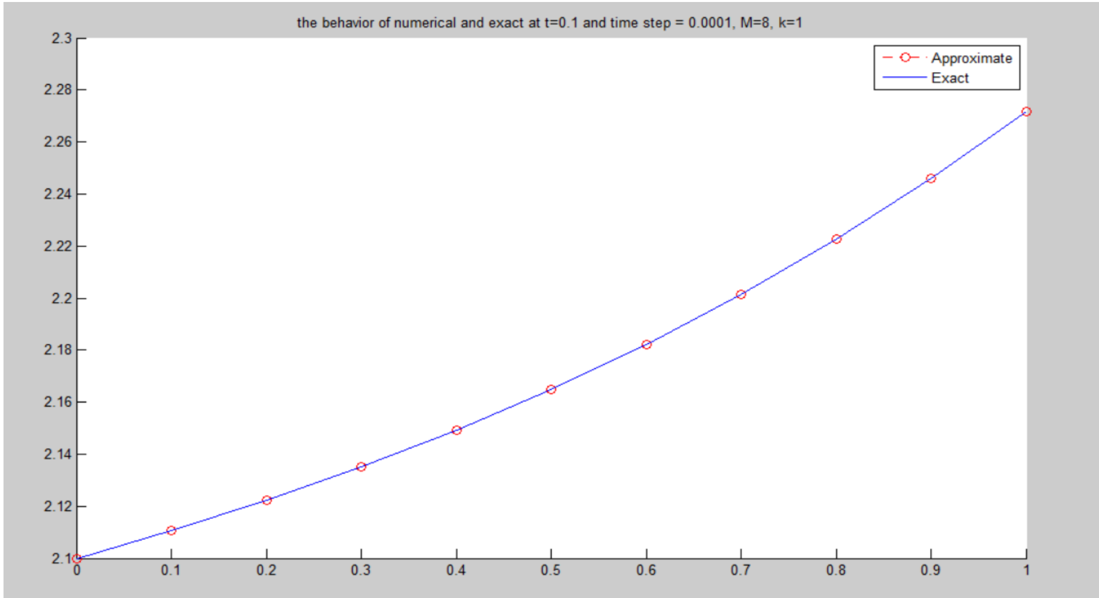

Figure (4.2): Comparison of numerical solutions and exact solution ofexample 2.

From the numerical results in Table 4.1 and 4.2 which shown in figures (4.1) and (4.2), it is easy to conclude that the obtained results by the proposed method are in good agreement with the exact solution

5 Conclusions

The aim of present work is to develop an efficient and accurate method for solving partial differential equation. The problem has been reduced to solving a system of linear algebraic equations. Two test problems were presented to demonstrate the validity of the technique.

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