Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/080105

New Results on the Generator of Conformable Semigroups

Sharifa Al-Sharif * *and Rahma Al-Shorman*

Department of Mathematics, Faculty of Science, Yarmouk University, Irbid, Jordan

Received: 2 Feb. 2020, Revised: 2 Apr. 2020, Accepted: 8 Jun. 2020
Published online: 1 Jan. 2022

Abstract: This paper is concerned with certain aspects of conformable semigroups. Mainly, we try to give an answer to the following question "when can a linear operator A generate a conformable semigroup?" To do this in the sequel, we introduce and prove new properties of conformable semigroups of operators similar to that of strongly continuous semigroups of operators.

Keywords: Conformable derivative, conformable semigroups, Hille-Yosida theorem.

1 Introduction and basic results

The theory of semigroups of bounded linear operators is used to solve differential equations. Semigroup theory had immediate applications to partial differential equations, Markov processes, and ergodic theory.

Let $\mathcal{L}(X,X)$ be the space of all bounded linear operators on a Banach space X. Then a family of $\{T(t)\}_{t\geq 0} \subseteq \mathcal{L}(X,X)$ is called a semigroup of operators if:

(i) $T(0) = I$,
(ii) $T(s+t) = T(s)T(t) \quad for\ all \quad s,\ t \geq 0.$

The semigroup $\{T(t)\}_{t\geq 0}$ is called a strongly continuous c_0-semigroup if for each fixed $x \in X$, $\|T(t)x - x\| \to 0$ as $t \to 0^+$, see [1].

Fractional calculus has a useful applications in engineering and science, physics, bioengineering, and dynamics of particles [2,3,4,5,6,7,8,9]. In the last decades, fractional calculus became an interesting subject in the area of mathematical analysis. The idea came from L'Hopital in 1695 when he asked what it means $\frac{d^n f}{dx^n}$ if $n = \frac{1}{2}$. Then, many researchers tried to put a definition of a fractional derivative [10,11,12]. Most of them used an integral form.

The authors In [12] define the fractional derivative as a natural extension to the usual definition of the first derivative as follows:
Let $f : [0,\infty) \to \mathbb{R}$ be a given function. Then for all $t > 0$, $\alpha \in (0,1)$, let

$$D_\alpha(f)(t) = f^{(\alpha)}(t) = \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

$D_\alpha f$ is called the Conformable derivative of f of order α. If f is $\alpha-$differentiable in some $(0,a)$, $a > 0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then we define

$$f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t).$$

It is easy to see that this new definition of fractional derivative satisfy all the usual properties of the first derivative. The $\alpha-$fractional integral of a function f starting from $a \geq 0$ is defined as

$$I^a_\alpha(f)(t) = \int_a^b f(t)d_\alpha t := \int_a^b f(t)t^{\alpha-1}dt.$$

* Corresponding author e-mail: sharifa@yu.edu.jo

According to [12], it is known that for a continuous function f such that $I_\alpha^a f$ exists, then

$$D_\alpha(I_\alpha^a f)(t) = f(t), \quad for\, t \geq a \geq 0.$$

Fore more on Conformable derivative and fractional integral we refer the reader to [12,13,14,15,16,17,18,19].

Conformable derivatives or their modifications in [15] as a type of local fractional derivatives of fractal ones are further important since they can be used to generate new generalized nonlocal fractional operators, see [13,20,21].

Conformable semigroups are related to semigroups generated by fractional powers of closed operators which is developed by Bochner, see [22]. In [23], the authors studied the problem of fractional powers of closed operators and the semigroups generated by them. The fractional Cauchy problem associated with a Feller semigroup was studied by Popescu, see [24].

The first study of conformable semigroups appeared in 2015, see [25]. The authors used the new definition of Conformable derivative to obtain a new basic definition of conformable semigroups of linear operators which is a natural extension to the usual semigroups as follows:
Let $\mathcal{L}(X,X)$ be the space of all bounded linear operators on a Banach space X. Then a family of $\{T(t)\}_{t\geq 0} \subseteq \mathcal{L}(X,X)$ is called a fractional $\alpha-$semigroup (or $\alpha-$semigroup) of operators if:

(i) $T(0) = I$,
(ii) $T(s+t)^{\frac{1}{\alpha}} = T(s^{\frac{1}{\alpha}})T(t^{\frac{1}{\alpha}})$ for all $s, t \in [0,\infty)$.

If $\alpha = 1$, this definition leads to the usual semigroups of operators.
The semigroup $\{T(t)\}_{t\geq 0}$ is called a strongly continuous $c_0 - \alpha-$semigroup if for each fixed $x \in X$, $\|T(t)x - x\| \to 0$ as $t \to 0^+$.

Example 1. For a bounded linear operator A on X, define $T(t) = e^{\frac{t^\alpha}{\alpha}A}$. It is easy to see that $\{T(t)\}_{t\geq 0}$ is an $\alpha-$semigroup. Indeed:

(i) $T(0) = e^{0A} = I$.
(ii) $T(s+t)^{\frac{1}{\alpha}} = e^{\frac{(s+t)}{\alpha}A} = e^{\frac{s}{\alpha}A}e^{\frac{t}{\alpha}A} = T(s^{\frac{1}{\alpha}})T(t^{\frac{1}{\alpha}})$.

Example 2. (See [25]). Let $X = C[0,\infty)$. Define $(T(t)f)(s) = f(s+2\sqrt{t})$. Then $\{T(t)\}_{t\geq 0}$ is a $\frac{1}{2}-$semigroup.

The linear operator A defined by $Ax = \lim_{t\to 0^+} T^{(\alpha)}(t)x$ provided that the limit exists is called the **infinitesimal generator** of the $\alpha-$semigroup $T(t)$, see [25].

Definition 1. *An $\alpha-$semigroup of bounded linear operators $T(t)$ is called uniformly continuous if*

$$\lim_{t\to 0^+} \|T(t^{\frac{1}{\alpha}}) - I\| = 0.$$

Now, the important question: Under what conditions can a linear operator A generate conformable $\alpha-$semigroup? The following theorem answered the question for case of uniformly continuous $\alpha-$semigroup. Section 3 provides proof for this theorem.

Theorem 1. *A linear operator A is the $\alpha-$infinitesimal generator of a uniformly continuous $\alpha-$semigroup if and only if A is a bounded linear operator.*

For the case of $c_0 - \alpha-$semigroup of contractions $T(t)$, that means $\|T(t)\| < 1$ for all $t \in (0,\infty)$, we have the following generlazation of the Hille-Yosida theorem, and we called it modified Hille-Yosida type theorem. Section 2 provides proof for this theorem.

Theorem 2. *(Modified Hille-Yosida Type Theorem). An unbounded linear operator A generates a strongly continuous $\alpha-$differentiable c_0- semigroup $\{T(t)\}_{t\geq 0}$ of contractions if and only if:*

(i) A is closed and $\mathcal{D}(A)$ is dense in X.
(ii) The resolvent set $\rho(A) \supseteq \mathbb{R}^+$ and for every $\lambda > 0$, we have

$$\|R(\lambda : A)\| \leq \frac{1}{\lambda}. \tag{1.1}$$

The goal of this article is to present a new and attractive properties of conformable semigroups and to prove the modified Hille-Yosida type theorem for conformable semigroups of operators.

This paper is organized as follows: Section 2 states and proves Hille-Yosida type theorem for conformable semigroups. Section 3 addresses the uniformly continuous $\alpha-$semigroups of bounded linear operator. Conclusion is presented in Section 4.

2 Modified Hille Yosida type theorem

Throughout this section, $\{T(t)\}_{t\geq 0}$ is an $\alpha-$conformable semigroup of operators on a Banach space X, with $0<\alpha\leq 1$. The generator will be denoted by A. To prove Theorem 2, we need first to prove the following classical results.

Lemma 1. *Let $\{T(t)\}_{t\geq 0}$ be a $c_0-\alpha-$semigroup, $\alpha\in(0,1)$, a, $h>0$. Then*

a) If $f(h)x=\int_a^{(t^\alpha+h^\alpha)^{\frac{1}{\alpha}}} T(u)x\,d_\alpha u$, then

$$D_\alpha(f)(t)x=f^{(\alpha)}(h)x=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{(t^\alpha+h^\alpha)^{\frac{1}{\alpha}}}^{((h+\varepsilon h^{1-\alpha})^\alpha+t^\alpha)^{\frac{1}{\alpha}}} T(u)x\,d_\alpha u.$$

b) If $g(h)$ is $\alpha-$differentiable at $h>0$ and $f(h)x=\int_a^{g(h)} T(u)x\,d_\alpha u$, then

$$D_\alpha(f)(t)x=f^{(\alpha)}(h)x=\frac{T(g(h))x}{g^{1-\alpha}(h)}g^{(\alpha)}(h).$$

Proof. a)

$$\begin{aligned}
f^{(\alpha)}(h)x&=\lim_{\varepsilon\to 0}\frac{f(h+\varepsilon h^{1-\alpha})x-f(h)x}{\varepsilon}\\
&=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\int_a^{((h+\varepsilon h^{1-\alpha})^\alpha+t^\alpha)^{\frac{1}{\alpha}}} T(u)x\,d_\alpha u-\int_a^{(t^\alpha+h^\alpha)^{\frac{1}{\alpha}}} T(u)x\,d_\alpha u\right]\\
&=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\int_a^{((h+\varepsilon h^{1-\alpha})^\alpha+t^\alpha)^{\frac{1}{\alpha}}} T(u)x\,d_\alpha u+\int_{(t^\alpha+h^\alpha)^{\frac{1}{\alpha}}}^{a} T(u)x\,d_\alpha u\right]\\
&=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{(t^\alpha+h^\alpha)^{\frac{1}{\alpha}}}^{((h+\varepsilon h^{1-\alpha})^\alpha+t^\alpha)^{\frac{1}{\alpha}}} T(u)x\,d_\alpha u.
\end{aligned}$$

b) Let

$$f(h)x=\int_a^{g(h)} T(u)x\,d_\alpha u=L(g(h))x,$$

where

$$L(h)x=\int_a^{h} T(u)x\,d_\alpha u.$$

Using semi-chain rule, (see [14]), we get

$$\begin{aligned} L^{(\alpha)}(g(h))x &= L'(g(h))xg^{(\alpha)}(h) \ \text{ where } \ L'(h)x = \frac{T(h)x}{h^{1-\alpha}} \\ &= \frac{T(g(h))x}{g^{1-\alpha}(h)} g^{(\alpha)}(h). \end{aligned}$$

□

Theorem 3. *(See [25]). Let A be the infinitesimal generator of a $c_0-\alpha-$semigroup $\{T(t)\}_{t\geq 0}$, $0<\alpha\leq 1$. If $T(t)$ is continuously $\alpha-$differentiable and $x\in\mathscr{D}(A)$, then*

$$T^{(\alpha)}(t)x = AT(t)x = T(t)Ax.$$

Theorem 4. *Let $\alpha\in(0,1)$, a, h, $t>0$, and $\{T(t)\}_{t\geq 0}$ be a $c_0-\alpha-$semigroup. Then*

a) For $x\in X$,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon t^{1-\alpha}} T(s)x\, d_\alpha s = T(t)x.$$

b) $\forall x\in X$, $\int_a^t T(s)x\, d_\alpha s \in \mathscr{D}(A)$.

c) If $T(t)$ is continuously $\alpha-$differentiable, then for $x\in\mathscr{D}(A)$

$$T(t+\varepsilon\, t^{1-\alpha})x - T(t)x = \int_t^{t+\varepsilon\, t^{1-\alpha}} T(u)\,A\,x\, d_\alpha u = \int_t^{t+\varepsilon\, t^{1-\alpha}} A\,T(u)x\, d_\alpha u.$$

Proof. a)

$$\begin{aligned} \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon t^{1-\alpha}} T(s)x\, d_\alpha s &= \lim_{\varepsilon\to 0}\frac{\int_0^{t+\varepsilon t^{1-\alpha}} T(s)x\, d_\alpha s - \int_0^t T(s)x\, d_\alpha s}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{G(t+\varepsilon t^{1-\alpha})x - G(t)x}{\varepsilon} \quad \text{where } G(u)x = \int_0^u T(s)x\, d_\alpha s \\ &= G^{(\alpha)}(t)x \\ &= T(t)x. \end{aligned}$$

b)

$$\frac{T(h+\varepsilon h^{1-\alpha}) - T(h)}{\varepsilon}\int_a^t T(s)x\, d_\alpha s = \frac{1}{\varepsilon}\left[\int_a^t T(s^\alpha + (h+\varepsilon h^{1-\alpha})^\alpha)^{\frac{1}{\alpha}} x\, d_\alpha s \right. \\ \left. - \int_a^t T(s^\alpha + h^\alpha)^{\frac{1}{\alpha}} x d_\alpha s\right].$$

Using change of variables, we get

$$\frac{T(h+\varepsilon h^{1-\alpha}) - T(h)}{\varepsilon}\int_a^t T(s)x\, d_\alpha s = \frac{1}{\varepsilon}\left[\int_{(a^\alpha+(h+\varepsilon h^{1-\alpha})^\alpha)^{\frac{1}{\alpha}}}^{(t^\alpha+(h+\varepsilon h^{1-\alpha})^\alpha)^{\frac{1}{\alpha}}} T(u)x\, d_\alpha u - \int_{(a^\alpha+h^\alpha)^{\frac{1}{\alpha}}}^{(t^\alpha+h^\alpha)^{\frac{1}{\alpha}}} T(u)x\, d_\alpha u\right]$$

$$= \frac{1}{\varepsilon}\left[\int_{(a^\alpha+(h+\varepsilon h^{1-\alpha})^\alpha)^{\frac{1}{\alpha}}}^{(a^\alpha+h^\alpha)^{\frac{1}{\alpha}}} T(u)x\, d_\alpha u + \int_{(a^\alpha+h^\alpha)^{\frac{1}{\alpha}}}^{(t^\alpha+h^\alpha)^{\frac{1}{\alpha}}} T(u)x\, d_\alpha u \right.$$
$$\left. + \int_{(t^\alpha+h^\alpha)^{\frac{1}{\alpha}}}^{(t^\alpha+(h+\varepsilon h^{1-\alpha})^\alpha)^{\frac{1}{\alpha}}} T(u)x\, d_\alpha u - \int_{(a^\alpha+h^\alpha)^{\frac{1}{\alpha}}}^{(t^\alpha+h^\alpha)^{\frac{1}{\alpha}}} T(u)x\, d_\alpha u \right].$$

If $\varepsilon \to 0$ Lemma 1 (a) implies

$$\lim_{\varepsilon\to 0}\frac{T(h+\varepsilon h^{1-\alpha})-T(h)}{\varepsilon}\int_a^t T(s)x\, d_\alpha s = -D_\alpha\left(\int_{a_2}^{(a^\alpha+h^\alpha)^{\frac{1}{\alpha}}} T(u)x\, d_\alpha u\right)$$
$$+ D_\alpha\left(\int_{a_1}^{(t^\alpha+h^\alpha)^{\frac{1}{\alpha}}} T(u)x\, d_\alpha u\right) \text{ where } a_1,\ a_2 > 0.$$

Using Lemma 1 (b) to get

$$\lim_{\varepsilon\to 0}\frac{T(h+\varepsilon h^{1-\alpha})-T(h)}{\varepsilon}\int_a^t T(s)x\, d_\alpha s = -T(a^\alpha+h^\alpha)^{\frac{1}{\alpha}}x + T(t^\alpha+h^\alpha)^{\frac{1}{\alpha}}x.$$

As $h \to 0^+$, we have

$$A\int_a^t T(s)x\, d_\alpha s = -T(a)x + T(t)x$$
$$= T(t)x - T(a)x.$$

c) Take $\alpha-$integral of $\left[T^{(\alpha)}(u)x = A\,T(u)x = T(u)\,A\,x\right]$ from t to $t+\varepsilon\, t^{1-\alpha}$.

$$\int_t^{t+\varepsilon\, t^{1-\alpha}} T^{(\alpha)}(u)x\, d_\alpha u = \int_t^{t+\varepsilon\, t^{1-\alpha}} A\,T(u)x\, d_\alpha u = \int_t^{t+\varepsilon\, t^{1-\alpha}} T(u)\,A\,x\, d_\alpha u.$$

Consequently,

$$\int_t^{t+\varepsilon\, t^{1-\alpha}} T^{(\alpha)}(u)x\, d_\alpha u = \int_a^{t+\varepsilon\, t^{1-\alpha}} T^{(\alpha)}(u)x\, d_\alpha u - \int_a^t T^{(\alpha)}(u)x\, d_\alpha u$$
$$= T(t+\varepsilon\, t^{1-\alpha})x - T(t)x.$$

Therefore,

$$T(t+\varepsilon\, t^{1-\alpha})x - T(t)x = \int_t^{t+\varepsilon\, t^{1-\alpha}} T(u)A\,x\, d_\alpha u = \int_t^{t+\varepsilon\, t^{1-\alpha}} A\,T(u)\,x\, d_\alpha u.$$

The proof is complete. □

Corollary 1. *If A is $\alpha-$infinitesimal generator of a $c_0-\alpha-$semigroup $T(t)$ and $T(t)$ is continuously $\alpha-$ differentiable then $\mathscr{D}(A)$ is dense in X and A is closed operator.*

Proof. Let $x \in X$. *Let* $x_{h,\varepsilon} = \frac{1}{\varepsilon}\int_h^{h+\varepsilon h^{1-\alpha}} T(u)x\, d_\alpha u \in \mathscr{D}(A)$.

$$\lim_{\substack{\varepsilon\to 0\\ h\to 0^+}} x_{h,\varepsilon} = \lim_{\substack{\varepsilon\to 0\\ h\to 0^+}} \frac{1}{\varepsilon}\int_h^{h+\varepsilon h^{1-\alpha}} T(u)x\, d_\alpha u = \lim_{h\to 0^+} T(h)x = T(0)x = x.$$

Thus $\overline{\mathscr{D}(A)} = X$.
To prove the closedness let $x_n \in \mathscr{D}(A)$, $x_n \to x$ and $Ax_n \to y$ as n$\to \infty$.
From Theorem 4 (c), we have

$$T(h+\varepsilon h^{1-\alpha})x_n - T(h)x_n = \int_h^{h+\varepsilon h^{1-\alpha}} T(u)Ax_n d_\alpha u.$$

Dividing by ε and taking $\varepsilon \to 0$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left[T(h+\varepsilon h^{1-\alpha})x_n - T(h)x_n\right] = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_h^{h+\varepsilon h^{1-\alpha}} T(u)Ax_n\, d_\alpha u.$$

Equivalently

$$T^{(\alpha)}(h)x_n = T(h)Ax_n.$$

As $n \to \infty$, we get

$$T^{(\alpha)}(h)x = T(h)y.$$

Also if $h \to 0^+$, we have

$$Ax = T(0)y = y.$$

Thus $Ax = y$ and $x \in \mathscr{D}(A)$. □

Now, we are able to prove Theorem 2. First, we proof necessity of the theorem.
Proof. (Necessity). If A is the infinitesimal generator of a $c_0 - \alpha-$semigroup then it is closed and $\overline{\mathscr{D}(A)} = X$ by Corollary 1. For $\lambda > 0$ and $x \in X$ let

$$R_\varepsilon(\lambda)x = \int_\varepsilon^\infty e^{-\lambda\frac{t^\alpha}{\alpha}} T(t)x\, d_\alpha t,$$

and

$$R(\lambda)x = \int_0^\infty e^{-\lambda\frac{t^\alpha}{\alpha}} T(t)x\, d_\alpha t.$$

Since $\|R_\varepsilon(\lambda) - R(\lambda)\| \longrightarrow 0$ as $\varepsilon \longrightarrow 0$, we get

$$R(\lambda)x = \lim_{\varepsilon\longrightarrow 0}\int_\varepsilon^\infty e^{-\lambda\frac{t^\alpha}{\alpha}} T(t)x\, d_\alpha t.$$

Provided that the map $t \to T(t)x$ is uniformly bounded and continuous, the improper integral exists as a Riemann integral and defines a bounded linear operator $R(\lambda)$ that satisfies the inequality

$$\|R(\lambda)x\| = \|\int_0^\infty e^{-\lambda\frac{t^\alpha}{\alpha}} T(t)x\, d_\alpha t\| \le \int_0^\infty e^{-\lambda\frac{t^\alpha}{\alpha}} \|T(t)x\| d_\alpha t \le \frac{1}{\lambda}\|x\|.$$

Therefore,

$$\|R(\lambda)\| \le \frac{1}{\lambda}.$$

Now, for $h > 0$

$$\frac{T(h+\varepsilon h^{1-\alpha}) - T(h)}{\varepsilon}\int_0^\infty e^{-\lambda\frac{t^\alpha}{\alpha}} T(t)x\, d_\alpha t$$
$$= \frac{1}{\varepsilon}\int_0^\infty e^{-\lambda\frac{t^\alpha}{\alpha}} T(t^\alpha + (h+\varepsilon h^{1-\alpha})^\alpha)^{\frac{1}{\alpha}} x\, d_\alpha t - \frac{1}{\varepsilon}\int_0^\infty e^{-\lambda\frac{t^\alpha}{\alpha}} T(t^\alpha + h^\alpha)^{\frac{1}{\alpha}} x\, d_\alpha t.$$

Using change of variables, we get

$$\begin{aligned}\frac{T(h+\varepsilon h^{1-\alpha})-T(h)}{\varepsilon}R(\lambda)x &= \frac{1}{\varepsilon}\int\limits_{h+\varepsilon h^{1-\alpha}}^{\infty} e^{-\lambda\frac{u^{\alpha}-(h+\varepsilon h^{1-\alpha})^{\alpha}}{\alpha}}T(u)x\,d_{\alpha}u - \frac{1}{\varepsilon}\int\limits_{h}^{\infty} e^{-\lambda\frac{u^{\alpha}-h^{\alpha}}{\alpha}}T(u)x\,d_{\alpha}u \\ &= \frac{e^{\lambda\frac{(h+\varepsilon h^{1-\alpha})^{\alpha}}{\alpha}}}{\varepsilon}\int\limits_{h+\varepsilon h^{1-\alpha}}^{\infty} e^{-\lambda\frac{u^{\alpha}}{\alpha}}T(u)x\,d_{\alpha}u - \frac{e^{\lambda\frac{h^{\alpha}}{\alpha}}}{\varepsilon}\int\limits_{h}^{\infty} e^{-\lambda\frac{u^{\alpha}}{\alpha}}T(u)x\,d_{\alpha}u \\ &= \frac{e^{\lambda\frac{(h+\varepsilon h^{1-\alpha})^{\alpha}}{\alpha}}}{\varepsilon}\int\limits_{h+\varepsilon h^{1-\alpha}}^{h} e^{-\lambda\frac{u^{\alpha}}{\alpha}}T(u)x\,d_{\alpha}u \\ &\quad + \frac{e^{\lambda\frac{(h+\varepsilon h^{1-\alpha})^{\alpha}}{\alpha}}}{\varepsilon}\int\limits_{h}^{\infty} e^{-\lambda\frac{u^{\alpha}}{\alpha}}T(u)x\,d_{\alpha}u - \frac{e^{\lambda\frac{h^{\alpha}}{\alpha}}}{\varepsilon}\int\limits_{h}^{\infty} e^{-\lambda\frac{u^{\alpha}}{\alpha}}T(u)x\,d_{\alpha}u.\end{aligned}$$

As $\varepsilon \to 0$, and using L'Hopital's rule with respect to ε, we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{T(h+\varepsilon h^{1-\alpha})-T(h)}{\varepsilon}R(\lambda)x &= \lim_{\varepsilon\to 0}\frac{e^{\lambda\frac{(h+\varepsilon h^{1-\alpha})^{\alpha}}{\alpha}}\frac{\lambda}{\alpha}\alpha(h+\varepsilon h^{1-\alpha})^{\alpha-1}h^{1-\alpha}}{1}\int\limits_{h}^{\infty} e^{-\lambda\frac{u^{\alpha}}{\alpha}}T(u)x\,d_{\alpha}u \\ &\quad + \lim_{\varepsilon\to 0}\frac{e^{\lambda\frac{(h+\varepsilon h^{1-\alpha})^{\alpha}}{\alpha}}}{\varepsilon}\int\limits_{h+\varepsilon h^{1-\alpha}}^{h} e^{-\lambda\frac{u^{\alpha}}{\alpha}}T(u)x\,d_{\alpha}u. \\ &= \lambda e^{\lambda\frac{h^{\alpha}}{\alpha}}\int\limits_{h}^{\infty} e^{-\lambda\frac{u^{\alpha}}{\alpha}}T(u)x\,d_{\alpha}u \\ &\quad - \lim_{\varepsilon\to 0}e^{\lambda\frac{(h+\varepsilon h^{1-\alpha})^{\alpha}}{\alpha}}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\limits_{h}^{h+\varepsilon h^{1-\alpha}} e^{-\lambda\frac{u^{\alpha}}{\alpha}}T(u)x\,d_{\alpha}u.\end{aligned}$$

Using Theorem 4 (a), we get

$$\lim_{\varepsilon\to 0}\frac{T(h+\varepsilon h^{1-\alpha})-T(h)}{\varepsilon}R(\lambda)x = \lambda e^{\lambda\frac{h^{\alpha}}{\alpha}}\int\limits_{h}^{\infty} e^{-\lambda\frac{u^{\alpha}}{\alpha}}T(u)x\,d_{\alpha}u - e^{\lambda\frac{h^{\alpha}}{\alpha}}e^{-\lambda\frac{h^{\alpha}}{\alpha}}T(h)x.$$

Also, for $h \to 0^{+}$, we get

$$\begin{aligned}AR(\lambda)x &= \lambda\int\limits_{0}^{\infty} e^{-\lambda\frac{u^{\alpha}}{\alpha}}T(u)x\,d_{\alpha}u - x \\ &= \lambda R(\lambda)x - x.\end{aligned}$$

This implies that for every $x \in X$ and $\lambda > 0$, $R(\lambda)x \in \mathscr{D}(A)$ and $AR(\lambda) = \lambda R(\lambda) - I$, or

$$(\lambda I - A)R(\lambda) = I. \tag{2.2}$$

For $x \in \mathscr{D}(A)$, we have

$$\begin{aligned}R(\lambda)Ax &= \int\limits_{0}^{\infty} e^{-\lambda\frac{t^{\alpha}}{\alpha}}T(t)Ax\,d_{\alpha}t \\ &= \int\limits_{0}^{\infty} e^{-\lambda\frac{t^{\alpha}}{\alpha}}AT(t)x\,d_{\alpha}t.\end{aligned}$$

Since A is closed, we get

$$R(\lambda)Ax = A(\int_0^\infty e^{-\lambda\frac{t^\alpha}{\alpha}}T(t)x\,d_\alpha t). \\ = AR(\lambda)x. \tag{2.3}$$

From (2.2) and (2.3), it follows that

$$R(\lambda)(\lambda I - A)x = x \qquad for \quad x \in \mathscr{D}(A).$$

Thus, $R(\lambda)$ is the inverse of $(\lambda I - A)$, it exists for all $\lambda > 0$ and satisfies the desired estimate (1.1). Thus, conditions (**i**) and (**ii**) are necessary.

To prove that the conditions (**i**) and (**ii**) of Theorem 2 are sufficient for A to be the $\alpha-$infinitesimal generator of a continuously $\alpha-$differentiable $c_0 - \alpha-$semigroup of contractions, we need the following lemmas:

Lemma 2. *Let A satisfy the conditions (**i**) and (**ii**) of Theorem 2 and let $R(\lambda : A) = (\lambda I - A)^{-1}$. Then*

$$\lim_{\lambda\to\infty} \lambda R(\lambda : A)x = x \qquad for \quad x \in X. \tag{2.4}$$

Proof. Let $x \in \mathscr{D}(A)$, then

$$\begin{aligned} \|\lambda R(\lambda : A)x - x\| &= \|AR(\lambda : A)x\| \\ &= \|R(\lambda : A)Ax\| \\ &\le \frac{1}{\lambda}\|Ax\| \to 0 \qquad as \quad \lambda \to \infty. \end{aligned}$$

Claim: $\lambda R(\lambda : A)x \to x$ as $\lambda \to \infty$ for every $x \in X$. To prove the claim, let $x \in X$. Since $\mathscr{D}(A)$ is dense X $\exists x_n \in \mathscr{D}(A)$ such that $x_n \to x$ and $\|\lambda R(\lambda : A)x_n - x_n\| \to 0$ as $\lambda \to \infty$. Then

$$\begin{aligned} \|\lambda R(\lambda : A)x - x\| &\le \|\lambda R(\lambda : A)x - x_n\| + \|x_n - x\| \\ &= \|\lambda R(\lambda : A)x - \lambda R(\lambda : A)x_n + \lambda R(\lambda : A)x_n - x_n\| + \|x_n - x\| \\ &\le \|\lambda R(\lambda : A)x - \lambda R(\lambda : A)x_n\| + \|\lambda R(\lambda : A)x_n - x_n\| + \|x_n - x\| \\ &\le \lambda \cdot \frac{1}{\lambda}\|x_n - x\| + \|\lambda R(\lambda : A)x_n - x_n\| + \|x_n - x\| \to 0. \end{aligned}$$

□

Define the sequences of operators

$$A_\lambda = \lambda AR(\lambda : A) = \lambda^2 R(\lambda : A) - \lambda I. \tag{2.5}$$

Lemma 3. *Let A satisfy the conditions (**i**) and (**ii**) of Theorem 2. Then*

$$\lim_{\lambda\to\infty} A_\lambda x = Ax \qquad for \quad x \in \mathscr{D}(A). \tag{2.6}$$

Proof. For $x \in \mathscr{D}(A)$ by Lemma 2 and the definition of A_λ, we have

$$\begin{aligned} \lim_{\lambda\to\infty} A_\lambda x &= \lim_{\lambda\to\infty} \lambda AR(\lambda : A)x \\ &= \lim_{\lambda\to\infty} \lambda R(\lambda : A)Ax \\ &= Ax. \end{aligned}$$

The proof is complete. □

Lemma 4. *Let A satisfy the conditions (**i**) and (**ii**) of Theorem 2. Then A_λ is the $\alpha-$infinitesimal generator of a uniformly continuous $\alpha-$semigroup of contractions $e^{\frac{t^\alpha}{\alpha}A_\lambda}$. Furthermore, for every $x \in X$, λ, $\mu > 0$ we have*

$$\|e^{\frac{t^\alpha}{\alpha}A_\lambda}x - e^{\frac{t^\alpha}{\alpha}A_\mu}x\| \le \frac{t^\alpha}{\alpha}\|A_\lambda x - A_\mu x\|. \tag{2.7}$$

Proof. From (2.5), it is clear that A_λ is a bounded linear operator and so it is the infinitesimal generator of a uniformly continuous $\alpha-$semigroup $e^{\frac{t^\alpha}{\alpha}A_\lambda}$ of bounded linear operators Theorem 1. Also,

$$\begin{aligned}\|e^{\frac{t^\alpha}{\alpha}A_\lambda}\| = \|e^{\frac{t^\alpha}{\alpha}(\lambda^2R(\lambda:A)-\lambda I)}\| &\le \|e^{\frac{t^\alpha}{\alpha}\lambda^2R(\lambda:A)-\frac{t^\alpha}{\alpha}\lambda}\| \\ &\le \|e^{\frac{t^\alpha}{\alpha}\lambda^2R(\lambda:A)}\|e^{-\frac{t^\alpha}{\alpha}\lambda} \\ &\le e^{\frac{t^\alpha}{\alpha}\lambda^2\|R(\lambda:A)\|}e^{-\frac{t^\alpha}{\alpha}\lambda} \\ &= e^0 = 1.\end{aligned} \tag{2.8}$$

So $e^{\frac{t^\alpha}{\alpha}A_\lambda}$ is an $\alpha-$semigroup of contractions. Consequently,

$$\begin{aligned}\|e^{\frac{t^\alpha}{\alpha}A_\lambda}x - e^{\frac{t^\alpha}{\alpha}A_\mu}x\| &= \|\int_0^1 \frac{d}{ds}(e^{\frac{t^\alpha}{\alpha}sA_\lambda}e^{\frac{t^\alpha}{\alpha}(1-s)A_\mu})x\,ds\| \\ &\le \int_0^1 \|\frac{d}{ds}(e^{\frac{t^\alpha}{\alpha}sA_\lambda}e^{\frac{t^\alpha}{\alpha}(1-s)A_\mu})x\|ds \\ &= \int_0^1 \|e^{\frac{t^\alpha}{\alpha}sA_\lambda}e^{\frac{t^\alpha}{\alpha}(1-s)A_\mu}(-\frac{t^\alpha}{\alpha})A_\mu x + e^{\frac{t^\alpha}{\alpha}(1-s)A_\mu}e^{\frac{t^\alpha}{\alpha}sA_\lambda}(\frac{t^\alpha}{\alpha})A_\lambda x\|ds \\ &\le \frac{t^\alpha}{\alpha}\int_0^1 \|e^{\frac{t^\alpha}{\alpha}sA_\lambda}\|\|e^{\frac{t^\alpha}{\alpha}(1-s)A_\mu}\|\|A_\lambda x - A_\mu x\|ds \\ &\le \frac{t^\alpha}{\alpha}(1)(1)\|A_\lambda x - A_\mu x\| \\ &= \frac{t^\alpha}{\alpha}\|A_\lambda x - A_\mu x\|.\end{aligned}$$

The proof is complete. □

Now, we will prove the sufficiency of the theorem.

Proof. Theorem 2 (Sufficiency). Let $x \in \mathscr{D}(A)$. Then

$$\begin{aligned}\|e^{\frac{t^\alpha}{\alpha}A_\lambda}x - e^{\frac{t^\alpha}{\alpha}A_\mu}x\| &\le \frac{t^\alpha}{\alpha}\|A_\lambda x - A_\mu x\| \\ &\le \frac{t^\alpha}{\alpha}\|A_\lambda x - Ax\| + \frac{t^\alpha}{\alpha}\|Ax - A_\mu x\|.\end{aligned} \tag{2.9}$$

Consequently using Lemma 3 and (2.9), we have $x \in \mathscr{D}(A)$ and $e^{\frac{t^\alpha}{\alpha}A_\lambda}$ converges uniformly on bounded intervals as $\lambda \to \infty$. Since $\|e^{\frac{t^\alpha}{\alpha}A_\lambda}\| \le 1$ and $\overline{\mathscr{D}(A_\lambda)} = X$, we have

$$\lim_{\lambda\to\infty} T_\lambda(t)x = \lim_{\lambda\to\infty} e^{\frac{t^\alpha}{\alpha}A_\lambda}x = T(t)x \qquad \text{for every} \quad x \in X. \tag{2.10}$$

Notice that the limits $T(t)$ satisfies the $\alpha-$semigroup property, that $T(0)x = \lim_{\lambda\to\infty} e^{A_\lambda(0)}x = x$, and

$$\begin{aligned}T(s+t)^{\frac{1}{\alpha}} &= \lim_{\lambda\to\infty} e^{\frac{(s+t)}{\alpha}A_\lambda} \\ &= \lim_{\lambda\to\infty} e^{\frac{s}{\alpha}A_\lambda}e^{\frac{t}{\alpha}A_\lambda} \\ &= \lim_{\lambda\to\infty} e^{\frac{(s^{\frac{1}{\alpha}})^\alpha}{\alpha}A_\lambda}e^{\frac{(t^{\frac{1}{\alpha}})^\alpha}{\alpha}A_\lambda} \\ &= T(s^{\frac{1}{\alpha}})T(t^{\frac{1}{\alpha}}).\end{aligned}$$

From (2.8) and (2.10), we have

$$\begin{aligned}\|T(t)x\| &= \|\lim_{\lambda\to\infty} e^{\frac{t^\alpha}{\alpha}A_\lambda}x\| \\ &= \lim_{\lambda\to\infty} \|e^{\frac{t^\alpha}{\alpha}A_\lambda}x\| \\ &\leq \|x\|.\end{aligned}$$

Therefore, $\|T(t)\| \leq 1$. Also, $t \to T(t)x$ is continuous for $t \geq 0$ as a uniform limit of the continuous functions $t \to e^{\frac{t^\alpha}{\alpha}A_\lambda}x$. Thus $T(t)$ is a $c_0-\alpha-$semigroup of contractions on X.
To complete the proof, we need to show that A is the infinitesimal generator of an $\alpha-$semigroup $T(t)$. Let B be the infinitesimal generator of an $\alpha-$semigroup $T(t)$ and $x \in \mathscr{D}(A)$. Then, using Theorem 4 and (2.10) we have,

$$\begin{aligned}T(t+\varepsilon t^{1-\alpha})x - T(t)x &= \lim_{\lambda\to\infty}(T_\lambda(t+\varepsilon t^{1-\alpha})x - T_\lambda(t)x) \\ &= \lim_{\lambda\to\infty} \int_t^{t+\varepsilon t^{1-\alpha}} T_\lambda(u)A_\lambda x\, d_\alpha u \\ &= \int_t^{t+\varepsilon t^{1-\alpha}} T(u)Ax\, d_\alpha u.\end{aligned}$$

Therefore,

$$T(t+\varepsilon t^{1-\alpha})x - T(t)x = \int_t^{t+\varepsilon t^{1-\alpha}} T(u)Ax\, d_\alpha u.$$

Dividing by $\varepsilon > 0$ and letting $\varepsilon \to 0$, we get

$$\lim_{\varepsilon\to 0}\frac{T(t+\varepsilon t^{1-\alpha})x - T(t)x}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon t^{1-\alpha}} T(u)Ax\, d_\alpha u.$$

Equivalently,

$$T^{(\alpha)}(t)x = T(t)Ax.$$

As $t \to 0^+$, we have

$$Bx = Ax \;\; and \;\; x \in \mathscr{D}(B).$$

Thus $B \supseteq A$. From necessary conditions, we have $1 \in \rho(B)$, and by assumption (ii) $1 \in \rho(A)$. Since $B \supseteq A$,

$$(I-B)\mathscr{D}(A) = (I-A)\mathscr{D}(A) = X,$$

which implies $\mathscr{D}(B) = (I-B)^{-1}X = \mathscr{D}(A)$ and therefore $A = B$. □

3 Uniformly continuous $\alpha-$semigroups of bounded linear operator

This section is dedicated to the study of uniformly continuous $\alpha-$semigroups of bounded linear operator. Now, we are ready to prove the Theorem 1.

Proof. Define $T(t) = e^{\frac{t^\alpha}{\alpha}A}$, A is bounded linear operator. Then $T(t)$ is an $\alpha-$semigroup with A is an $\alpha-$infinitesimal generator.

$$\begin{aligned}\|T(t^{\frac{1}{\alpha}})-I\| &= \|e^{\frac{t}{\alpha}A}-I\| \\ &= \|\sum_{k=0}^{\infty}\frac{(\frac{t}{\alpha}A)^k}{k!}-I\| \\ &= \|\sum_{k=1}^{\infty}\frac{(\frac{t}{\alpha}A)^k}{k!}\| \\ &= \|\frac{t}{\alpha}A+\frac{(\frac{t}{\alpha}A)^2}{2!}+\ldots\| \\ &\le \frac{t}{\alpha}\|A\|\|I+\frac{\frac{t}{\alpha}A}{2!}+\ldots\| \\ &\le \frac{t}{\alpha}\|A\|\|I+\frac{t}{\alpha}A+\ldots\| \\ &= \frac{t}{\alpha}\|A\|e^{\frac{t}{\alpha}\|A\|}.\end{aligned}$$

$\lim_{t\to 0^+}\|T(t^{\frac{1}{\alpha}})-I\|=0.$

Now for the converse. Fix $a>0$, choose $\rho>0$ small enough such that $\|I-\lambda\int_a^{\rho}T(t)\,d_\alpha t\|<1$, then $\int_a^{\rho}T(t)\,d_\alpha t$ invertible and

$$\begin{aligned}\frac{T(h+\varepsilon h^{1-\alpha})-T(h)}{\varepsilon}\int_a^{\rho}T(t)d_\alpha t &= \frac{1}{\varepsilon}\left[\int_a^{\rho}T(t^\alpha+(h+\varepsilon h^{1-\alpha})^\alpha)^{\frac{1}{\alpha}}d_\alpha t-\int_a^{\rho}T(t^\alpha+h^\alpha)^{\frac{1}{\alpha}}d_\alpha t\right] \\ &= \frac{1}{\varepsilon}\left[\int_{(a^\alpha+(h+\varepsilon h^{1-\alpha})^\alpha)^{\frac{1}{\alpha}}}^{(\rho^\alpha+(h+\varepsilon h^{1-\alpha})^\alpha)^{\frac{1}{\alpha}}}T(u)d_\alpha u-\int_{(a^\alpha+h^\alpha)^{\frac{1}{\alpha}}}^{(\rho^\alpha+h^\alpha)^{\frac{1}{\alpha}}}T(u)\,d_\alpha u\right]\end{aligned}$$

$$\begin{aligned}&= \frac{1}{\varepsilon}\left[\int_{(a^\alpha+(h+\varepsilon h^{1-\alpha})^\alpha)^{\frac{1}{\alpha}}}^{(a^\alpha+h^\alpha)^{\frac{1}{\alpha}}}T(u)d_\alpha u+\int_{(a^\alpha+h^\alpha)^{\frac{1}{\alpha}}}^{(\rho^\alpha+h^\alpha)^{\frac{1}{\alpha}}}T(u)d_\alpha u\right. \\ &\qquad\left.+\int_{(\rho^\alpha+h^\alpha)^{\frac{1}{\alpha}}}^{(\rho^\alpha+(h+\varepsilon h^{1-\alpha})^\alpha)^{\frac{1}{\alpha}}}T(u)d_\alpha u-\int_{(a^\alpha+h^\alpha)^{\frac{1}{\alpha}}}^{(\rho^\alpha+h^\alpha)^{\frac{1}{\alpha}}}T(u)d_\alpha u\right].\end{aligned}$$

As $\varepsilon\to 0$ Lemma 1 (a) implies

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{T(h+\varepsilon h^{1-\alpha})-T(h)}{\varepsilon}\int_a^{\rho}T(t)d_\alpha t &= -D_\alpha\left(\int_{a_2}^{(a^\alpha+h^\alpha)^{\frac{1}{\alpha}}}T(u)d_\alpha u\right) \\ &\qquad+D_\alpha\left(\int_{a_1}^{(\rho^\alpha+h^\alpha)^{\frac{1}{\alpha}}}T(u)d_\alpha u\right)\text{ where } a_1,\ a_2>0.\end{aligned}$$

Now, using Lemma 1 (b) to get

$$\lim_{\varepsilon\to 0}\frac{T(h+\varepsilon h^{1-\alpha})-T(h)}{\varepsilon}\int_a^{\rho}T(t)d_\alpha=-T(a^\alpha+h^\alpha)^{\frac{1}{\alpha}}+T(\rho^\alpha+h^\alpha)^{\frac{1}{\alpha}}.$$

As $h \to 0^+$, we get

$$A\int_a^\rho T(t)d_\alpha t = -T(a) + T(\rho)$$
$$= T(\rho) - T(a).$$

Therefore, $A = (T(\rho) - T(a))(\int_a^\rho T(t)d_\alpha t)^{-1}$ and so A is bounded. □

4 Conclusion

This paper investigates the generator of conformable semigroups and its resolvent operator. Generally, this paper introduces and proves some of new properties of conformable semigroups of operators similar to those of strongly continuous semigroups of operators. Also, this paper defines a uniformly continuous conformable semigroups and proves some characterization of its generator. Based on these results, some further works could apply our theorems to solve certain problems and applications in fractional differential equations.

Acknowledgment

This research is supported in part by Yarmouk University.

Conflict of Interest

The authors declare that they have no conflict of interest.

References

[1] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Vol. **44** (Applied Mathematical Sciences), Springer Verlag Publishers, New York, 1993.
[2] B. J. West, M. Bologna and P. Grigolini, *Physics of fractal operators*, Springer, Berlin, Heidelberg and New York, 2003.
[3] D. Baleanu, K. Diethelm, E. Scalas and J. J. Trujillo, *Fractional calculus*: *models and numerical methods, Series on Complexity, Nonlinearity and Chaos*, Vol. **3**, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 2012.
[4] M. D. Ortigueira, *Fractional calculus for scientists and engineers*, Springer, Berlin, Heidelberg and New York, 2011.
[5] S. Hu, Y.-Q. Chen and T.-S. Qiu, *Fractional processes and fractional-order signal processing*: *techniques and applications*, Springer, Berlin, Heidelberg and New York, 2012.
[6] R. L. Magin, *Fractional calculus in bioengineering*, Begerll House, West Redding, Connecticut, 2006.
[7] F. Mainardi, *Fractional calculus and waves in linear viscoelasticity*: *an introduction to mathematical models*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 2010.
[8] J. Klafter, S.-C. Lim and R. Metzler, *Fractional calculus*: *recent advances*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 2011.
[9] V. E. Tarasov, *Fractional dynamics*: *applications of fractional calculus to dynamics of particles, fields and media*, Springer, Berlin, Heidelberg and New York, 2011.
[10] I. Podlubny, *Fractional differential equations*, Vol. **198** (Mathematics in Science and Engineering), Academic Press, San Diego, California, USA, 1999.
[11] A. Kilbas, H. Srivastava and J. Trujillo, *Theory and applications of fractional differential equations*, Vol. **204** (North-Holland Mathematics Studies), Elsevier (North-Holland) Science Publishers, New York, USA, 2006.
[12] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, A new definition of fractional derivative, *J. Comput. Appl. Math.* **264**, 65–70 (2014).
[13] K. Hosseini and R. Ansari, New exact solutions of nonlinear conformable time-fractional Boussinesq equations using the modified Kudryashov method, *Wave. Rand. Compl. Med.* **27**, 628–636 (2017).
[14] Sh. Al-Sharif and A. Malkawi, Modification of conformable fractional derivative with classical properties, *Ital. J. Pure Appl. Math.* **44**, (2020) to appear.
[15] R. Anderson and J. Ulness, Newly defined conformable derivatives, *Adv. Dyn. Sys. Appl.* **10**, 109–137 (2015).

[16] A. Atangana, D. Baleanu and A. Alsaedi, New properties of conformable derivative, *Open Math.* **13**, 889–898 (2015).
[17] T. Abdeljawad, On conformable fractional calculus, *J. Comput. Appl. Math.* **279**, 57-66 (2015).
[18] T. Abdeljawad, J. Alzabut and F. Jarad, A generalized Lyapunov-type inequality in the frame of conformable derivatives, *Adv. Differ. Equ.* (**2017**) 2017:321 DOI 10.1186/s13662-017-1383-z.
[19] M. Al-Rifae and T. Abdeljawad, Fundamental results of conformable SturmLiouville eigenvalue problems, *Complexity* 2017, Article ID 3720471, https://doi.org/10.1155/2017/3720471.
[20] F. Jarad, E. Ugurlu, T. Abdeljawad and D. Baleanu, On a new class of fractional operators, *Adv. Differ. Equ.* (**247**), (2017).
[21] F. Jarad, T. Abdeljawad and J. Alzabut, Generalized fractional derivatives generated by a class of local proportional derivatives,*Eur. Phys. J. Spec. Top.* (**1618**), 3457-3471 (2017).
[22] S. Bochner, Diffusion equation and stochastic processes, *PNAS* **35**, 368–370 (1949).
[23] A. Balakrishnan, Fractional powers of closed operators and the semigroups generated by them, *Pac. J. Math.* **10**, 419–439 (1960).
[24] E. Popescu, On the fractional Cauchy problem associated with a Feller Semigroup, *Math. Rep.* **12**, 181-188 (2010).
[25] T. Abdeljawad, M. Al- Horani and R. Khalil, Conformable fractional semigroup operators, *JSTA* **2015**, Article ID. 7, (2015).