Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/070405

On the Entropic Order Quantity Model Based on the Conformable Calculus

Samir B. Hadid[1], *Rabha W. Ibrahim*[2,*] *and Shaher Momani*[3,4]

[1] Department of Mathematics and Sciences, College of Humanities and Sciences, Ajman University, Ajman, UAE
[2] 1IEEE: 94086547, Kuala Lumpur, 59200, Malaysia
[3] Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, UAE
[4] Department of Mathematics, Faculty of Science, University of Jordan, Amman 11942, Jordan

Received: 5 Feb. 2020, Revised: 7 May 2020, Accepted: 14 Jun. 2020
Published online: 1 Oct. 2021

Abstract: In this paper, we present a dynamic system for economic order quantity (EOQ) by utilizing a special type of fractional calculus named conformable calculus. We present the generalized conformable entropy order quantity (C-EnOQ). In this situation, we supply the cost functions with reference to time in a recurring dated. In this system, we consider the linked optimization issue and improve an uninterrupted method for figuring a bounded interval casing the optimal arrangement expanse, exploiting the Tsallis fractional entropy. Moreover, for an astonishing class of transference functions, we explore these cost functions to compute the optimal magnitude.

Keywords: Fractional operator, conformable calculus, conformable differential operator, fractional calculus, economic order quantity, fractional entropy, cost function.

1 Introduction

Currently, entropy has been the most important quantity of chaos in all physical structures including computer studies specially image processing [1]. Some assembly investigators have utilized the idea of entropy rented from information theory employing the Shannon's entropy formula. Distinctly, the thermodynamic entropy notion has been smeared in studies containing result trees, work force schemes, logistics organization, business process management, inventory organization, price-quality relations, organizing orders in a quantity chain, opposite logistics and bio-economy (see[2,3, 4] for recent works).

The economic order quantity (EOQ) system established by Harris [5] is supposed to be the first available record management system and the foundation for the growth of many record systems. Investigators advised that substantial error or misconstructions during parameter effort purpose often lead to limited consequences. Ibrahim and Hadid [6] involved the marginal of EOQ formula. The study is based on the fixed-point theorems in some of compact sets. The minimization of EOQ is given by allowing a geometric setting for the full cost function through a time. The investigation correlated to the minimum norm result. This method allows EOQ difficulties to be under variable credit period, where the classical simulations are fulfilled Leray-Schauder theorem [7].

Jaber et al. [8] planned a correlation between the performance of manufacture structures as well as the behavior of physical scheme. They recommended a research work by employing the first and second formulas of the thermodynamics to decrease scheme entropy and increase progress in construction structure achievement. They familiarized the notion of entropy based on the cost function and its derivative. That is desirable to control the development procedure and enlarged union influence. Their consequences recommended that stuffs must be demanded in superior values than designated by the classical EOQ system. Also, Jaber et al. [8] examined their earlier system for organizing tips in a two-tier source chain lower than the molds of a constant rather than an increasing product. Lately, they modified the system for the

* Corresponding author e-mail: rabhaibrahim@yahoo.com

supposition of approved delay in costs. Many studies later appeared in the EnOQ. Pattnaik [9] presented and generalized a new style of replenishment policy in EnOQ system for a delicate invention with two component request under cash discounts predominantly over a finite time limit. Researchers investigated the estimating and lot sizing EnOQ system for a moldering thing under finite construction, exponential decay, limited back ordering and lost sales. Moreover, they considered EnOQ system with marketing price in need of demand and showing to discrete damaged quality producers. They found a significant reflection about the conclusion of entropy cost on the maximization of income, which shapes that the entropy cost has analogous conduct as the marketing price of the produce. Their results inform the visions of the entropic order record system and supplement the progress of the works of record system [10].

Anderson and Ulness suggested a novel class of a calculus, which is conformable (CC) [11] (extended in [12]) having a control construction, which is called the proportional-integral-derivative controller (PID). PID is a control loop mechanism using feedback generally utilized in industrial and economy control systems and a selection of other applications that require continuously tempered control. These PID controller approaches have different advantages, disadvantages, and concerns if one approach controls. Extreme relative action leads to falter the economy or hesitation in the growth of the economy, extreme integral activation implies overreaches, and the extreme derivative achievement causes an oscillatory attitude to set-point. Therefore, in such systems, it is confidential to build an entropy solution formula. Based on CC, we define the Tsallis conformable entropy (TCE) and employ the result to deliver a new look of EnOQ calling conformable entropy order quantity (C-EnOQ).

2 Processing

We process our method in this part.

2.1 CC

Definition 1. *A Conformable operator (CO) $\mathscr{D}^\beta$ is called conformable of order $\beta \in [0,1]$, if $\mathscr{D}^0$ is the identity operator and $\mathscr{D}^1$ is the ordinary differential operator. Particularly, the operator*

$$\mathscr{D}^\beta f(t) = \left(t^{1-\beta}\right) f'(t) \tag{1}$$

is conformable if for a differential function $f(t)$,

$$\mathscr{D}^0 f(t) = f(t) \ \ and \ \ \mathscr{D}^1 f(t) = f'(t). \tag{2}$$

Utilizing the control system with the controller u at time t with two tuning criteria has the setting

$$u(t) = \nu_p \Sigma(t) + \nu_d \Sigma'(t), \tag{3}$$

where ν_p is the proportional gain, ν_d is the derivative gain, and Σ is the error between the formal variable and the actual variable. Based on (3), Anderson and Ulness [11] suggested the following construction:

Definition 2. *Consider two continuous functions $\Bbbk_0, \Bbbk_1 : [0,1] \times \mathbb{R} \to (0,\infty)$ such that*

$$\mathscr{D}^\beta f(t) = \Bbbk_1(\ell,t) f(t) + \Bbbk_0(\ell,t) f'(t) \tag{4}$$

such that $\Bbbk_1(\ell,t) \neq -\Bbbk_0(\ell,t)$,

$$\lim_{\ell\to 0} \Bbbk_1(\ell,t) = 1, \quad \lim_{\ell\to 1} \Bbbk_1(\ell,t) = 0, \quad \Bbbk_1(\ell,t) \neq 0, \ \forall t, \ \ell \in (0,1), \tag{5}$$

and

$$\lim_{\ell\to 0} \Bbbk_0(\ell,t) = 0, \quad \lim_{\ell\to 1} \Bbbk_0(\ell,t) = 1, \quad \Bbbk_0(\ell,t) \neq 0, \ \forall t, \ell \in (0,1). \tag{6}$$

Definition 3. *Coordinately, the integral is defined, as follows:*

$$\int \mathscr{D}^\ell f(t) d_\ell t = f(t) + k e_0(t,t_0), \tag{7}$$

where $k \in \mathbb{R}$, $d_\ell t = \frac{dt}{\Bbbk_0(t)}, \nu \neq 0$ and

$$e_0(t,\kappa) = \exp\left(- \int_\kappa^t \frac{\Bbbk_1(\ell,\varsigma)}{\Bbbk_0(\ell,\varsigma))} d\varsigma \right). \tag{8}$$

In the sequel, we formulate the connections, as follows:

$$\Bbbk_1(\ell,t) = (1-\ell)t^{\ell}, \quad \Bbbk_0(\ell,t) = \ell t^{1-\ell}, \quad t \in (0,\infty), \tag{9}$$

$$\Bbbk_1(\ell,t) = (1-\ell)|t|^{\ell}, \quad \Bbbk_0(\ell,t) = \ell |t|^{1-\ell}, \tag{10}$$

$$\Bbbk_1(\ell,t) = \cos(\frac{\ell\pi}{2})t^{\ell}, \quad \Bbbk_0(\ell,t) = \sin(\frac{\ell\pi}{2})t^{1-\ell}, \quad t \in (0,\infty) \tag{11}$$

$$\Bbbk_1(\ell,t) = \cos(\frac{\ell\pi}{2})|t|^{\ell}, \quad \Bbbk_0(\ell,t) = \sin(\frac{\ell\pi}{2})|t|^{1-\ell} \quad t \in \mathbb{R}\backslash\{0\}. \tag{12}$$

2.2 EOQ

The construction of EOQ is formulated by Ford W. Harris [5], as follows:

$$Q = \sqrt{\frac{2\aleph\flat}{\mathsf{C}}}, \tag{13}$$

where $\aleph$ is the annual demand quantity, $\flat$ is the cost for each item and C is the annual holding cost for each item. This formula is generated, as follows [6]:

$$Q'(t) = \Theta\Big(Q(\aleph t), Q(\flat t), Q(\mathsf{C}t)\Big). \tag{14}$$

While the fractal of EnOQ was investigated in [10]. In the recent work, we aim to use CC to generalize (14), as follows:

$$\mathscr{D}^{\beta}Q(t) = \Bbbk_1(\ell,t)Q(t) + \Bbbk_0(\ell,t)Q'(t) \tag{15}$$

$$= \Bbbk_1(\ell,t)Q(t) + \Bbbk_0(\ell,t)\Theta\big(Q(\aleph t), Q(\flat t), Q(\mathsf{C}t)\big). \tag{16}$$

Different studies are presented in [6,7]. Here, we generalize derivative of Tsallis entropy in view of the (4) (C-EOQ) and show that the unique outcome of Eq.(15) is bounded by Tsallis conformable entropy under some hypotheses. It is well known that the boundedness by entropy formula leads to the stability of the system.

2.3 Tsallis Conformable Entropy (TCE)

Tsallis [13] designed an entropy, which is described by an index τ and denoted by $\mathbb{T}_{\tau}$. Tsallis formula is utilized in numerous scientific areas (theories and applications). The overall continuous arrangement of this entropy is formulated as follows:

$$\mathbb{T}_{\tau}(\sigma) = \frac{\int_t (\sigma(t))^{\tau}dt - 1}{1-\tau}, \quad \tau \neq 1, \tag{17}$$

where $\sigma(t)$ is a probability density function. Recently, Ibrahim and Darus [14] have extended the Tsallis entropy into the complex plane by suggesting the complex probability density function

$$T_{\tau}(z) := (\tau - 1)\mathbb{T}_{\tau}[P_c]_U = 1 - \int_0^z \big(P_c(w)\big)^{\tau} dw, \quad z \in U, \tag{18}$$

where P_c (complex probability density function) is analytic in U defined in the open unit disk U owning the formula

$$P_c(z) = \sum_{n=0}^{\infty} p_n z^n, \quad z \in U. \tag{19}$$

Using the CC in Eq.(4), Tsallis entropy (17) becomes

$$\mathscr{D}^{\beta}\mathbb{T}_{\tau}(\sigma)(t) = v_1(\beta,t)\mathbb{T}_{\tau}(\sigma)(t) + v_0(\beta,t)\mathbb{T}'_{\tau}(\sigma)(t) \tag{20}$$

where $\mathbb{T}'_{\tau}$ represents the first derivative of Tsallis entropy. Thus, using (7), we obtain

$$\int \mathscr{D}^{\beta}\mathbb{T}_{\tau}(\sigma)(t)d_{\beta}t = \mathbb{T}_{\tau}(\sigma)(t) + Ke_0(t,t_0), \tag{21}$$

where $e_0(t,t_0)$ is given in (8).

By the thermodynamical idea, the system in (20) organizes a connection between irrevocable and revocable domains. On one hand, the equation is time reverse invariant, and on the other it is not. Note that the Tsallis entropies are not extensive (unequal to the Shannon type), so the time derivative of them cannot mainly be recognized as a production rate. Some investigators indicated that the entropy production rate increases rather than decreases within the Shannon type if the entropy time derivative is considered to be the amount of deviation from equilibrium. Formula (20) might not indicate the above properties because of the definitions of v_0 and v_1. We shall use (20) as an upper bound of the unique solution of (15) in the next section.

2.4 *Fixed Point Theorem*

A class of fixed-point theorems is one of the significant and motivating subjects of nonlinear functional analysis that mergers geometry, analysis and applied mathematics. It is a solid tool to recognize the outcomes of many problems in physics, economics and computer science. In this effort, we shall apply a special type of fixed-point theorem called weakly contractually fixed point theorem. We need the following information.

Definition 4.([15]) *Let X be a nonempty set and $\mathfrak{S}: X^3 \to [0,\infty)$ be a function satisfying the following conditions for all $x, y, z, w \in X$:*
(H_1) $\mathfrak{S}(x,y,z) = 0$ if and only if $x = y = z$;
(H_2) $\mathfrak{S}(x,y,z) \leq \mathfrak{S}(x,x,w) + \mathfrak{S}(y,y,w) + \mathfrak{S}(z,z,w)$.
Then the function $\mathfrak{S}$ is called an $\mathfrak{S}$-metric on X and the pair $(X,\mathfrak{S})$ is called an $\mathfrak{S}$-metric space.

This space achieves the property $\mathfrak{S}(x,x,y) = \mathfrak{S}(y,y,x)$ for all $x, y \in X$.

Lemma 1.*Let (X; $\mathfrak{S}$) be a complete $\mathfrak{S}$-metric space and P : X →X be a contraction. Then P admits a unique fixed point in X.*

Next section addresses the main study in this paper. We present the results in two subsections: conformable calculus outcomes and classical fractional calculus to compare the calculus.

3 Results

Our main results are, as follows:

3.1 *Conformable Calculus Results*

Define an operator $P; X \to X$, as follows:

$$(PQ)(t) = \int \big(\Bbbk_1(\ell,t)Q(t) + \Bbbk_0(\ell,t)\,\Theta\big(Q(\aleph t),Q(\flat t),Q(\complement t)\big)\big)\,d_\ell t + Ke_0(t,t_0), \quad X := \mathbb{R}. \tag{22}$$

Since $Q \in X$, then P is a self-mapping. Also, let a function $\mathfrak{S} : X^3 \to \mathbb{R}^+$ by

$$\mathfrak{S}(\chi_1,\chi_2,\chi_3) = \max\{|\chi_t - \chi_| : t = 1,2,3, t \neq\},$$

where $\chi_1(t) = Q(t)$, $\chi_2(t) = Q(Q(t))$ and $\chi_3(t) := \Theta\big(Q(at),Q(bt),Q(ct)\big)$. Obviously, $\mathfrak{S}(\chi_1,\chi_2,\chi_3) = 0$ for $\chi_1 = \chi_2 = \chi_3$. Also, we have

$$\begin{aligned}
&\mathfrak{S}(\chi_1,\chi_1,\chi_i) + \mathfrak{S}(\chi_2,\chi_2,\chi_j) + \mathfrak{S}(\chi_3,\chi_3,\chi_k) && (23)\\
&= \max_{i=2,3}\{|\chi_1 - \chi_i|\} + \max_{j=1,3}\{|\chi_2 - \chi_j|\} + \max_{k=1,2}\{|\chi_3 - \chi_k|\} && (24)\\
&= \max\{|\chi_1 - \chi_2|, |\chi_1 - \chi_3\} + \max\{|\chi_2 - \chi_1|, |\chi_2 - \chi_3\} && (25)\\
&\quad + \max\{|\chi_3 - \chi_1|, |\chi_3 - \chi_2|\} && (26)\\
&= 2\max\{|\chi_1 - \chi_2|, |\chi_2 - \chi_3|, |\chi_3 - \chi_1|\} && (27)\\
&> \max\{|\chi_1 - \chi_2|, |\chi_2 - \chi_3|, |\chi_3 - \chi_1|\} && (28)\\
&= \max\{|\chi_t - \chi_| : t = 1,2,3, t \neq\} && (29)\\
&= \mathfrak{S}(\chi_1,\chi_2,\chi_3). && (30)
\end{aligned}$$

Hence, the metric $\mathfrak{S}(\chi_1,\chi_2,\chi_3)$ is an $\mathfrak{S}$-metric on the set X.

Theorem 1. *In (15), if*

$$|\Theta\big(Q_1(\aleph t),Q_1(\flat t),Q_1(\complement t)\big)-\Theta\big(Q_2(\aleph t),Q_2(\flat t),Q_2(\complement t)\big)|<\hbar|Q_1(t)-Q_2(t)| \tag{31}$$

for some positive constant

$$\hbar<\frac{1-(1-\ell)\mathfrak{T}^{\ell}}{\beta\mathfrak{T}^{1-\ell}},\quad \mathfrak{T}<\infty. \tag{32}$$

Then, P admits a unique fixed point in X.

Proof. Let

$$\Theta\big(Q(\aleph t),Q(\flat t),Q(\complement t)\big):=\Theta\big(Q(t)\big).$$

Let the functions $\Bbbk_0$ and $\Bbbk_1$ be given by

$$\Bbbk_1(\ell,t)=(1-\ell)t^{\beta},\quad \Bbbk_0(\ell,t)=\ell t^{1-\ell},\quad t\in(0,\mathfrak{T}),\mathfrak{T}<\infty. \tag{33}$$

Then, we have

$$\begin{aligned}
&\mathfrak{S}(PQ_1(t),PQ_2(t),PQ_3(t))=\max\{|PQ_{(t)}-PQ_{(t)}|:t=1,2,3,t\neq\}\\
&\leq\max\left\{|\Bbbk_1(\beta,t)Q_t(t)+\Bbbk_0(\ell,t)\Theta(Q_{(t)})-\big(\Bbbk_1(\ell,t)Q_{(t)}+\Bbbk_0(\ell,t)\Theta(Q_{(t)})\big)|\frac{\mathfrak{T}^{\ell}}{\ell^2}:t=1,2,3,t\neq\right\}\\
&\leq\max\left\{\Big(\Bbbk_1(\ell,t)|Q_t-Q_|+\Bbbk_0(\ell,t)\hbar|Q_t-Q_|\Big)\frac{\mathfrak{T}^{\ell}}{\ell^2}:t=1,2,3,t\neq\right\}\\
&\leq\max\left\{\frac{\mathfrak{T}^{\ell}}{\ell^2}[(1-\ell)\mathfrak{T}^{\ell}+\ell\mathfrak{T}^{1-\ell}\hbar]|Q_t-Q|:t=1,2,3,t\neq\right\}\\
&:=r\mathfrak{S}(Q_1,Q_2,Q_3).
\end{aligned}$$

Hence, P admits an optimal value in the unit ball B_r of radius $r\in(0,1)$. Let $t,\varsigma\in(0,\mathfrak{T})$ with $t>\varsigma$ so $Q(t)>Q(\varsigma)$ (increasing function). A calculation implies that

$$\begin{aligned}
&\mathfrak{S}\big(PQ_1(t),PQ_2(t),PQ_3(t)-(PQ_1(\varsigma),PQ_2(\varsigma),PQ_3(\varsigma)\big)\\
&=\mathfrak{S}\big(P\big(Q_1(t)-Q_1(\tau)\big),P\big(Q_2(t)-Q_2(\varsigma)\big),P\big(Q_3(t)-Q_3(\varsigma)\big)\big)\\
&=\mathfrak{S}\big(PQ_1(t-\varsigma),PQ_2(t-\varsigma),PQ_3(t-\varsigma)\big)\\
&\leq\mathfrak{S}\big(PQ_1(t),PQ_2(t),PQ_3(t)\big)\\
&\leq r\mathfrak{S}(Q_1,Q_2,Q_3).
\end{aligned}$$

Thus, P is equicontinuous inside the ball B_r. Also, assuming $Q_l(t)-\eta_l(t)=\xi_l(t)$, $l=1,2,3$, we attain that

$$\begin{aligned}
&\mathfrak{S}\big(P\big(Q_1(t)-\eta_1(t)\big),P\big(Q_2(t)-\eta_2(t)\big),P\big(Q_3(t)-\eta_3(t)\big)\big)\\
&=\mathfrak{S}\big(Q\big(\xi_1(t)\big),Q\big(\xi_2(t)\big),Q\big(\xi_3(t)\big)\big)\\
&\leq\max\left\{\frac{\mathfrak{T}^{\ell}}{\ell^2}|\Bbbk_1(\ell,t)\xi_{(t)}+\Bbbk_0(\ell,t)\Theta(\xi_{((t))}-\big(\Bbbk_1(\ell,t)\xi_{(t)}+\Bbbk_0(\ell,t)\Theta(\xi_{(t)})\big)|:t=1,2,3,t\neq\right\}\\
&\leq\max\left\{\Big((1-\ell)\mathfrak{T}^{\ell}|\xi_t-\xi_|+\ell\mathfrak{T}^{1-\ell}\ell\Big)\frac{\mathfrak{T}^{\ell}}{\ell^2}|\xi_t-\xi|:t=1,2,3,t\neq\right\}\\
&=\max\left\{[(1-\ell)\mathfrak{T}^{\ell}+\ell\mathfrak{T}^{1-\ell}\ell]\frac{\mathfrak{T}^{\ell}}{\ell^2}|\xi_t-\xi_|:t=1,2,3,t\neq\right\}\\
&=\frac{\mathfrak{T}^{\ell}}{\ell^2}[(1-\ell)\mathfrak{T}^{\ell}+\beta\mathfrak{T}^{1-\beta}\hbar]\max\left\{|\xi_t-\xi_|:t=1,2,3,t\neq\right\}\\
&=r\mathfrak{S}(\xi_1,\xi_2,\xi_3)\leq r\mathfrak{S}(Q_1,Q_2,Q_3).
\end{aligned}$$

Hence, P is continuous in B_r. Consequently, P admits a fixed point $PQ=Q$. Thus, according to Lemma 1, we have that P admits a unique fixed point in B_r, $r\in(0,1)$.

Theorem 2. *Consider the Eq.* (15) *such that* $0<\tau<1$, $0<\beta<1$. *If the solution* Q *is a quasi-norm that is achieving the inequality*

$$N_\tau(Q)=\int_{B_r}|Q|^\tau\,d\mu<\infty,$$

for some measure, $K_e>1/\tau$, *and*

$$\hbar<\left|\left(\frac{1}{1-\tau}-\Bbbk_1(\ell,t)\right)/\Bbbk_0(\ell,t)\right|,\quad t\in(0,\mathfrak{T}].$$

Then every solution of Eq. (15) *is bounded by conformable entropy satisfying* (20)-(21).

Proof. Assume that Q is a solution taking the following formula

$$Q(t)=\int\big(\Bbbk_1(\ell,t)Q(t)+\Bbbk_0(\ell,t)\,\Theta\big(Q(\aleph t),Q(\flat t),Q(\complement t)\big)\big)\,d_\ell t+Ke_0(t,t_0).$$

Let $K_e:=Ke_0(t,t_0)$. A calculation gives the following illustration:

$$\begin{aligned}Q(t)=\int\big(\Bbbk_1(\ell,t)Q(t)+\Bbbk_0(\ell,t)\,\Theta\big(Q(\aleph t),Q(\flat t),Q(\complement t)\big)\big)\,d_\ell t+Ke_0(t,t_0)&\\ \le K_e+\Bbbk_1(\ell,\mathfrak{T})N_\tau(Q)+\Bbbk_0(\ell,\mathfrak{T})\hbar N_\tau(Q)&\\ =K_e+(\nu_1(\beta,\mathfrak{T})+\nu_0(\beta,\mathfrak{T})\ell)N_\tau(Q)&\\ =K_e+(\Bbbk_1(\ell,\mathfrak{T})+\Bbbk_0(\ell,\mathfrak{T})\hbar)\int_{B_r}|Q|^\tau\,d\mu&\\ \le\frac{K_e-K_e\,\tau}{1-\tau}+\frac{\int_{B_r}|Q|^\tau\,d\mu}{1-\tau}&\\ \le\frac{K_e}{1-\tau}+\frac{\int_0^{\mathfrak{T}}Q^\tau(t)\,dt-K_e\,\tau}{1-\tau}.&\end{aligned}$$

Since $K_e\,\tau>1$, then we obtain the following inequality:

$$Q(t)\le\frac{K_e}{1-\tau}+\frac{\int_0^{\mathfrak{T}}|Q|^\tau(t)\,dt-1}{1-\tau}\tag{34}$$

$$=\frac{\tau K_e}{1-\tau}+\frac{\int_0^{\mathfrak{T}}|Q|^\tau(t)\,dt-1}{1-\tau}+K_e\tag{35}$$

$$=\frac{\tau Ke_0(t,t_0)}{1-\tau}+\mathbb{T}_\tau(Q)(t)+Ke_0(t,t_0)\tag{36}$$

Hence, the unique outcome of Eq. (15) has a bound based on the conformable entropy satisfying (20)-(21).

Remark.

– We note that the value of the upper bound of $\hbar$ in Theorem 1 is different from that in Theorem 2. This because we assume that the functions $\Bbbk_0$ and $\Bbbk_1$ are of the formula

$$\Bbbk_1(\ell,t)=(1-\ell)t^\ell,\quad \Bbbk_0(\ell,t)=\ell t^{1-\ell},\quad t\in(0,\mathfrak{T}),\mathfrak{T}<\infty.\tag{37}$$

If we change the above definition, we get another upper bound depending on the selection of Eq.s (9)-(**??**). While the upper bound in Theorem 2 is the general formula including the fractional power of the Tsallis entropy. Therefore, by using (37), we have the limit

$$\lim_{\tau\to0,t\to\mathfrak{T}}\left|\left(\frac{1}{1-\tau}-\Bbbk_1(\ell,t)\right)/\Bbbk_0(\ell,t)\right|=\frac{1-(1-\ell)\mathfrak{T}^\ell}{\ell\mathfrak{T}^{1-\ell}}.$$

Moreover, the other types of $\Bbbk_0$ and $\Bbbk_1$ are useful depending on the applications. For example, if one is looking for a periodic solution, the best selection is achieved using sin and cos functions.

– The entropy solution in Eq.(34) is finite ($Q(t)<\infty$) and analytic in $0<\tau<1$.

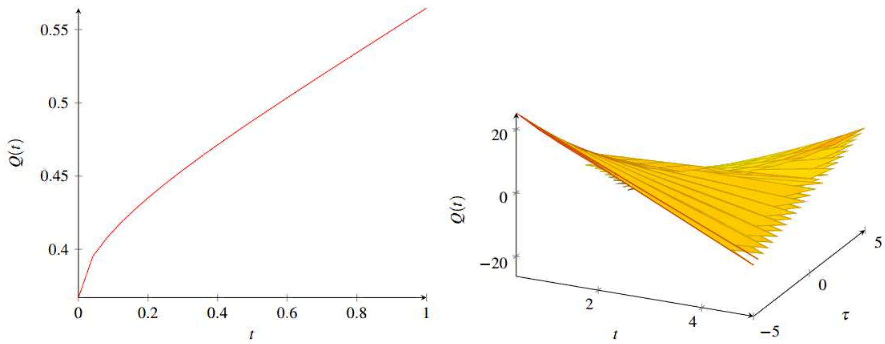

Fig. 1: The solution $Q(t)$ and 3D plot of the entropy solution in Eq. (34) with respect to τ in Example 1.

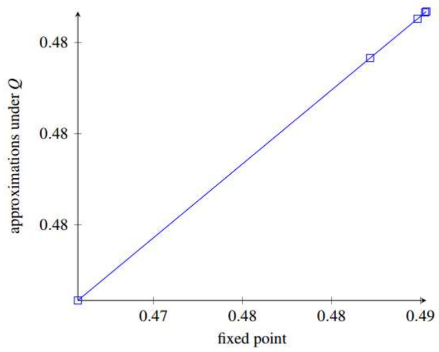

Fig. 2: The iteration of the fixed point of Example 1.

3.2 *Numerical Examples*

In this place, we aim to introduce an example to satisfy the conditions of Theorem 2.

*Example 1.*Adopt that a current state-run of the market satisfies the following data:

– $\mathfrak{T} = 1, \ell = 0.5$ and $\hbar = 0.5$;
– $\Theta(Q)(t) = 0.3Q(t)$;
– $0 < \tau < 1$.

Since $|\Theta(Q_1)(t) - \Theta(Q_2)(t)| < 0.3|Q_1 - Q_2|$ then in view of Theorem 1, Eq. (15) has a unique solution in a unit ball. Considering the formula of $e_0(t,0)$ we have

$$e_0(t,0) = \exp\left(-\int_0^t \frac{\mathbb{k}_1(\ell,\varsigma)}{\mathbb{k}_0(\ell,\varsigma))} d\varsigma\right) = 0.367;$$

and since $0 < \tau < 1$, we can approach to $e_0(t,0)\tau < 1$ for all τ and $0 < K < 1$. Hence, all the conditions of Theorem 2 are fulfilled, we conclude that the unique solution of Eq. (15) is bounded by a TCE. It is clear that the conditions of Theorem 2 are not difficult to satisfy. We proceed to compute the formula of the solution $Q(t)$. From Eq. (15), we get the solution (see Fig.1),

$$Q(t) \approx \frac{0.367}{1 - 0.35\sqrt{t}}, \quad t \in (0,1].$$

Moreover, with the help of Mathematica 11.2, the exact value of the fixed point is $p = 0.48534$ after five iteration with error $e = 0.00230$ (see Fig.2)

*Example 2.*Consider the following data

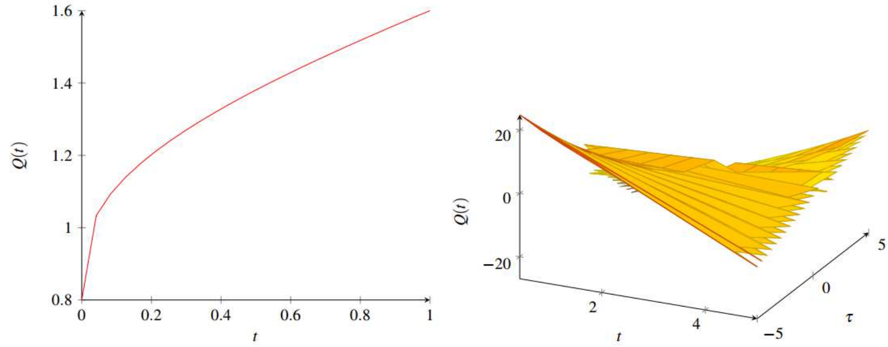

Fig. 3: The solution $Q(t)$ and 3D plot of the entropy solution in Example 2.

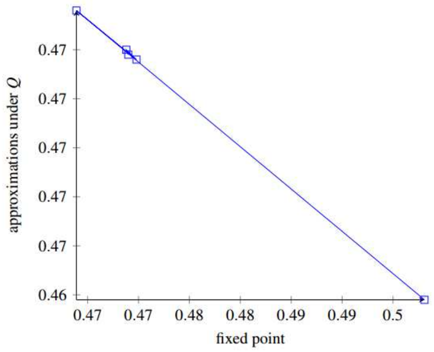

Fig. 4: The iteration of the fixed point of Example 2.

– $\mathfrak{T} = 1, \beta = 0.75$;
– $\Theta(Q)(t) = \frac{1}{2}\sqrt{Q^2+5}$;
– $0 < \tau < 1$.

It is clear that Θ is is Lipschitz continuous with the Lipschitz constant $\ell = 0.5 < \frac{1-(1-0.75)}{0.75}$. Then in view of Theorem 1, Eq. (15) has a unique solution in a unit ball. The formula of $e_0(t,0)$ can be calculated as follows:

$$e_0(t,0) = \exp\left(-\int_0^t \frac{\nu_1(0.75,\varsigma)}{\nu_0(0.75,\varsigma))} d\varsigma\right) = 0.8007;$$

and since $0 < \tau < 1$, we have $e_0(t,0)\tau < 1$ with $0 < K < 1$. Therefore, in virtue of Theorem 2, we attain that the unique solution of Eq. 15 is bounded by a TCE such that (see Fig 3)

$$Q(t) \approx \frac{0.8}{1-0.5t^{0.25}}, \quad t \in (0,1].$$

The exact value of the fixed point is $p = 0.46898$ after five iteration with error $e = 0.006$

Moreover, with the help of Mathematica 11.2, the exact value of the fixed point is $p = 0.48534$ after five iteration with error $e = 0.00230$. It is clear that Θ is is Lipschitz continuous with the Lipschitz constant $\hbar = 0.5 < \frac{1-(1-0.75)}{0.75}$. Then in view of Theorem 1, Eq. (15) has a unique solution in a unit ball. Th formula of $e_0(t,0)$ can be calculated as follows:

$$e_0(t,0) = \exp\left(-\int_0^t \frac{\nu_1(0.75,\varsigma)}{\Bbbk_0(0.75,\varsigma))} d\varsigma\right) = 0.8007;$$

and since $0 < \tau < 1$, we have $e_0(t,0)\tau < 1$ with $0 < K < 1$. Therefore, in virtue of Theorem 2, we attain that the unique solution of Eq. 15 is bounded by a TCE such that (see Fig 3)

$$Q(t) \approx \frac{0.8}{1-0.5t^{0.25}}, \quad t \in (0,1].$$

The exact value of the fixed point is $p = 0.46898$ after five iteration with error $e = 0.006$

3.3 Comparison

Recently, CCs have been studied and modified by many researchers , one can see for example [16]-[18]. Our comparison is based on the new development given by Baleanu et al. [16]. They presented a hybrid integral and differential operators found by the well known Caputo fractional derivative and integral as follows:

$$^{C}\mathscr{D}^{\beta}Q(t) = \frac{1}{\Gamma(1-\beta)}\int_0^t \left(\nu_1(\beta,\tau)Q(\tau)+\nu_0(\beta,\tau)Q'(\tau)\right)(t-\tau)^{-\beta}d\tau \tag{38}$$

$$= \left(\frac{t^{-\beta}}{\Gamma(1-\beta)}\right) * \left(\nu_1(\beta,t)Q(t)+\nu_0(\beta,t)Q'(t)\right), \tag{39}$$

where

$$\lim_{\beta\to 0} {^{C}\mathscr{D}^{\beta}}Q(t) = \int_0^t Q(\tau)d\tau, \quad \lim_{\beta\to 1} {^{C}\mathscr{D}^{\beta}}Q(t) = Q'(t).$$

Clearly, (38) implies that

$$^{C}\mathscr{D}^{\beta}Q(t) = \frac{1}{\Gamma(1-\beta)}\int_0^t \left(\nu_1(\beta,\tau)\chi(\tau)+\nu_0(\beta,\tau)\Theta(Q(\tau))\right)(t-\tau)^{-\beta}d\tau \tag{40}$$

$$= \left(\frac{1}{t^{\beta}\Gamma(1-\beta)}\right) * \left(\nu_1(\beta,t)Q(t)+\nu_0(\beta,t)\Theta(Q(t))\right), \tag{41}$$

which is corresponding to the hybrid conformable integral

$$^{C}\mathscr{I}^{\beta}Q(t) = \int_0^t e_0(t,\tau)\frac{^{RL}\mathscr{D}^{1-\beta}Q(\tau)}{\nu_0(\beta,\tau)}d\tau,$$

where the operator $^{RL}\mathscr{D}^{1-\beta}$ indicated the the Riemann-Liouville differ-integrals operator. Moreover, by Proposition 2 in [16], the corresponding integral satisfies the relation

$$^{C}\mathscr{I}^{\beta}\, {^{C}\mathscr{D}^{\beta}}Q(t) = Q(t) - e_0(t,\tau)Q(0). \tag{42}$$

Note that the initial solution of (40) is $Q(0) = 0$.
Define an operator $Z : X \to X$ by the following construction

$$(ZQ)(t) := \int_0^t \left(\frac{1}{\tau^{\beta}\Gamma(1-\beta)}\right) * \left(\nu_1(\beta,\tau)Q(\tau)+\nu_0(\beta,\tau)\Theta(\tau,Q)\right)d\tau.$$

We have the following existence result:

Theorem 3. *Consider the hybrid conformable equation (40). If*

$$|\Theta(Q(t)) - \Theta(\eta(t))| < \ell|Q(t)-\eta(t)|$$

for some positive constant $\ell < \frac{\Gamma(1-\beta)-(1-\beta)\mathfrak{T}^{\beta}}{\beta\mathfrak{T}^{1-\beta}}$, $\mathfrak{T} \in (0,\infty)$. *Then Z has a unique fixed point in X.*

Proof

In the same manner of Theorem 1, we let ν_0 and ν_1 to be as follows:

$$\nu_1(\beta,t) = (1-\beta)t^{\beta}, \quad \nu_0(\beta,t) = \beta t^{1-\beta}, \quad t \in (0,\mathfrak{T}), \mathfrak{T} < \infty.$$

$$
\begin{aligned}
\mathfrak{S}(ZQ_1(t),ZQ_2(t),ZQ_3(t)) &= \max\{|ZQ_t(t)-ZQ_(t)| : t=1,2,3,t\neq\}\\
&\leq \max\{|\nu_1(\beta,t)Q_t(t)+\nu_0(\beta,t)\Theta(Q_t(Q_{(t)})\\
-\left(\nu_1(\beta,t)Q_(t)+\nu_0(\beta,t)\Theta(Q_(Q_(t))\right)| &\frac{\mathfrak{T}}{\beta^2\Gamma(1-\beta)} : t=1,2,3,t\neq\}\\
\leq \max\{\left(\nu_1(\beta,t)|Q_t-\chi_|+\nu_0(\beta,t)\ell|Q_t-\chi|\right) &\frac{\mathfrak{T}}{\beta^2\Gamma(1-\beta)} : t=1,2,3,t\neq\}\\
\leq \max\{\left((1-\beta)\mathfrak{T}^\beta|Q_t-Q|+\beta\mathfrak{T}^{1-\beta}\ell|Q_t-Q_|\right) &\frac{\mathfrak{T}}{\beta^2\Gamma(1-\beta)} : t=1,2,3,t\neq\}\\
=\max\{[(1-\beta)\mathfrak{T}^\beta+\beta\mathfrak{T}^{1-\beta}\ell]\left(\frac{\mathfrak{T}}{\beta^2\Gamma(1-\beta)}\right)& |Q_t-Q_| : t=1,2,3,t\neq\}\\
&:= r\mathfrak{S}(Q_1,Q_2,Q_3).
\end{aligned}
$$

Since $[(1-\beta)\mathfrak{T}^\beta+\beta\mathfrak{T}^{1-\beta}\ell]\left(\frac{\mathfrak{T}}{\beta^2\Gamma(1-\beta)}\right)<1 \Rightarrow \frac{[(1-\beta)\mathfrak{T}^\beta+\beta\mathfrak{T}^{1-\beta}\ell]}{\Gamma(1-\beta)}<\frac{\beta^2}{\mathfrak{T}}\leq 1$. Hence, Z is bounded in the unit ball B_r of radius $0<r<1$. In the similar manner of Theorem 1, we confirm that the operator Z has a unique fixed point according to Lemma 1.

Remark

As a comparison, we indict that the upper bound of ℓ in Theorem 1 is equal to 1 for all $\beta\in(0,1)$. However, in Theorem 3, the maximum value appears at $\beta=0.9$, which approaches 12.

4 Conclusion

Generally, the economic system can be achieved by considering a collection of fixed points. Therefore, fixed-point proposals can afford the equilibrium of economic composed with the conformable fractional entropy (Theorem 2). The setting of this effort established a new procedure of EnOQ system. We presented a method of decreasing it utilizing the notion of fixed-point philosophy. We industrialized this technique to be appropriate for the typical (Theorem 1) applications. Applications are illustrated for a linear system. For future work, one can employ the complex conformable derivative operator that is given in [12] to extend the study conformable Tsallis' entropy into a complex domain to generalize the results in [14].

Conflict of Interest

The authors declare that they have no conflict of interest.

References

[1] A. Sagheer, M. Zidan and M. Abdelsamea, A novel autonomous perceptron model for pattern classification applications, *Entropy* **763**, 1-24 (2019).
[2] D. Blumberga, B. Chen, A. Ozarska, Z. Indzere and D. Lauka, Energy, bioeconomy, climate changes and environment nexus, *Environ. Clim. Technol.* **23**(3), 370-92123 (2019).
[3] M. Giampietro, On the circular bioeconomy and decoupling: implications for sustainable growth, *Ecol. Econ.* **1**(162), 143-56 (2019).
[4] S. Mohan, S. Dahiya, K. Amulya, R. Katakojwala and T. Vanitha, Can circular bioeconomy be fueled by waste biorefineries-A closer look, *Bioresour. Technol. Rep.* **1**(7), 1-44 (2019).
[5] F. Harris, How many parts to make at once, *Oper. Res.* **38**(6), 947-50 (1990).
[6] R. W. Ibrahim and S. Hadid, The EOQ model: a differential cyclic system for calculating economic order quantity, *Int. J. Anal. Appl.* **16.3**, 437-444 (2018).
[7] R. W. Ibrahim, H.A. Jalab and A. Gani, Entropy solution of fractional dynamic cloud computing system associated with finite boundary condition, *Bound. Value Prob.* **1**, 1-12 (2016).
[8] M. Jaber, R. Y. Nuwayhid and M. A. Rosen, Price-driven economic order systems from a thermodynamic point of view, *Int. J. Prod. Res.* **42**(24), 5167-84 (2004).

[9] M. Pattnaik, Optimization in an instantaneous economic order quantity (EOQ) model incorporated with promotional effort cost, variable ordering cost and units lost due to deterioration, *Uncert. Supply Chain Manag.* **1**(2), 57-66 (2013).
[10] R. W. Ibrahim, H. Jafari, H. A. Jalab and S. B. Hadid, Local fractional system for economic order quantity using entropy solution, *Adv. Differ. Equ.* **2019**(1), 1-11 (2019).
[11] D. Anderson and D. Ulness, Newly defined conformable derivatives, *Adv. Dyn. Syst. Appl.* **10**(2), 109-37 (2015).
[12] R. W. Ibrahim and J. M. Jahangiri, Conformable differential operator generalizes the Briot-Bouquet differential equation in a complex domain, *AIMS Math.* **4**(6), 1582-1595 (2019).
[13] C. Tsallis, Possible generalization of Boltzmann-Gibbs statistics, *J. Stat. Phys.* **52**(1-2), 479-87 (1988).
[14] R. W. Ibrahim and M. Darus, Analytic study of complex fractional Tsallis' entropy with applications in CNNs, *Entropy* **20**(10), 1-11 (2018).
[15] S. Sedghi, N. Shobe and A. Aliouche, A generalization of fixed point theorems in S-metric spaces, *Mat. Vesnik* **64**(3), 258-266 (2012).
[16] D. Baleanu, A. Fernandez and A. Akgul, On a fractional operator combining proportional and classical differintegrals, *Mathematics* **8**(3), 1-13 (2020).
[17] D. Baleanu, F. Jarad and E. Ugurlu, Singular conformable sequential differential equations with distributional potentials, *Quaestiones Math.* **42**(3), 277-287 (2019).
[18] Z. Korpinar, A. Alshomrani and D. Baleanu, The deterministic and stochastic solutions of the Schrodinger equation with time conformable derivative in birefrigent fibers, *AIMS Math.* **5**(3), 1-20 (2020).