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A New Nonsingular Fractional Model for a Biological Snap Oscillator with Chaotic Attractors

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Abstract: The research aims to propose a new non-integer order model to analyse the chaotic behaviour of a biological snap oscillator. The suggested model consists of a newly developed fractional derivative with Mittag-Leffler kernel. To investigate the model, the time-domain response and the phase portrait are considered. In addition, a powerful numerical method is employed to implement the model in an appropriate precise manner. The existence of chaotic attractors are shown by some simulations and experiments. Finally, to control the chaos, a stabilizing controller is designed and its effectiveness is illustrated and verified.

Keywords: Biological oscillator, Chaos control, Chaotic system, Fractional calculus, Mittag-Leffler kernel.

1 Introduction

Chaos is a natural phenomenon found in many nonlinear systems, such as biological, economical, mechanical, or physical processes [1–6]. Nowadays, a lot of outstanding researches have been done to analyze chaotic behaviours in biology. For instance, a biological snap oscillator was employed in [7] to describe the interaction between substrate and enzyme in a brain waves model. In [8], the authors investigated a system of substrate-enzyme reaction with brain waves ferroelectric behaviours in which the associate second-order differential equations were imposed by a sinusoidal function as an external input. In [9], a chaotic model of heartbeat activity was investigated. It used the Van der Pol equations as a starting point considering a periodic force for heart electrical stimulation. In [10], using a modified Van der Pol oscillator connected with time-delay couplings, a mathematical model was extended to generate ECG signals.

The fractional calculus, as a branch of mathematical analysis, studies the integral/derivative operators with non-integer order. The application of the fractional calculus can be found in many old and new publications. Recently, the fractional models with chaotic attractors have appeared in different fields of study, such as biology, mechanics, finance, and physics [11–13]. In recent years, many researchers have illustrated that the hereditary nature of fractional derivatives can accurately extract the hidden features of many realistic systems [14–23]. Moreover, new studies proved that the complex behaviours of chaotic phenomena can be described properly by the fractional-order mathematical models [24, 25]. However, to perform an accurate analysis for such systems, we need more efficient fractional models capable of describing nonlocal dynamics, e.g. the models described by nonsingular derivative operators. Inspired by the aforementioned discussion, this research aims to propose a new non-integer order model to exhibit the chaotic behaviour a biological snap oscillator. The fractional model consists of a nonsingular derivative with Mittag-Leffler (ML) function as its kernel [26]. To analyse the model, the time-domain response and the phase portrait are investigated. In addition, a powerful approximation scheme is employed to implement the new model properly. Furthermore, a state-feedback controller is employed to stabilize the chaotic system. Finally, simulation results are used to show the validity of the theoretical analyses.

The rest of this paper is structured as follows: Section 2 introduces some preliminaries and notations. In Section 3, a chaotic snap oscillator is modeled by a fractional differential equation with nonsingular derivative operator. Section 4 provides a numerical method to implement the new non-integer order model. Simulation results of the chaotic attractors are presented in Section 5. Next, the notion of chaos control via a state-feedback controller is examined in Section 6. Conclusion is presented in the last section.

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2 Basic definitions

This section defines a nonsingular fractional derivative with ML kernel and addresses some basic properties associated with this operator [26].

Definition 1.*Let* $t\in(0,T)$, $x\in\mathbb{H}^1(0,T)$, *and* $\alpha\in(0,1)$ *be the fractional order. Then, the AB-Caputo derivative of* x *is defined by*

$$ {}^{ABC}_{\ 0}\mathscr{D}^{\alpha}_{t}x(t)=\frac{\mathscr{M}(\alpha)}{1-\alpha}\int_0^t \mathscr{E}_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-\rho)^{\alpha}\right)\dot{x}(\rho)d\rho, \tag{1}$$

where $\mathscr{E}_{\alpha}$ *denotes the ML function, and* $\mathscr{M}(\alpha)$, *satisfying* $\mathscr{M}(0)=\mathscr{M}(1)=1$, *is a normalizing function. The fractional integral corresponding to Eq. (1) is also expressed as*

$$ {}^{ABC}_{\ 0}\mathscr{I}^{\alpha}_{t}x(t)=\frac{1-\alpha}{\mathscr{M}(\alpha)}x(t)+\frac{\alpha}{\mathscr{M}(\alpha)\Gamma(\alpha)}\int_0^t(t-\rho)^{\alpha-1}x(\rho)d\rho. \tag{2}$$

Hereinafter, some basic properties of the AB-Caputo operator are reviewed [26].

*Property 1.*For a constant function $x(t)\equiv x_c$ we have ${}^{ABC}_{\ 0}\mathscr{D}^{\alpha}_{t}x_c=0$.

*Property 2.*For each $k_1,k_2\in\mathbb{R}$ and $x_1,x_2\in\mathbb{H}^1(0,T)$ we can write

$$ {}^{ABC}_{\ 0}\mathscr{D}^{\alpha}_{t}(k_1x_1(t)+k_2x_2(t))=k_1\,{}^{ABC}_{\ 0}\mathscr{D}^{\alpha}_{t}x_1(t)+k_2\,{}^{ABC}_{\ 0}\mathscr{D}^{\alpha}_{t}x_2(t). \tag{3}$$

*Property 3.*The anti-derivative property holds for the operators (1) and (2) in the way

$$ {}^{ABC}_{\ 0}\mathscr{I}^{\alpha}_{t}\left[{}^{ABC}_{\ 0}\mathscr{D}^{\alpha}_{t}x(t)\right]=x(t)-x(0). \tag{4}$$

*Property 4.*The Laplace transform of the AB-Caputo differential operator is given by

$$\mathfrak{L}\left\{{}^{ABC}_{\ 0}\mathscr{D}^{\alpha}_{t}x(t)\right\}(s)=\frac{\mathscr{M}(\alpha)}{1-\alpha}\frac{s^{\alpha}X(s)-s^{\alpha-1}x(0)}{s^{\alpha}+\frac{\alpha}{1-\alpha}}, \tag{5}$$

where $X(s)=\mathfrak{L}\{x(t)\}(s)$.

*Property 5.*For each $x_1,x_2\in\mathbb{H}^1(0,T)$, the Lipschitz condition is satisfied by the AB-Caputo operator, i.e.

$$\left\|{}^{ABC}_{\ 0}\mathscr{D}^{\alpha}_{t}x_1(t)-{}^{ABC}_{\ 0}\mathscr{D}^{\alpha}_{t}x_2(t)\right\|\leq L\|x_1(t)-x_2(t)\|,\ \|x(t)\|=\max_{0\leq t\leq T}|x(t)|, \tag{6}$$

where $L>0$ is the Lipschitz constant.

*Property 6.*The AB-Caputo derivative satisfies the inequality

$$\left\|{}^{ABC}_{\ 0}\mathscr{D}^{\alpha}_{t}x(t)\right\|\leq\frac{\mathscr{M}(\alpha)}{1-\alpha}\|x(t)\|,\ x\in\mathbb{H}^1(0,T), \tag{7}$$

where $\|x(t)\|=\max_{0\leq t\leq T}|x(t)|$.

The interested reader can see [26–28] for more information.

3 The new model of a biological snap oscillator

In [7], an autonomous model was introduced for the interaction between substrate and enzyme mechanism. By considering the architectonics of snap, the proposed model is given by

$$\begin{cases}\dot{x_1}=x_2,\\ \dot{x_2}=ax_3,\\ \dot{x_3}=x_4,\\ \dot{x_4}=-x_1-bx_3+cx_2-x_4+dx_2(1-x_1^2+ex_1^4-fx_1^6),\end{cases} \tag{8}$$

where a,b,c,d,e,f are real constants. As stated in [7], the model (8) demonstrates chaotic behaviours for certain values of parameters as $(a,b,c,d,e,f) = (5,24,-0.05,2.001,2.55,1.7)$. However, the integer-order model (8) suffers from lack of memory effects as an underlying characteristic of many complex biological systems. To overcome this drawback, we modify this model in the form of a system of fractional-order differential equations. To do so, we use the AB-Caputo derivative in the model (8) instead of ordinary time-derivatives and consider an auxiliary parameter σ to avoid dimensional mismatching. Therefore, the new fractional snap oscillator is formulated, as follows:

$$\begin{cases} \frac{1}{\sigma^{1-\alpha}} {}^{ABC}_{0}\mathscr{D}_t^{\alpha} x_1 = x_2, \\ \frac{1}{\sigma^{1-\alpha}} {}^{ABC}_{0}\mathscr{D}_t^{\alpha} x_2 = a x_3, \\ \frac{1}{\sigma^{1-\alpha}} {}^{ABC}_{0}\mathscr{D}_t^{\alpha} x_3 = x_4, \\ \frac{1}{\sigma^{1-\alpha}} {}^{ABC}_{0}\mathscr{D}_t^{\alpha} x_4 = -x_1 - b x_3 + c x_2 - x_4 + d x_2 (1 - x_1^2 + e x_1^4 - f x_1^6), \end{cases} \tag{9}$$

where ${}^{ABC}_{0}\mathscr{D}_t^{\alpha} x_i$ is the AB-Caputo derivative of x_i specified by (1). The system (9) is an extended version of a generalised Van der Pol oscillator. Thus, it can be used as a model for many bio-physical and biological systems.

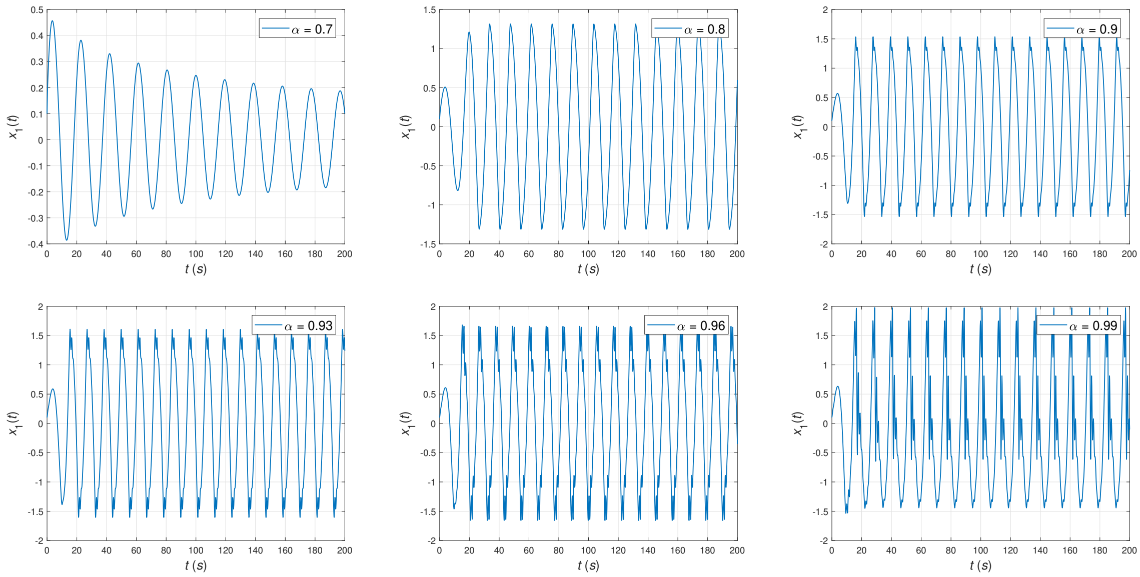

Fig. 1: The time-domain response of the state variable $x_1(t)$ for $\alpha = 0.7, 0.8, 0.9, 0.93, 0.96, 0.99$.

4 The proposed numerical method

In this section, the product-integration (PI) rule [29] is employed to design a powerful numerical method solving the fractional biological model under consideration. To do so, consider the problem

$$\begin{cases} \frac{1}{\sigma^{1-\alpha}} {}^{ABC}_{0}\mathscr{D}_t^{\alpha} x(t) = f(x(t)), \\ x(0) = x_0, \end{cases} \tag{10}$$

where $f(\cdot)$ represents a continuous function. From Eq. (10), we obtain the following Volterra integral equation by applying the integral operator (2) and taking into account the anti-derivative property (4)

$$\frac{1}{\sigma^{1-\alpha}} (x(t) - x(0)) = \frac{1-\alpha}{\mathscr{M}(\alpha)} f(x(t)) + \frac{\alpha}{\mathscr{M}(\alpha)\Gamma(\alpha)} \int_0^t (t-\rho)^{\alpha-1} f(x(\rho)) d\rho. \tag{11}$$

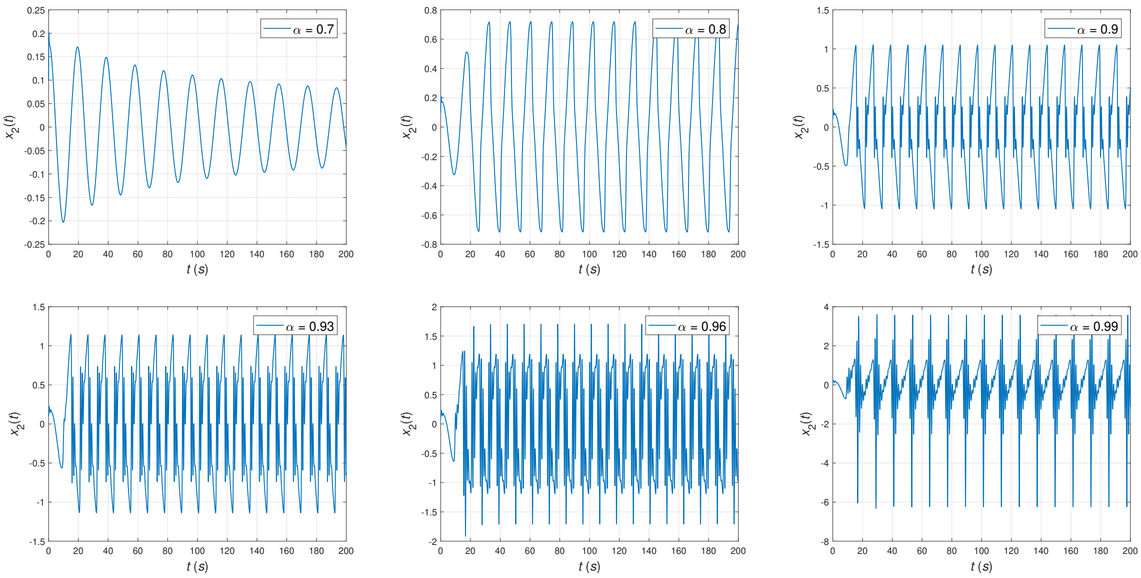

Fig. 2: The time-domain response of the state variable $x_2(t)$ for $\alpha = 0.7, 0.8, 0.9, 0.93, 0.96, 0.99$.

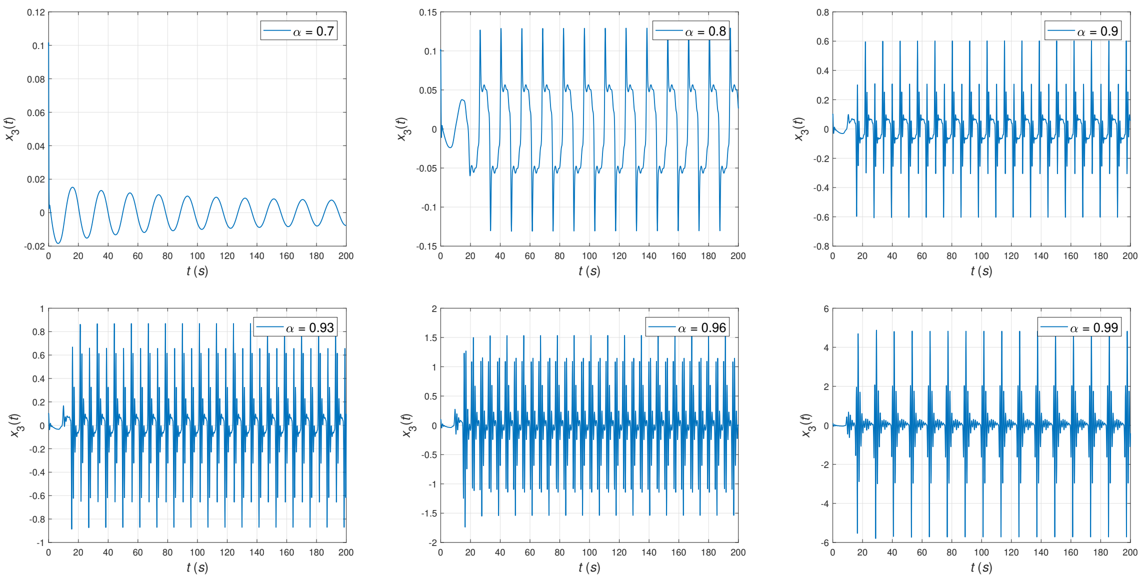

Fig. 3: The time-domain response of the state variable $x_3(t)$ for $\alpha = 0.7, 0.8, 0.9, 0.93, 0.96, 0.99$.

Setting $t = t_k = kh$ with h as a time step size, we get

$$\frac{1}{\sigma^{1-\alpha}}x(t_k) = \frac{1}{\sigma^{1-\alpha}}x(0) + \frac{1-\alpha}{\mathscr{M}(\alpha)}f(x(t_k)) + \frac{\alpha}{\mathscr{M}(\alpha)\Gamma(\alpha)}\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}(t_k-\rho)^{\alpha-1}f(x(\rho))d\rho. \tag{12}$$

Now, we approximate the function $f(x(\rho))$ in Eq. (12), $\rho \in [t_j, t_{j+1}]$ by using the first-order Lagrange interpolation

$$f(x(\rho)) \approx f(x_{j+1}) + \frac{\rho - t_{j+1}}{h}\left(f(x_{j+1}) - f(x_j)\right), \tag{13}$$

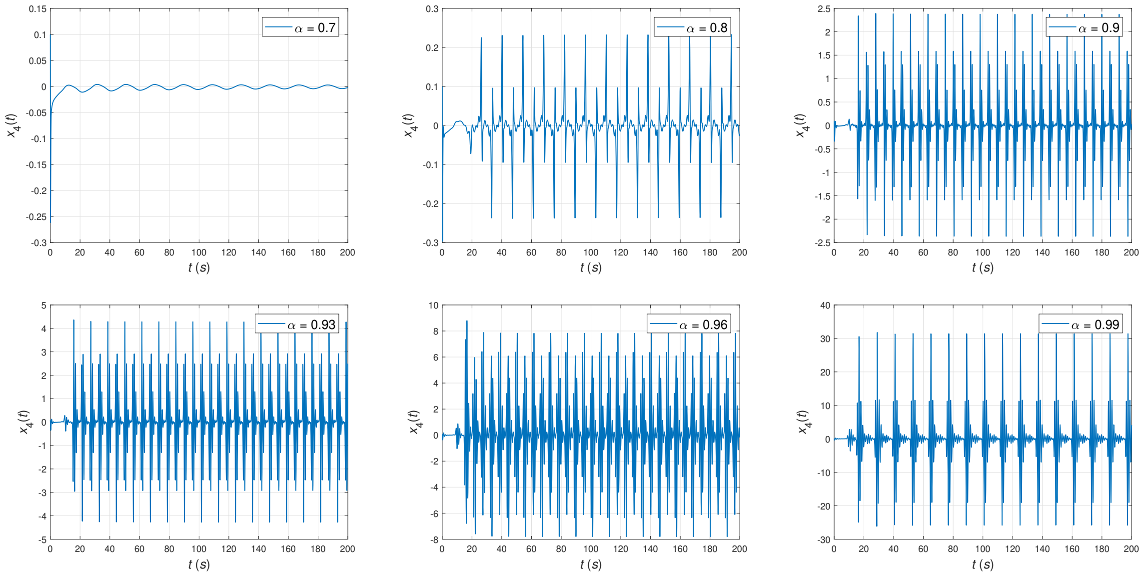

Fig. 4: The time-domain response of the state variable $x_4(t)$ for $\alpha = 0.7, 0.8, 0.9, 0.93, 0.96, 0.99$.

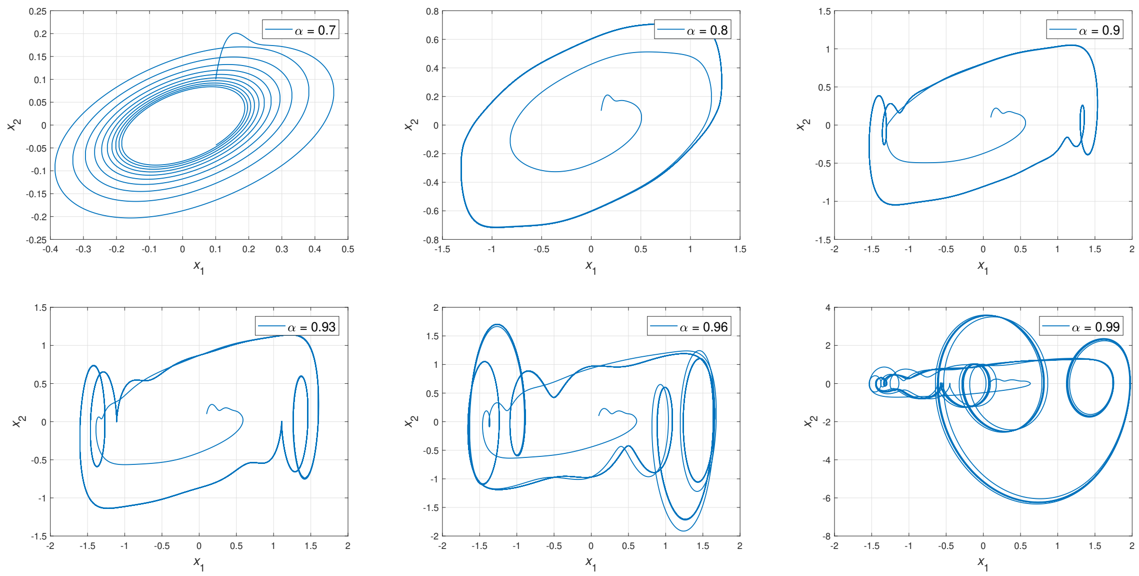

Fig. 5: The phase-portrait in (x_1, x_2)-plane for $\alpha = 0.7, 0.8, 0.9, 0.93, 0.96, 0.99$.

where the approximation of $x(t)$ at $t = t_j$ is denoted by x_j. The approximation (13) is used in (12). Then, by some algebraic manipulations, the AB-Caputo PI formula is attained as follows

$$x_k = x_0 + \frac{(\sigma^{1-\alpha})\alpha h^{\alpha}}{\mathscr{M}(\alpha)} \left(a_k f(x_0) + \sum_{j=1}^{k} b_{k-j} f(x_j) \right), k \geq 1, \tag{14}$$

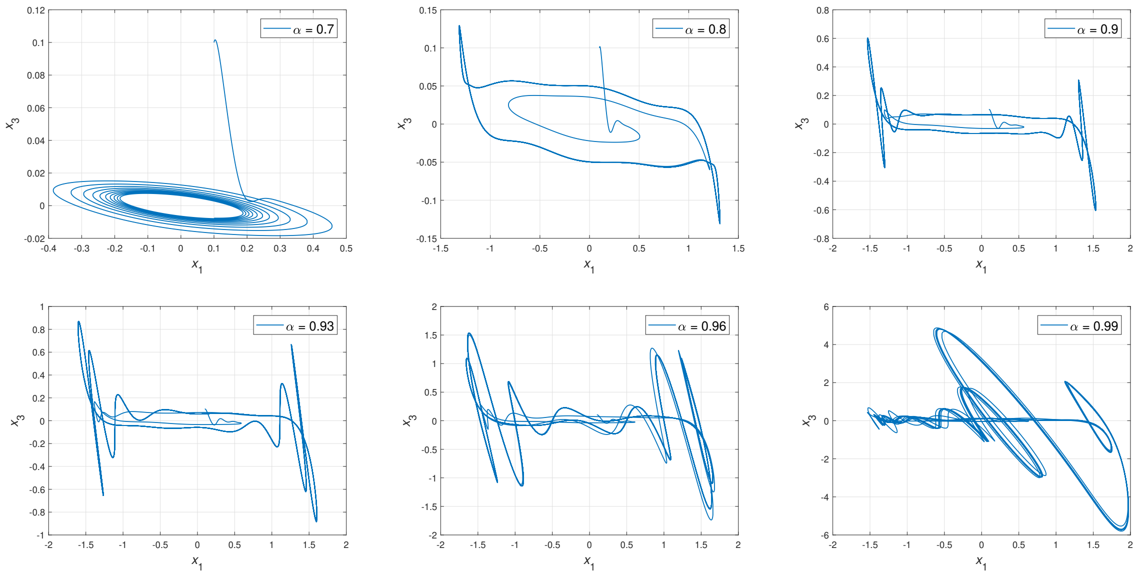

Fig. 6: The phase-portrait in (x_1, x_3)-plane for $\alpha = 0.7, 0.8, 0.9, 0.93, 0.96, 0.99$.

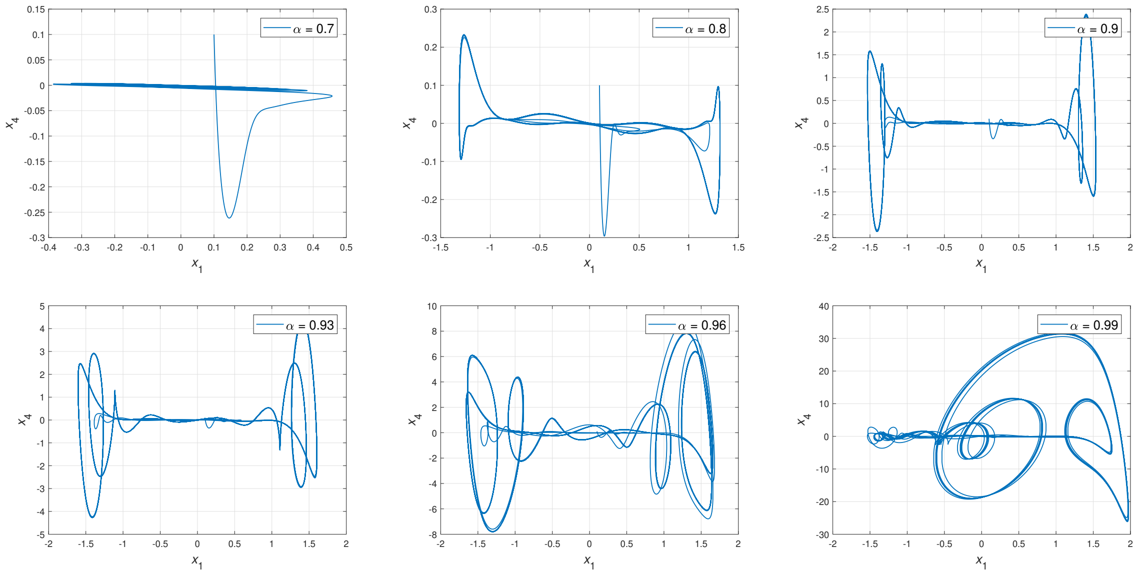

Fig. 7: The phase-portrait in (x_1, x_4)-plane for $\alpha = 0.7, 0.8, 0.9, 0.93, 0.96, 0.99$.

where

$$a_k = \frac{(k-1)^{\alpha+1} - k^{\alpha}(k-\alpha-1)}{\Gamma(\alpha+2)}, \tag{15}$$

$$b_i = \begin{cases} \frac{1}{\Gamma(\alpha+2)} + \frac{1-\alpha}{\alpha h^{\alpha}}, & i = 0, \\ \frac{(i-1)^{\alpha+1} - 2i^{\alpha+1} + (i+1)^{\alpha+1}}{\Gamma(\alpha+2)}, & i = 1, 2, \ldots, k-1. \end{cases} \tag{16}$$

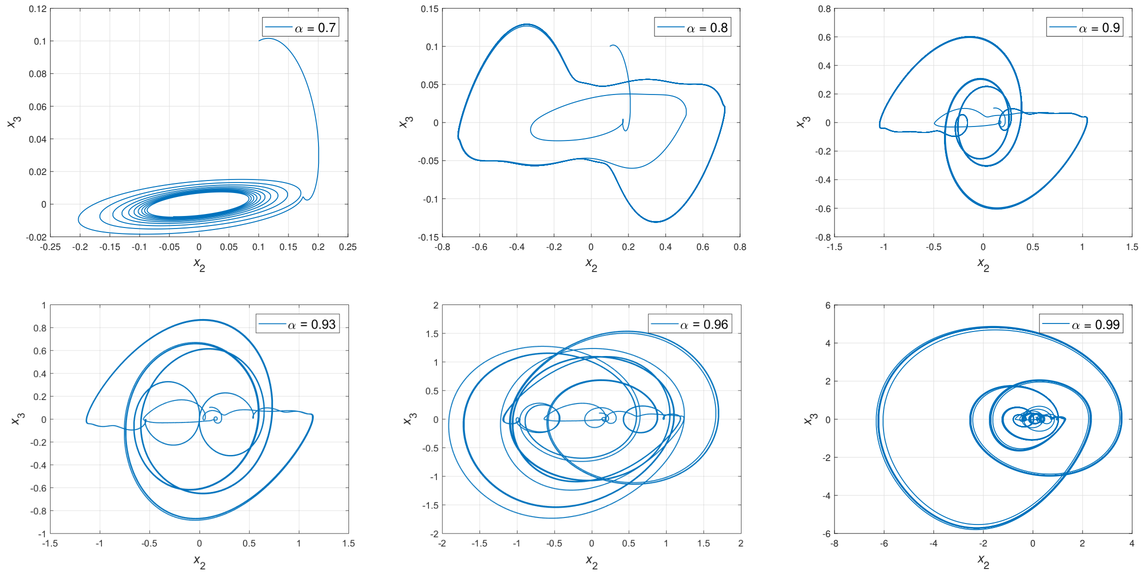

Fig. 8: The phase-portrait in (x_2,x_3)-plane for $\alpha = 0.7, 0.8, 0.9, 0.93, 0.96, 0.99$.

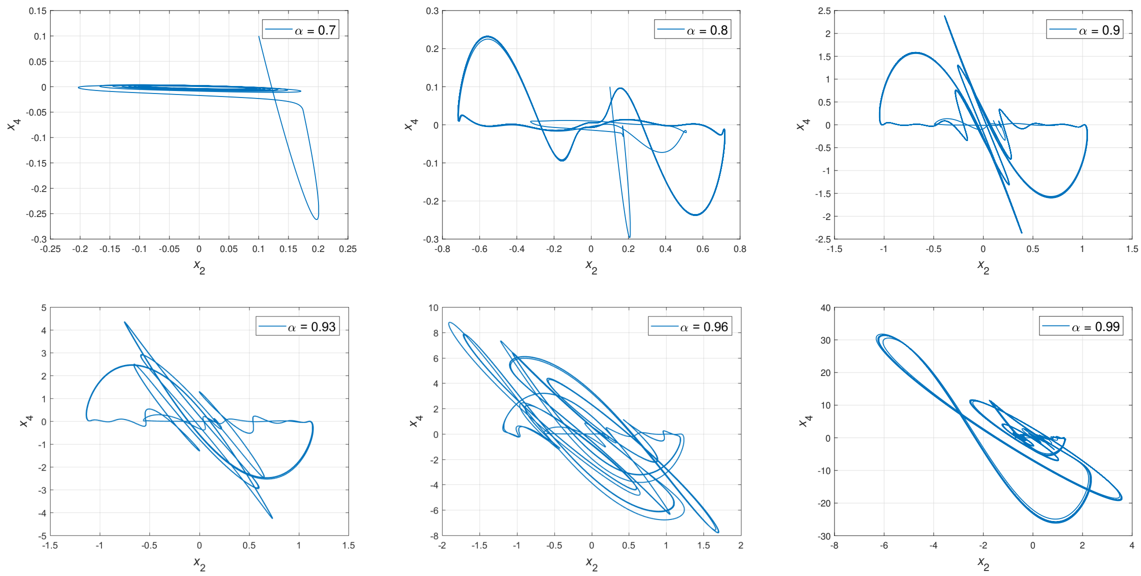

Fig. 9: The phase-portrait in (x_2,x_4)-plane for $\alpha = 0.7, 0.8, 0.9, 0.93, 0.96, 0.99$.

According to the analysis in [30], the convergence order here is $1+\alpha$, i.e. the error satisfies $|x(t_k)-x_k| = \mathscr{O}(h^{1+\alpha})$. Finally, we obtain the following recursive formulas by applying the proposed approach to the system (9)

$$x_{1,k} = x_{1,0} + \frac{(\sigma^{1-\alpha})\alpha h^{\alpha}}{\mathscr{M}(\alpha)} \left(a_k f_1(x_{1,0},x_{2,0},x_{3,0},x_{4,0}) + \sum_{j=0}^{k} b_{k-j} f_1(x_{1,j},x_{2,j},x_{3,j},x_{4,j}) \right), \tag{17}$$

$$x_{2,k} = x_{2,0} + \frac{(\sigma^{1-\alpha})\alpha h^{\alpha}}{\mathscr{M}(\alpha)} \left(a_k f_2(x_{1,0},x_{2,0},x_{3,0},x_{4,0}) + \sum_{j=0}^{k} b_{k-j} f_2(x_{1,j},x_{2,j},x_{3,j},x_{4,j}) \right), \tag{18}$$

$$x_{3,k} = x_{2,0} + \frac{(\sigma^{1-\alpha})\alpha h^{\alpha}}{\mathscr{M}(\alpha)} \left(a_k f_3(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}) + \sum_{j=0}^{k} b_{k-j} f_3(x_{1,j}, x_{2,j}, x_{3,j}, x_{4,j}) \right), \tag{19}$$

$$x_{4,k} = x_{4,0} + \frac{(\sigma^{1-\alpha})\alpha h^{\alpha}}{\mathscr{M}(\alpha)} \left(a_k f_4(x_{1,0}, x_{2,0}, x_{3,0}, x_{4,0}) + \sum_{j=0}^{k} b_{k-j} f_4(x_{1,j}, x_{2,j}, x_{3,j}, x_{4,j}) \right), \tag{20}$$

where

$$\begin{aligned} f_1(x_1, x_2, x_3, x_4,) &= x_2, \\ f_2(x_1, x_2, x_3, x_4,) &= a x_3, \\ f_3(x_1, x_2, x_3, x_4,) &= x_4, \\ f_4(x_1, x_2, x_3, x_4,) &= -x_1 - b x_3 + c x_2 - x_4 + d x_2(1 - x_1^2 + e x_1^4 - f x_1^6), \end{aligned} \tag{21}$$

5 Simulation results

This section discusses the complex behavior of the AB-Caputo fractional snap oscillator (9) through some simulations and figures. The method we used to implement the model is the AB-Caputo PI scheme developed in the previous section. The initial conditions are taken into account

$$x(0) = (x_1(0), x_2(0), x_3(0), x_4(0), x_5(0)) = (0.1, 0.1, 0.1, 0.1), \tag{22}$$

the parameter values are considered as $(a,b,c,d,e,f) = (5, 24, -0.05, 2.001, 2.55, 1.7)$, in which the integer-order model (8) exhibits chaotic attractors, the modification parameter is chosen to be $\sigma = 0.99$, and the fractional order is taken from $\alpha = 0.7, 0.8, 0.9, 0.93, 0.96, 0.99$. The time-domain responses are shown in figures 1-4 whereas the phase portraits are depicted in figures 5-9. These figures indicate that totally different dynamical behaviours are appeared by changing the parameter α. This confirms that the fractional order itself provides a degree of flexibility affecting the performance of non-integer order models. This fact helps us exhibit the hidden features of complex dynamical systems properly.

6 Chaos control

In this section, a state-feedback control is employed to overcome the undesirable behaviour of the fractional snap oscillator (9). Thus, the fractional model (9) is rewritten as follows

$$\begin{cases} \frac{1}{\sigma^{1-\alpha}} {}^{ABC}_{0}\mathscr{D}^{\alpha}_{t} x_1 = x_2 - p_1 x_1, \\ \frac{1}{\sigma^{1-\alpha}} {}^{ABC}_{0}\mathscr{D}^{\alpha}_{t} x_2 = a x_3 - p_2 x_2, \\ \frac{1}{\sigma^{1-\alpha}} {}^{ABC}_{0}\mathscr{D}^{\alpha}_{t} x_3 = x_4 - p_3 x_3, \\ \frac{1}{\sigma^{1-\alpha}} {}^{ABC}_{0}\mathscr{D}^{\alpha}_{t} x_4 = -x_1 - b x_3 + c x_2 - x_4 + d x_2(1 - x_1^2 + e x_1^4 - f x_1^6) - p_4 x_4, \end{cases} \tag{23}$$

where the constant $p_i > 0$ is the state-feedback gain. Also, we select $\alpha = 0.96$ in which the uncontrolled system (9) exhibits chaotic behavior. The initial conditions and the parameter values are considered as in the previous section. Simulating the controlled model with $p_i = 0.25$, we depict the results in figures 10-11. Our numerical findings verify the efficiency of the proposed stabilizing controller to diminish the undesirable effect of chaotic attractors.

7 Conclusion

In this paper, a new fractional chaotic model was introduced for a biological snap oscillator. The proposed model consisted of the recently introduced AB-Caputo fractional derivative. The model was also implemented by applying a powerful approximation scheme formulated by the PI rule. The chaotic attractors of the AB-Caputo snap oscillator were portrayed through some simulations and figures. The results in figures 1-9 indicated that the fractional-order model provides a degree of flexibility affecting the performance of complex systems. This fact helps us exhibit the hidden features of real-world phenomena in a proper manner. Finally, we designed a state-feedback controller to diminish the undesirable effect of chaotic attractors. Simulation results in figures 10-11 confirmed the validity of the proposed stabilizing controller.

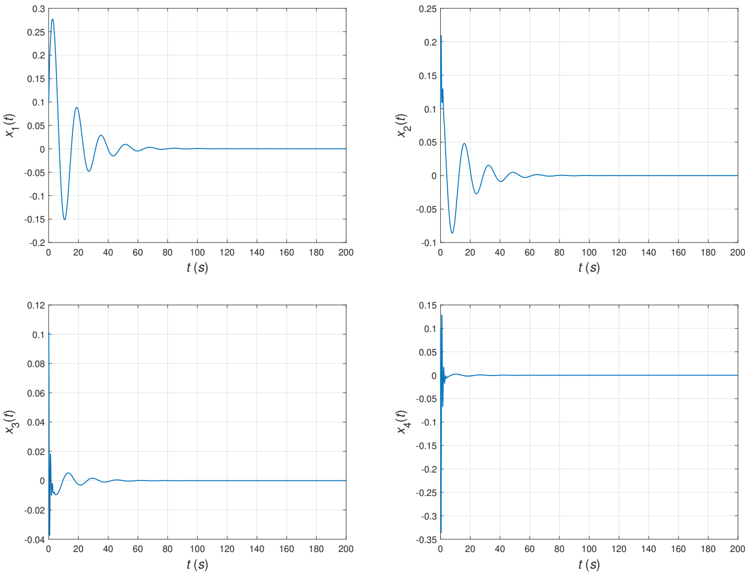

Fig. 10: The state trajectories of the controlled system (23) with $\alpha = 0.96$ and $p_i = 0.25$.

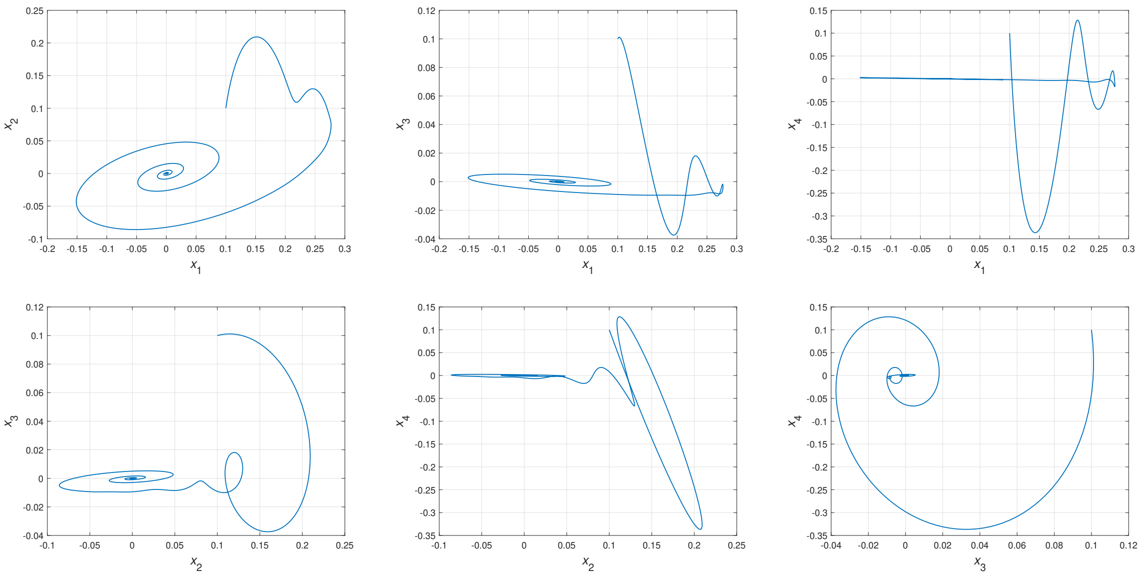

Fig. 11: The phase-portraits of the controlled system (23) with $\alpha = 0.96$ and $p_i = 0.25$.

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