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Parameter Estimation for a Mixture of Inverse Chen and Inverse Compound Rayleigh Distribution Based on Type-I Hybrid Censoring Scheme

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Abstract: In this paper, we consider the statistical inference of the unknown parameters of a mixture of the inverse Chen and inverse compound Rayleigh distributions (ICICRD) based on the Type-I hybrid censored scheme. The parameters are estimated using maximum likelihood and Bayesian methods. However, since Bayes estimators do not exist in an explicit form for the parameters, Tierney-Kadane (T-K) approximation is used to obtain the Bayes estimators with loss functions: the symmetric square error (SE), asymmetric linear exponential (LINEX) and general entropy (GE) loss functions . An extensive simulation study is conducted to compare the performances of different methods.

Keywords: Mixture model, Hybrid censored sample, Maximum likelihood estimation, Bayesian estimation, T-K approximation.

1 Introduction

Life testing experiments are concerned with studies of reliability data. In the recent years, the hybrid censoring scheme has received a considerable attention in the reliability and life testing experiments. This type is a mixture of Type-I and Type-II censoring schemes. In this type, the test is terminated as soon as prefixed (T) time has been reached. In Type-I hybrid censoring, the test is terminated at a time $T^* = min(x_{R:n}, T)$, where $x_{R:n}$ represents the failure time of the R^{th} item and T is the prefixed maximum allowable time of test. However, in hybrid Type-II censoring, the test is terminated at a time $T^* = max(x_{R:n}, T)$, i.e. the test has at least R failure items in Type-II hybrid censoring scheme, and the test can never be reached beyond the time T. This scheme, which is known as Type-I hybrid censoring scheme, has been introduced by Epstein [1]. The Type-I hybrid censoring scheme has been used as a reliability acceptance test in MIL-STD-781C [2]. Fairbanks et al. [3] have slightly modified the proposition of Epstein [1] and suggested a simple set of confidence intervals. Chen and Bhattacharya [4] have provided an exact confidence interval band for a parameter using the distribution of the corresponding maximum likelihood estimator. Draper and Guttman [5] have obtained Bayesian estimators and the two sided credible intervals of the mean lifetime using inverted gamma prior. Many authors have used hybrid censoring schemes (see, Gupta and Kundu [6], Ebrahimi [7], Kundu [8], Kundu and Pradhan [9], Rastogi and Tripathi[10], Singh et al. [11], Hyun et al. [12] and Sultana et al. [13]). Under the Type-I hybrid censoring scheme, we can observe the following:

Case I:$\{x_{1:n} < x_{2:n} < \cdots < x_{R:n}\}$ if $x_{R:n} < T$.
Case II:$\{x_{1:n} < x_{2:n} < \cdots < x_{d:n}\}$ if $d < R$ and $x_{d:n} < T < x_{d+1:n}$.

Chen [14] has proposed a new two parameter lifetime distribution with bathtub shaped or increasing failure rate function. The bathtub shape hazard function provides an appropriate conceptual model for some electronic and mechanical products. Inferences on the Chen distribution have been addressed by Wu et al. [15], Wu [16], Sarhan et al. [17] as well as Rastogi and Tripathi [10]. Srivastava and Srivastava [18] have derived a new distribution called Inverse Chen (IC) distribution with its maximum likelihood estimators (MLEs) and their asymptotic confidence intervals, survival function and hazard rate.

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The Inverse Chen (IC) distribution has the following cumulative distribution function (cdf) and the density function (pdf) given, respectively, by

$$F_1(x) = e^{\lambda_1\left(1-e^{x^{-\beta_1}}\right)}, \qquad x>0,\ \lambda_1>0, \beta_1>0 \tag{1}$$

$$f_1(x) = \lambda_1\beta_1 x^{-(\beta_1+1)} e^{\left[x^{-\beta_1}+\lambda_1\left(1-e^{x^{-\beta_1}}\right)\right]}, \qquad x>0,\ \lambda_1>0, \beta_1>0 \tag{2}$$

and, the reliability function is given by

$$R_1(x) = 1-e^{\lambda_1\left(1-e^{x^{-\beta_1}}\right)}, \qquad x>0,\ \lambda_1>0, \beta_1>0 \tag{3}$$

The compound Rayleigh distribution provides a population model which is useful in several areas of statistics, including life testing, reliability and survival analysis. This distribution is a special case of the three parameter Burr XII distribution. In the last couple of decades, several statisticians have paid attention for the development of this distribution [see Abushal[19], Al-Hossain [20], and Abd-Elmougod and Mahmoud [21]. The introduced model will be named Inverse Compound Rayleigh (ICR) distribution. Its cdf and pdf are given, respectively, by

$$F_2(x) = \beta_2^{\lambda_2}\left(\beta_2+\frac{1}{x^2}\right)^{-\lambda_2}, \qquad x>0,\ \lambda_2>0, \beta_2>0 \tag{4}$$

$$f_2(x) = 2\lambda_2\beta_2^{\lambda_2}x^{-3}\left(\beta_2+\frac{1}{x^2}\right)^{-(\lambda_2+1)}, \qquad x>0,\ \lambda_2>0, \beta_2>0 \tag{5}$$

Then, the reliability of ICR distribution is given by

$$R_2(x) = 1-\beta_2^{\lambda_2}\left(\beta_2+\frac{1}{x^2}\right)^{-\lambda_2}, \qquad x>0,\ \lambda_2>0, \beta_2>0 \tag{6}$$

Mixture distributions have gained great interest for the analysis because they play a vital role in many practical applications. Direct applications of finite mixture models are medicine , botany, life testing, reliability, ... etc. Indirect applications include outliers, cluster analysis, latent structure models, modeling prior densities, empirical Bayes method and nonparametric density estimation. Finite mixture models are studied both theoretically and practically by many authors [see Everitt and Hand [22], Titterington et al.[23], McLachlan and Basford [24], Lindsay [25] as well as McLachlan and Peel [26]. Also, mixture distributions have been considered extensively by the researchers using both classical and Bayesian techniques [for example , Abu-Zinadah [27], Erisoglu et al.[28], Feroze and Aslam [29], Daniyal and Rajab [30], Mahmoud et al.[31], and Zhu et al. [32].

If the population consists of a mixture of two independent subpopulation representing failure types, then the distribution function of the mixed population can be expressed by

$$F(x) = \sum_{j=1}^{2} p_j F_j(x), \qquad j=1,2 \tag{7}$$

where $F(x)$ is the cdf of the mixed population, $F_j(x)$ is the cdf of the j-th subpopulation defined by (1) for j=1 and (4) for j=2, and the mixing proportions p_j are such that $0 \le p_j \le 1, \quad j=1,2$, $p_1 + p_2 = 1$.

Also, the corresponding density function is given by

$$f(x) = \sum_{j=1}^{2} p_j f_j(x), \qquad j=1,2 \tag{8}$$

where $f_j(x)\,,\ j=1,2$, are given by (2) and (5).

Thus, the reliability function is given by

$$R(x) = \sum_{j=1}^{2} p_j R_j(x), \qquad j=1,2 \tag{9}$$

where $R_j(x)\,,\ j=1,2$, are given by (3) and (6).

The present paper aims to estimate the unknown parameter of the mixture Inverse Chen and Inverse compound Raleigh distributions based on Type-I hybrid censoring schemes. The rest of the paper is arranged, as follows: The maximum likelihood estimators (MLEs) is obtained in Section 2. In Section 3, we apply Tierney and Kadane's approximation to compute the Bayes estimators. Simulation study is presented in Section 4. Conclusion is presented in Section 5.

2 Maximum Likelihood Estimation

Suppose that n identical units from population with pdf (8) are placed on a life-test. In the type-I hybrid censoring scheme, R, T are known in advance and the termination time of experiment is $t = min\left(x_{R:n}, T\right)$, where $x_{R:n}$ is the Rth order statistic of the sample of size n.Suppose r units have failed during the interval (0,t): r_1 units from the first subpopulation and r_2 units from the second subpopulation, such that $r = r_1 + r_2$. Also, assume that x_{ij} denotes the failure time of the j^{th} unit belonging to the i^{th} subpopulation, where $i = 1,2 \ and \ j = 1,2,\ldots,r_i$. For a two component mixture model, the likelihood function is given by

$$L\left(\vartheta|\underline{x}\right) = \frac{n!}{(n-r)!}\left[\prod_{j=1}^{r_1} p_1 f_1\left(x_{1j}\right)\right]\left[\prod_{j=1}^{r_2} p_2 f_2\left(x_{2j}\right)\right]\left[1-F\left(t\right)\right]^{(n-r)} \tag{10}$$

where ϑ is the vector of parameters; $\vartheta = (\lambda_1, \lambda_2, \beta_1, \beta_2, p)$,

$$r = r_1 + r_2, F\left(t\right) = \sum_{j=1}^{2} p_j F_j\left(t\right), \qquad j = 1,2$$

$$\mathrm{r} = \begin{cases} \mathrm{R} & \text{for case I} \\ \mathrm{d} & \text{for case II} \end{cases} \qquad \mathrm{t} = \begin{cases} \mathrm{x_{R:n}} & \text{for case I} \\ \mathrm{T} & \text{for case II} \end{cases}$$

The likelihood function can be written as

$$L\left(\vartheta|\underline{x}\right) = \frac{n!}{(n-r)!}\prod_{j=1}^{r_1} p_1\lambda_1\beta_1 x_{1j}^{-(\beta_1+1)} e^{\left[x_{1j}^{-\beta_1}+\lambda_1\left(1-e^{x_{1j}^{-\beta_1}}\right)\right]}\prod_{j=1}^{r_2} p_2(2\lambda_2)\beta_2^{\lambda_2} x_{2j}^{-3}\left(\beta_2+\frac{1}{x_{2j}^2}\right)^{-(\lambda_2+1)}\left[G\left(t\right)\right]^{(n-r)} \tag{11}$$

where

$$G\left(t\right) = 1-\left\{p_1 e^{\lambda_1\left(1-e^{t^{-\beta_1}}\right)} + p_2\beta_2^{\lambda_2}\left(\beta_2+\frac{1}{t^2}\right)^{-\lambda_2}\right\}, \qquad p_1 = p,\ p_2 = 1-p.$$

Then, the log-likelihood function can be expressed as

$$\begin{aligned}\ln L &= \ln L\left(\vartheta|\underline{x}\right) \\ &\propto r_1\ln p_1 + r_1\ln\lambda_1 + r_1\ln\beta_1 - (\beta_1+1)\sum_{j=1}^{r_1}\ln x_{1j} + \sum_{j=1}^{r1} x_{1j}^{-\beta_1} + \lambda_1\sum_{j=1}^{r_1}\left(1-e^{x_{1j}^{-\beta_1}}\right) + r_2\ln p_2 + r_2\ln\left(2\lambda_2\right) + r_2\lambda_2\ln\beta_2 \\ &- 3\sum_{j=1}^{r_2}\ln x_{2j} - (\lambda_2+1)\sum_{j=1}^{r_2}\ln\left(\beta_2+\frac{1}{x_{2j}^2}\right) + (n-r)\ln\left[G\left(t\right)\right]\end{aligned} \tag{12}$$

Taking derivatives with respect to λ_1, λ_2,β_1,β_2 and p in Equation (12), we obtain the following

$$\frac{\partial \ln L}{\partial \lambda_1} = \frac{r_1}{\lambda_1} + \sum_{j=1}^{r_1}\left(1-e^{x_{1j}^{-\beta_1}}\right) + \frac{(n-r)}{G\left(t\right)}\left(\frac{\partial G\left(t\right)}{\partial \lambda_1}\right)$$

$$\frac{\partial \ln L}{\partial \lambda_2} = \frac{r_2}{\lambda_2} + r_2\ln\left(\beta_2\right) - \sum_{j=1}^{r_2}\ln\left(\beta_2+\frac{1}{x_{2j}^2}\right) + \frac{(n-r)}{G\left(t\right)}\left(\frac{\partial G\left(t\right)}{\partial \lambda_2}\right)$$

$$\frac{\partial \ln L}{\partial \beta_1}=\frac{r_1}{\beta_1}-\sum_{j=1}^{r_1}\ln\left(x_{1j}\right)-\sum_{j=1}^{r_1}x_{1j}^{-\beta_1}\ln\left(x_{1j}\right)\ +\lambda_1\sum_{j=1}^{r_1}e^{x_{1j}^{-\beta_1}}x_{1j}^{-\beta_1}\ln\left(x_{1j}\right)\ +\frac{(n-r)}{G(t)}\left(\frac{\partial G(t)}{\partial \beta_1}\right)$$

$$\frac{\partial \ln L}{\partial \beta_2}=\frac{r_2\lambda_2}{\beta_2}-(\lambda_2+1)\sum_{j=1}^{r_2}\frac{1}{\beta_2+\frac{1}{x_{2j}^2}}+\frac{(n-r)}{G(t)}\left(\frac{\partial G(t)}{\partial \beta_2}\right)$$

$$\frac{\partial \ln L}{\partial p}=\frac{r_1}{p_1}-\frac{r_2}{p_2}+\frac{(n-r)}{G(t)}\left(\frac{\partial G(t)}{\partial p}\right)$$

where,

$$\frac{\partial G(t)}{\partial \lambda_1}=-p_1e^{\lambda_1\left(1-e^{t^{-\beta_1}}\right)}\left(1-e^{t^{-\beta_1}}\right),$$

$$\frac{\partial G(t)}{\partial \lambda_2}=p_2\beta_2^{\lambda_2}\left(\beta_2+\frac{1}{t^2}\right)^{-\lambda_2}\left[\ln\left(\beta_2+\frac{1}{t^2}\right)-\ln\left(\beta_2\right)\right]$$

$$\frac{\partial G(t)}{\partial \beta_1}=-p_1e^{t^{-\beta_1}+\lambda_1\left(1-e^{t^{-\beta_1}}\right)}t^{-\beta_1}\ln(t)\ \lambda_1$$

$$\frac{\partial G(t)}{\partial \beta_2}=p_2\lambda_2\beta_2^{\lambda_2}\left(\beta_2+\frac{1}{t^2}\right)^{-\lambda_2}\left[\left(\beta_2+\frac{1}{t^2}\right)^{-1}-\beta_2^{-1}\right]$$

$$\frac{\partial G(t)}{\partial p}=-e^{\lambda_1\left(1-e^{t^{-\beta_1}}\right)}+\beta_2^{\lambda_2}\left(\beta_2+\frac{1}{t^2}\right)^{-\lambda_2}$$

The maximum likelihood estimators of the five parameters are obtained by solving the above equations simultaneously after equating to zero.

3 Bayesian Estimation

In this section, we handle the problem of Bayesian estimation of the unknown parameters of the mixture ICICR distribution under the Type-I hybrid censored data. Assuming that the prior distribution for the parameters follow the independent gamma prior, the joint prior for ϑ; $\vartheta=(\lambda_1,\lambda_2,\beta_1,\beta_2,p)$, is thus given by:

$$\pi(\vartheta)=\left[\prod_{i=1}^{2}\pi_i(\lambda_i)\,g_i(\beta_i)\right]\times\pi_3(p)$$

Suppose that $\pi_i(\lambda_i)$, $g_i(\beta_i)$ have conjugated gamma prior distribution and the prior distribution of the proportion parameter p is uniform over the interval (0,1).

$$\pi_i(\lambda_i)=\frac{b_i^{a_i}}{\Gamma(a_i)}\lambda_i^{a_i-1}e^{-b_i\lambda_i},\quad \lambda_i>0,\ a_i,b_i>0;i=1,2.$$

$$g_i(\beta_i)=\frac{d_i^{c_i}}{\Gamma(c_i)}\beta_i^{c_i-1}e^{-d_i\beta_i},\quad \beta_i>0,\ c_i,d_i>0;i=1,2$$

and

$$\pi_3(p)=1$$

Thus, the joint prior distribution of ϑ can be written, as follows:

$$\pi(\vartheta)\propto\prod_{i=1}^{2}\lambda_i^{a_i-1}e^{-b_i\lambda_i}\ \beta_i^{c_i-1}e^{-d_i\beta_i}\tag{13}$$

The joint posterior density function ϑ can be written in the form

$$P(\vartheta|\underline{x}) = \frac{L(\vartheta|\underline{x})\,\pi(\theta)}{\int_\vartheta L(\vartheta|\underline{x})\,\pi(\theta)\,d\vartheta} \tag{14}$$

where $\int_\vartheta = \int_{\lambda_1}\int_{\lambda_2}\int_{\beta_1}\int_{\beta_2}\int_p$ and $d\vartheta = d\lambda_1 d\lambda_2 d\beta_1 d\beta_2 dp$.

The Bayes estimator under squared error loss function of any function ϕ is the posterior mean and has been given by:

$$\widehat{\phi}_{SE}(\vartheta) = E_{\vartheta|\underline{x}}[\phi(\vartheta)] = \frac{\int_\vartheta \phi(\vartheta)L(\vartheta|\underline{x})\,\pi(\theta)\,d\vartheta}{\int_\vartheta L(\vartheta|\underline{x})\,\pi(\theta)\,d\vartheta} \tag{15}$$

Also, the Bayes estimator of $\phi(\vartheta)$ using LINEX and general entropy loss function are

$$\widehat{\phi}_{LINEX}(\vartheta) = -\frac{1}{q}\ln\left[E_{\vartheta|\underline{x}}\left[e^{-q\phi(\vartheta)}\right]\right], \quad q \neq 0 \tag{16}$$

where

$$E_{\vartheta|\underline{x}}\left[e^{-q\phi(\vartheta)}\right] = \frac{\int_\vartheta \; e^{-q\phi(\vartheta)}L(\vartheta|\underline{x})\,\pi(\vartheta)\,d\vartheta}{\int_\vartheta \; L(\vartheta|\underline{x})\,\pi(\vartheta)\,d\vartheta} \tag{17}$$

and

$$\widehat{\phi}_{GE}(\vartheta) = \left[E_{\vartheta|\underline{x}}\left[\phi(\vartheta)^{-h}\right]\right]^{-\frac{1}{h}}, \qquad h \neq 0 \tag{18}$$

where

$$E_{\vartheta|\underline{x}}\left[\phi(\vartheta)^{-h}\right] = \frac{\int_\vartheta \phi(\vartheta)^{-h}L(\vartheta|\underline{x})\,\pi(\vartheta)\,d\vartheta}{\int_\vartheta L(\vartheta|\underline{x})\,\pi(\vartheta)\,d\vartheta} \tag{19}$$

The ratio of the integrals in Equations $(15), (17)$ *and* (19) cannot be obtained in a closed form. Therefore, in this case, Tierney-Kadane approximation can be used to compute the Bayes estimators for the parameters.

3.1 Tierney_Kadane's approximation Method

Here, we compute the Bayes estimators of unknown parameters using the method given by Tierney and Kadane [33]. The Bayes estimator of $\phi(\vartheta)$ can be expressed as

$$E[\phi(\vartheta)|x] = \frac{\int_\vartheta \; e^{n\delta^*(\vartheta)}d\vartheta}{\int_\vartheta \; e^{n\delta(\vartheta)}d\vartheta} \tag{20}$$

We consider the following functions

$$\delta(\vartheta) = \frac{1}{n}[L(\vartheta|\underline{x}) + \rho(\vartheta)], \qquad \delta^*(\vartheta) = \delta(\vartheta) + \frac{1}{n}\ln\phi(\vartheta)$$

where, $L(\vartheta|x)$ is the log-likelihood function and $\rho(\vartheta) = \ln\pi\,(\vartheta)$, and assuming that $\widehat{\vartheta}_{\delta^*}$ and $\widehat{\vartheta}_\delta$ maximize functions $\delta^*(\vartheta)$ and $\delta(\vartheta)$, respectively, where $\widehat{\vartheta}_{\delta^*} = \left(\widehat{\lambda}_{1\delta^*}, \widehat{\lambda}_{2\delta^*}, \widehat{\beta}_{1\delta^*}, \widehat{\beta}_{2\delta^*}, \hat{p}_{\delta^*}\right)$, and $\widehat{\vartheta}_\delta = (\widehat{\lambda}_{1\delta}, \widehat{\lambda}_{2\delta}, \widehat{\beta}_{1\delta}, \widehat{\beta}_{2\delta}, \hat{p}_\delta)$.

Following Tierney and Kadane [33], Equation (20) can be approximated in the following form

$$\widehat{\phi}(\vartheta) = \sqrt{\frac{|\Sigma^*|}{|\Sigma|}}exp\left[n\left\{\delta^*\left(\widehat{\vartheta}_{\delta^*}\right) - \delta\left(\widehat{\vartheta}_\delta\right)\right\}\right] \tag{21}$$

where Σ^* and Σ are the negatives of the inverse Hessians of $\delta^*(\vartheta)$ and $\delta(\vartheta)$ at $\widehat{\vartheta}_{\delta^*}$ and $\widehat{\vartheta}_\delta$, respectively. We have

$$\delta(\vartheta)=\frac{1}{n}[r_1\ln p_1+(r_1+a_1-1)\ln\lambda_1+(r_1+c_1-1)\ln\beta_1-(\beta_1+1)\sum_{j=1}^{r_1}\ln x_{1j}+\sum_{j=1}^{r1}x_{1j}^{-\beta_1}+\lambda_1\sum_{j=1}^{r_1}\left(1-e^{x_{1j}^{-\beta_1}}\right)+r_2\ln p_2$$
$$+r_2\ln(2)+(r_2+a_2-1)\ln\lambda_2+(r_2\lambda_2+c_2-1)\ln\beta_2-3\sum_{j=1}^{r_2}\ln x_{2j}-(\lambda_2+1)\sum_{j=1}^{r_2}\ln\left(ß_2+\frac{1}{x_{2j}^2}\right)$$
$$+(n-r)\ln[G(t)]-b_1\lambda_1-b_2\lambda_2-d_1\beta_1-d_2\beta_2]$$

Then, $(\widehat{\lambda}_{1\delta},\widehat{\lambda}_{2\delta},\widehat{\beta}_{1\delta},\widehat{\beta}_{2\delta},\hat{p}_\delta)$ are computed by solving the following non-linear equations

$$\frac{\partial\delta}{\partial\lambda_1}=\frac{1}{n}\left[\frac{r_1+a_1-1}{\lambda_1}+\sum_{j=1}^{r_1}\left(1-e^{x_{1j}^{-\beta_1}}\right)+\frac{(n-r)}{G(t)}\left(\frac{\partial G(t)}{\partial\lambda_1}\right)-b_1\right]=0,$$

$$\frac{\partial\delta}{\partial\lambda_2}=\frac{1}{n}\left[\frac{r_2+a_2-1}{\lambda_2}+r_2\ln(\beta_2)-\sum_{j=1}^{r_2}\ln\left(\beta_2+\frac{1}{x_{2j}^2}\right)+\frac{(n-r)}{G(t)}\left(\frac{\partial G(t)}{\partial\lambda_2}\right)-b_2\right]=0,$$

$$\frac{\partial\delta}{\partial\beta_1}=\frac{1}{n}\left[\frac{r_1+c_1-1}{\beta_1}-\sum_{j=1}^{r_1}\ln(x_{1j})-\sum_{j=1}^{r_1}x_{1j}^{-\beta_1}\ln(x_{1j})+\lambda_1\sum_{j=1}^{r_1}e^{x_{1j}^{-\beta_1}}x_{1j}^{-\beta_1}\ln(x_{1j})+\frac{(n-r)}{G(t)}\left(\frac{\partial G(t)}{\partial\beta_1}\right)-d1\right]=0,$$

$$\frac{\partial\delta}{\partial\beta_2}=\frac{1}{n}\left[\frac{r_2\lambda_2+c_2-1}{\beta_2}-(\lambda_2+1)\sum_{j=1}^{r_2}\frac{1}{\beta_2+\frac{1}{x_{2j}^2}}+\frac{(n-r)}{G(t)}\left(\frac{\partial G(t)}{\partial\beta_2}\right)-d2\right]=0,$$

$$\frac{\partial\delta}{\partial p}=\frac{1}{n}\left[\frac{r_1}{p_1}-\frac{r_2}{p_2}+\frac{(n-r)}{G(t)}\left(\frac{\partial G(t)}{\partial p}\right)\right]=0$$

where, the derivatives of G with respect to the five parameters are given previously in Section 2.

We also obtain

$$\delta_{\lambda_1\lambda_1}=\frac{\partial^2\delta}{\partial\lambda_1^2}=\frac{1}{n}\left[-\frac{r_1+a_1-1}{\lambda_1^2}+\frac{(n-r)}{G(t)}\left\{\frac{\partial^2G(t)}{\partial\lambda_1^2}-\left(\frac{\partial G(t)}{\partial\lambda_1}\right)^2\frac{1}{G(t)}\right\}\right]$$

$$\delta_{\lambda_2\lambda_2}=\frac{\partial^2\delta}{\partial\lambda_2^2}=\frac{1}{n}\left[-\frac{r_2+a_2-1}{\lambda_2^2}+\frac{(n-r)}{G(t)}\left\{\frac{\partial^2G(t)}{\partial\lambda_2^2}-\left(\frac{\partial G(t)}{\partial\lambda_2}\right)^2\frac{1}{G(t)}\right\}\right]$$

$$\delta_{\beta_1\beta_1}=\frac{\partial^2\delta}{\partial\beta_1^2}=\frac{1}{n}\left[-\frac{r_1+c_1-1}{\beta_1^2}+\sum_{j=1}^{r_1}x_{1j}^{-\beta_1}[\ln(x_{1j})]^2+\lambda_1\sum_{j=1}^{r_1}\left(-e^{x_{1j}^{-\beta_1}}x_{1j}^{-\beta_1}[\ln(x_{1j})]^2\left(x_{1j}^{-\beta_1}+1\right)\right)\right.$$
$$\left.+\frac{(n-r)}{G(t)}\left\{\frac{\partial^2G(t)}{\partial\beta_1^2}-\left(\frac{\partial G(t)}{\partial\beta_1}\right)^2\frac{1}{G(t)}\right\}\right]$$

$$\delta_{\beta_2\beta_2}=\frac{\partial^2\delta}{\partial\beta_2^2}=\frac{1}{n}\left[-\frac{r_2\lambda_2+c_2-1}{\beta_2^2}+(\lambda_2+1)\sum_{j=1}^{r_2}\frac{1}{\left(\beta_2+\frac{1}{x_{2j}^2}\right)^2}+\frac{(n-r)}{G(t)}\left\{\frac{\partial^2G(t)}{\partial\beta_2^2}-\left(\frac{\partial G(t)}{\partial\beta_2}\right)^2\frac{1}{G(t)}\right\}\right]$$

$$\delta_{pp}=\frac{\partial^2\delta}{\partial p^2}=\frac{1}{n}\left[-\frac{r_1}{p_1^2}-\frac{r_2}{p_2^2}-\frac{(n-r)}{(G(t))^2}\left(\frac{\partial G(t)}{\partial p}\right)^2\right]$$

$$\delta_{\lambda_1\lambda_2}=\delta_{\lambda_2\lambda_1}=\frac{\partial^2\delta}{\partial\lambda_1\partial\lambda_2}=\frac{1}{n}\left[-\frac{(n-r)}{(G(t))^2}\left(\frac{\partial G(t)}{\partial\lambda_1}\right)\left(\frac{\partial G(t)}{\partial\lambda_2}\right)\right]$$

$$\delta_{\lambda_1\beta_1}=\delta_{\beta_1\lambda_1}=\frac{\partial^2\delta}{\partial\lambda_1\partial\beta_1}=\frac{1}{n}\left[\sum_{j=1}^{r_1}e^{x_{1j}^{-\beta_1}}x_{1j}^{-\beta_1}\ln(x_{1j})+\frac{(n-r)}{G(t)}\left\{\frac{\partial^2 G(t)}{\partial\lambda_1\partial\beta_1}-\left(\frac{\partial G(t)}{\partial\lambda_1}\right)\left(\frac{\partial G(t)}{\partial\beta_1}\right)\frac{1}{G(t)}\right\}\right]$$

$$\delta_{\lambda_1\beta_2}=\delta_{\beta_2\lambda_1}=\frac{\partial^2\delta}{\partial\lambda_1\partial\beta_2}=\frac{1}{n}\left[-\frac{(n-r)}{(G(t))^2}\left(\frac{\partial G(t)}{\partial\lambda_1}\right)\left(\frac{\partial G(t)}{\partial\beta_2}\right)\right]$$

$$\delta_{\lambda_1 p}=\delta_{p\lambda_1}=\frac{\partial^2\delta}{\partial\lambda_1\partial p}=\frac{1}{n}\left[\frac{(n-r)}{G(t)}\left\{\frac{\partial^2 G(t)}{\partial\lambda_1\partial p}-\left(\frac{\partial G(t)}{\partial\lambda_1}\right)\left(\frac{\partial G(t)}{\partial p}\right)\frac{1}{G(t)}\right\}\right]$$

$$\delta_{\lambda_2\beta_1}=\delta_{\beta_1\lambda_2}=\frac{\partial^2\delta}{\partial\lambda_2\partial\beta_1}=\frac{1}{n}\left[-\frac{(n-r)}{(G(t))^2}\left(\frac{\partial G(t)}{\partial\lambda_2}\right)\left(\frac{\partial G(t)}{\partial\beta_1}\right)\right]$$

$$\delta_{\lambda_2\beta_2}=\delta_{\beta_2\lambda_2}=\frac{\partial^2\delta}{\partial\lambda_2\partial\beta_2}=\frac{1}{n}\left[\frac{r_2}{\beta_2}-\sum_{j=1}^{r_2}\frac{1}{\beta_2+\frac{1}{x_{2j}^2}}+\frac{(n-r)}{G(t)}\left\{\frac{\partial^2 G(t)}{\partial\lambda_2\partial\beta_2}-\left(\frac{\partial G(t)}{\partial\lambda_2}\right)\left(\frac{\partial G(t)}{\partial\beta_2}\right)\frac{1}{G(t)}\right\}\right]$$

$$\delta_{\lambda_2 p}=\delta_{p\lambda_2}=\frac{\partial^2\delta}{\partial\lambda_2\partial p}=\frac{1}{n}\left[\frac{(n-r)}{G(t)}\left\{\frac{\partial^2 G(t)}{\partial\lambda_2\partial p}-\left(\frac{\partial G(t)}{\partial\lambda_2}\right)\left(\frac{\partial G(t)}{\partial p}\right)\frac{1}{G(t)}\right\}\right]$$

$$\delta_{\beta_1\beta_2}=\delta_{\beta_2\beta_1}=\frac{\partial^2\delta}{\partial\beta_1\partial\beta_2}=\frac{1}{n}\left[-\frac{(n-r)}{(G(t))^2}\left(\frac{\partial G(t)}{\partial\beta_1}\right)\left(\frac{\partial G(t)}{\partial\beta_2}\right)\right]$$

$$\delta_{\beta_1 p}=\delta_{p\beta_1}=\frac{\partial^2\delta}{\partial\beta_1\partial p}=\frac{1}{n}\left[\frac{(n-r)}{G(t)}\left\{\frac{\partial^2 G(t)}{\partial\beta_1\partial p}-\left(\frac{\partial G(t)}{\partial\beta_1}\right)\left(\frac{\partial G(t)}{\partial p}\right)\frac{1}{G(t)}\right\}\right]$$

$$\delta_{\beta_2 p}=\delta_{p\beta_2}=\frac{\partial^2\delta}{\partial\beta_2\partial p}=\frac{1}{n}\left[\frac{(n-r)}{G(t)}\left\{\frac{\partial^2 G(t)}{\partial\beta_2\partial p}-\left(\frac{\partial G(t)}{\partial\beta_2}\right)\left(\frac{\partial G(t)}{\partial p}\right)\frac{1}{G(t)}\right\}\right]$$

and

$$\frac{\partial^2 G(t)}{\partial\lambda_1^2}=-p_1e^{\lambda_1\left(1-e^{t^{-\beta_1}}\right)}\left(1-e^{t^{-\beta_1}}\right)^2$$

$$\frac{\partial^2 G(t)}{\partial\lambda_2^2}=p_2\beta_2^{\lambda_2}\left(\beta_2+\frac{1}{t^2}\right)^{-\lambda_2}\left[-[\ln(\beta_2)]^2+2\ln(\beta_2)\ln\left(\beta_2+\frac{1}{t^2}\right)-\left[\ln\left(\beta_2+\frac{1}{t^2}\right)\right]^2\right]$$

$$\frac{\partial^2 G(t)}{\partial\beta_1^2}=p_1e^{t^{-\beta_1}+\lambda_1\left(1-e^{t^{-\beta_1}}\right)}t^{-\beta_1}[\ln(t)]^2\lambda_1\left[1+t^{-\beta_1}-e^{t^{-\beta_1}}t^{-\beta_1}\lambda_1\right]$$

$$\frac{\partial^2 G(t)}{\partial\beta_2^2}=p_2\beta_2^{\lambda_2}\left(\beta_2+\frac{1}{t^2}\right)^{-\lambda_2}\lambda_2\left[-\left(\beta_2+\frac{1}{t^2}\right)^{-2}(\lambda_2+1)-\beta_2^{-2}(\lambda_2-1)+2\beta_2^{-1}\left(\beta_2+\frac{1}{t^2}\right)^{-1}\lambda_2\right]$$

$$\frac{\partial^2 G(t)}{\partial\lambda_1\partial\beta_1}=-p_1e^{t^{-\beta_1}+\lambda_1\left(1-e^{t^{-\beta_1}}\right)}t^{-\beta_1}\ln(t)\left[1+\left(1-e^{t^{-\beta_1}}\right)\lambda_1\right]$$

$$\frac{\partial^2 G(t)}{\partial\lambda_1\partial p}=-e^{\lambda_1\left(1-e^{t^{-\beta_1}}\right)}\left(1-e^{t^{-\beta_1}}\right)$$

$$\frac{\partial^2 G(t)}{\partial \lambda_2 \partial \beta_2} = p_2 \beta_2^{\lambda_2} \left(\beta_2 + \frac{1}{t^2}\right)^{-\lambda_2} \left(1 + \ln(\beta_2)\lambda_2 - \ln\left(\beta_2 + \frac{1}{t^2}\right)\lambda_2\right) \left[\left(\beta_2 + \frac{1}{t^2}\right)^{-1} - \beta_2^{-1}\right]$$

$$\frac{\partial^2 G(t)}{\partial \lambda_2 \partial p} = \beta_2^{\lambda_2} \left(\beta_2 + \frac{1}{t^2}\right)^{-\lambda_2} \left[\ln(\beta_2) - \ln\left(\beta_2 + \frac{1}{t^2}\right)\right]$$

$$\frac{\partial^2 G(t)}{\partial \beta_1 \partial p} = -e^{t^{-\beta_1} + \lambda_1\left(1 - e^{t^{-\beta_1}}\right)} t^{-\beta_1} \ln(t)\, \lambda_1$$

$$\frac{\partial^2 G(t)}{\partial \beta_2 \partial p} = \beta_2^{\lambda_2} \left(\beta_2 + \frac{1}{t^2}\right)^{-\lambda_2} \lambda_2 \left[\beta_2^{-1} - \left(\beta_2 + \frac{1}{t^2}\right)^{-1}\right]$$

The matrix $\sum$ takes the form:

$$\sum = \left(-\frac{\partial^2 \delta}{\partial \omega_i \partial \omega_l}\right), \qquad i,l = 1,2,\cdots,5 \tag{22}$$

where $\omega_1 = \lambda_1$, $\omega_2 = \lambda_2$, $\omega_3 = \beta_1$, $\omega_4 = \beta_2$ and $\omega_5 = p$.

Now, the Bayes estimators of $\lambda_1, \lambda_2, \beta_1, \beta_2$ *and* p are computed, as follows:

If $\phi(\vartheta) = \lambda_1$, $\delta^*(\vartheta)$ becomes

$$\delta^*_{\lambda_1}(\vartheta) = \delta(\vartheta) + \frac{1}{n}\ln\lambda_1$$

and then $\left(\widehat{\lambda}_{1\delta^*}, \widehat{\lambda}_{2\delta^*}, \widehat{\beta}_{1\delta^*}, \widehat{\beta}_{2\delta^*}, \hat{p}_{\delta^*}\right)$ are obtained by solving the following non-linear equations:

$$\frac{\partial \delta^*_{\lambda_1}}{\partial \lambda_1} = \frac{\partial \delta}{\partial \lambda_1} + \frac{1}{n\lambda_1} = 0, \frac{\partial \delta^*_{\lambda_1}}{\partial \lambda_2} = \frac{\partial \delta}{\partial \lambda_2} = 0, \frac{\partial \delta^*_{\lambda_1}}{\partial \beta_1} = \frac{\partial \delta}{\partial \beta_1} = 0, \frac{\partial \delta^*_{\lambda_1}}{\partial \beta_2} = \frac{\partial \delta}{\partial \beta_2} = 0, \frac{\partial \delta^*_{\lambda_1}}{\partial p} = \frac{\partial \delta}{\partial p} = 0.$$

Using the following expressions

$$\delta^*_{\lambda_1\lambda_1} = \frac{\partial^2 \delta^*_{\lambda_1}}{\partial \lambda_1^2} = \frac{\partial^2 \delta}{\partial \lambda_1^2} - \frac{1}{n\lambda_1^2}, \delta^*_{\lambda_1\lambda_2} = \delta_{\lambda_1\lambda_2}, \delta^*_{\lambda_1\beta_1} = \delta_{\lambda_1\beta_1}, \delta^*_{\lambda_1\beta_2} = \delta_{\lambda_1\beta_2}, \delta^*_{\lambda_1 p} = \delta_{\lambda_1 p}, \delta^*_{\lambda_2\lambda_2} = \delta_{\lambda_2\lambda_2}, \delta^*_{\lambda_2\beta_1} = \delta_{\lambda_2\beta_1},$$
$$\delta^*_{\lambda_2\beta_2} = \delta_{\lambda_2\beta_2}, \delta^*_{\lambda_2 p} = \delta_{\lambda_2 p}, \delta^*_{\beta_1\beta_1} = \delta_{\beta_1\beta_1}, \delta^*_{\beta_1\beta_2} = \delta_{\beta_1\beta_2}, \delta^*_{\beta_1 p} = \delta_{\beta_1 p}, \delta^*_{\beta_2\beta_2} = \delta_{\beta_2\beta_2}, \delta^*_{\beta_2 p} = \delta_{\beta_2 p}, \delta^*_{pp} = \delta_{pp}.$$

Hence,

$$\sum\nolimits^*_{\lambda_1} = \left(-\frac{\partial^2 \delta^*_{\lambda_1}}{\partial \omega_i \partial \omega_l}\right), \qquad i,l = 1,2,\cdots,5. \tag{23}$$

where $\omega_1 = \lambda_1$, $\omega_2 = \lambda_2$, $\omega_3 = \beta_1$, $\omega_4 = \beta_2$ and $\omega_5 = p$.

Thus, the approximate Bayes estimator of λ_1 under square error loss function is given by

$$\widehat{\lambda}_{1SE\_TK} = \sqrt{\frac{|\Sigma^*_{\lambda_1}|}{|\Sigma|}} exp\left[n\left\{\delta^*_{\lambda_1}\left(\widehat{\vartheta}_{\delta^*}\right) - \delta\left(\widehat{\vartheta}_{\delta}\right)\right\}\right]$$

Similarly, the Bayes estimator of $\lambda_2, \beta_1, \beta_2$ *and* p, respectively

$$\widehat{\lambda}_{2SE\_TK} = \sqrt{\frac{|\Sigma^*_{\lambda_2}|}{|\Sigma|}} exp\left[n\left\{\delta^*_{\lambda_2}\left(\widehat{\vartheta}_{\delta^*}\right) - \delta\left(\widehat{\vartheta}_{\delta}\right)\right\}\right]$$

$$\widehat{\beta}_{1SE\_TK} = \sqrt{\frac{|\Sigma^*_{\beta_1}|}{|\Sigma|}} exp\left[n\left\{\delta^*_{\beta_1}\left(\widehat{\vartheta}_{\delta^*}\right) - \delta\left(\widehat{\vartheta}_{\delta}\right)\right\}\right]$$

$$\widehat{\beta}_{2SE\_TK} = \sqrt{\frac{|\Sigma^*_{\beta_2}|}{|\Sigma|}} exp\left[n\left\{\delta^*_{\beta_2}\left(\widehat{\vartheta}_{\delta^*}\right) - \delta\left(\widehat{\vartheta}_{\delta}\right)\right\}\right]$$

and

$$\hat{p}_{SE\_TK} = \sqrt{\frac{|\Sigma_p^*|}{|\Sigma|}} exp\left[n\left\{\delta_p^*\left(\widehat{\vartheta}_{\delta^*}\right) - \delta\left(\widehat{\vartheta}_{\delta}\right)\right\}\right]$$

If $\phi(\vartheta) = e^{-q\lambda_1}$, $\delta^*(\vartheta)$ becomes

$$\delta_{\lambda_1}^*(\vartheta) = \delta(\vartheta) - \frac{1}{n} q\lambda_1$$

and then $\left(\widehat{\lambda}_{1\delta^*}, \widehat{\lambda}_{2\delta^*}, \widehat{\beta}_{1\delta^*}, \widehat{\beta}_{2\delta^*}, \hat{p}_{\delta^*}\right)$ are obtained by solving the following non-linear equations:

$$\frac{\partial\delta_{\lambda_1}^*}{\partial\lambda_1} = \frac{\partial\delta}{\partial\lambda_1} - \frac{1}{n}q = 0, \frac{\partial\delta_{\lambda_1}^*}{\partial\lambda_2} = \frac{\partial\delta}{\partial\lambda_2} = 0, \frac{\partial\delta_{\lambda_1}^*}{\partial\beta_1} = \frac{\partial\delta}{\partial\beta_1} = 0, \frac{\partial\delta_{\lambda_1}^*}{\partial\beta_2} = \frac{\partial\delta}{\partial\beta_2} = 0, \frac{\partial\delta_{\lambda_1}^*}{\partial p} = \frac{\partial\delta}{\partial p} = 0.$$

Again, one can obtain $\sum_{\lambda_1}^*$ from (23), where $\delta_{\lambda_1\lambda_1}^* = \frac{\partial^2\delta_{\lambda_1}^*}{\partial\lambda_1^2} = \frac{\partial^2\delta}{\partial\lambda_1^2}$, $\delta_{\lambda_1\lambda_2}^* = \delta_{\lambda_1\lambda_2}$, $\delta_{\lambda_1\beta_1}^* = \delta_{\lambda_1\beta_1}$, $\delta_{\lambda_1\beta_2}^* = \delta_{\lambda_1\beta_2}$, $\delta_{\lambda_1 p}^* = \delta_{\lambda_1 p}$, $\delta_{\lambda_2\lambda_2}^* = \delta_{\lambda_2\lambda_2}$, $\delta_{\lambda_2\beta_1}^* = \delta_{\lambda_2\beta_1}$, $\delta_{\lambda_2\beta_2}^* = \delta_{\lambda_2\beta_2}$, $\delta_{\lambda_2 p}^* = \delta_{\lambda_2 p}$, $\delta_{\beta_1\beta_1}^* = \delta_{\beta_1\beta_1}$, $\delta_{\beta_1\beta_2}^* = \delta_{\beta_1\beta_2}$, $\delta_{\beta_1 p}^* = \delta_{\beta_1 p}$, $\delta_{\beta_2\beta_2}^* = \delta_{\beta_2\beta_2}$, $\delta_{\beta_2 p}^* = \delta_{\beta_2 p}$, $\delta_{pp}^* = \delta_{pp}$.

The approximate Bayes estimator of λ_1 under LINEX loss function is given by

$$\widehat{\lambda}_{1LINEX\_TK} = -\frac{1}{q}\ln\left[\sqrt{\frac{|\Sigma_{\lambda_1}^*|}{|\Sigma|}} exp\left[n\left\{\delta_{\lambda_1}^*\left(\widehat{\vartheta}_{\delta^*}\right) - \delta\left(\widehat{\vartheta}_{\delta}\right)\right\}\right]\right]$$

Also, similarly, the Bayes estimator of $\lambda_2, \beta_1, \beta_2$ *and* p under LINEX loss function, respectively, are

$$\widehat{\lambda}_{2LINEX\_TK} = -\frac{1}{q}\ln\left[\sqrt{\frac{|\Sigma_{\lambda_2}^*|}{|\Sigma|}} exp\left[n\left\{\delta_{\lambda_2}^*\left(\widehat{\vartheta}_{\delta^*}\right) - \delta\left(\widehat{\vartheta}_{\delta}\right)\right\}\right]\right]$$

$$\widehat{\beta}_{1LINEX\_TK} = -\frac{1}{q}\ln\left[\sqrt{\frac{|\Sigma_{\beta_1}^*|}{|\Sigma|}} exp\left[n\left\{\delta_{\beta_1}^*\left(\widehat{\vartheta}_{\delta^*}\right) - \delta\left(\widehat{\vartheta}_{\delta}\right)\right\}\right]\right]$$

$$\widehat{\beta}_{2LINEX\_TK} = -\frac{1}{q}\ln\left[\sqrt{\frac{|\Sigma_{\beta_2}^*|}{|\Sigma|}} exp\left[n\left\{\delta_{\beta_2}^*\left(\widehat{\vartheta}_{\delta^*}\right) - \delta\left(\widehat{\vartheta}_{\delta}\right)\right\}\right]\right]$$

and

$$\hat{p}_{LINEX\_TK} = -\frac{1}{q}\ln\left[\sqrt{\frac{|\Sigma_p^*|}{|\Sigma|}} exp\left[n\left\{\delta_p^*\left(\widehat{\vartheta}_{\delta^*}\right) - \delta\left(\widehat{\vartheta}_{\delta}\right)\right\}\right]\right]$$

If $\phi(\vartheta) = \lambda_1^{-h}$, $\delta^*(\vartheta)$ becomes

$$\delta_{\lambda_1}^*(\vartheta) = \delta(\vartheta) - \frac{1}{n} h \ln\lambda_1$$

and then $\left(\widehat{\lambda}_{1\delta^*}, \widehat{\lambda}_{2\delta^*}, \widehat{\beta}_{1\delta^*}, \widehat{\beta}_{2\delta^*}, \hat{p}_{\delta^*}\right)$ are obtained by solving the following non-linear equations:

$$\frac{\partial\delta_{\lambda_1}^*}{\partial\lambda_1} = \frac{\partial\delta}{\partial\lambda_1} - \frac{h}{n\lambda_1} = 0, \frac{\partial\delta_{\lambda_1}^*}{\partial\lambda_2} = \frac{\partial\delta}{\partial\lambda_2} = 0, \frac{\partial\delta_{\lambda_1}^*}{\partial\beta_1} = \frac{\partial\delta}{\partial\beta_1} = 0, \frac{\partial\delta_{\lambda_1}^*}{\partial\beta_2} = \frac{\partial\delta}{\partial\beta_2} = 0, \frac{\partial\delta_{\lambda_1}^*}{\partial p} = \frac{\partial\delta}{\partial p} = 0.$$

And obtain $\sum_{\lambda_1}^*$ from (23), where $\delta_{\lambda_1\lambda_1}^* = \frac{\partial^2\delta_{\lambda_1}^*}{\partial\lambda_1^2} = \frac{\partial^2\delta}{\partial\lambda_1^2} + \frac{h}{n\lambda_1^2}$, $\delta_{\lambda_1\lambda_2}^* = \delta_{\lambda_1\lambda_2}$, $\delta_{\lambda_1\beta_1}^* = \delta_{\lambda_1\beta_1}$, $\delta_{\lambda_1\beta_2}^* = \delta_{\lambda_1\beta_2}$, $\delta_{\lambda_1 p}^* = \delta_{\lambda_1 p}$, $\delta_{\lambda_2\lambda_2}^* = \delta_{\lambda_2\lambda_2}$, $\delta_{\lambda_2\beta_1}^* = \delta_{\lambda_2\beta_1}$, $\delta_{\lambda_2\beta_2}^* = \delta_{\lambda_2\beta_2}$ $\delta_{\lambda_2 p}^* = \delta_{\lambda_2 p}$, $\delta_{\beta_1\beta_1}^* = \delta_{\beta_1\beta_1}$, $\delta_{\beta_1\beta_2}^* = \delta_{\beta_1\beta_2}$, $\delta_{\beta_1 p}^* = \delta_{\beta_1 p}$, $\delta_{\beta_2\beta_2}^* = \delta_{\beta_2\beta_2}$, $\delta_{\beta_2 p}^* = \delta_{\beta_2 p}$, $\delta_{pp}^* = \delta_{pp}$.

The approximate Bayes estimator of λ_1 under general entropy loss function is given by

$$\widehat{\lambda}_{1GE\_TK} = \left[\sqrt{\frac{|\Sigma^{*}_{\lambda_1}|}{|\Sigma|}} exp\left[n\left\{\delta^{*}_{\lambda_1}\left(\widehat{\vartheta}_{\delta^*}\right) - \delta\left(\widehat{\vartheta}_{\delta}\right)\right\}\right]\right]^{-\frac{1}{h}}$$

Also, similarly, the Bayes estimator of $\lambda_2, \beta_1, \beta_2$ *and* p under general entropy loss function, respectively, are

$$\widehat{\lambda}_{2GE\_TK} = \left[\sqrt{\frac{|\Sigma^{*}_{\lambda_2}|}{|\Sigma|}} exp\left[n\left\{\delta^{*}_{\lambda_2}\left(\widehat{\vartheta}_{\delta^*}\right) - \delta\left(\widehat{\vartheta}_{\delta}\right)\right\}\right]\right]^{-\frac{1}{h}}$$

$$\widehat{\beta}_{1GE\_TK} = \left[\sqrt{\frac{|\Sigma^{*}_{\beta_1}|}{|\Sigma|}} exp\left[n\left\{\delta^{*}_{\beta_1}\left(\widehat{\vartheta}_{\delta^*}\right) - \delta\left(\widehat{\vartheta}_{\delta}\right)\right\}\right]\right]^{-\frac{1}{h}}$$

$$\widehat{\beta}_{2GE\_TK} = \left[\sqrt{\frac{|\Sigma^{*}_{\beta_2}|}{|\Sigma|}} exp\left[n\left\{\delta^{*}_{\beta_2}\left(\widehat{\vartheta}_{\delta^*}\right) - \delta\left(\widehat{\vartheta}_{\delta}\right)\right\}\right]\right]^{-\frac{1}{h}}$$

and

$$\hat{p}_{GE\_TK} = \left[\sqrt{\frac{|\Sigma^{*}_{p}|}{|\Sigma|}} exp\left[n\left\{\delta^{*}_{p}\left(\widehat{\vartheta}_{\delta^*}\right) - \delta\left(\widehat{\vartheta}_{\delta}\right)\right\}\right]\right]^{-\frac{1}{h}}$$

Noting that when value $h = -1$, the general entropy loss function is the same as the squared error loss function.

4 Simulation Study

In this section, a Monte Carlo simulation study is conducted to compare the performance of maximum likelihood and Bayes estimators. The performance of the competitive estimates has been compared on the basis of their mean square error (MSE). We estimate the unknown parameters using MLE and Bayes estimation obtained by Tierney and Kadane approximation. For simulation, the samples are generated for a mixture of the ICICR distributions using inverse CDF transformation method for fixed values of $\lambda_1 = 1, \lambda_2 = 2, \beta_1 = 1.5,\ \beta_2 = 1$ *and* $p = 0.45$, and with different choice of (n, R, T).

For computing Bayes estimators, we have used informative gamma prior for the parameters $\lambda_1, \lambda_2, \beta_1$ *and* β_2 in which the hyper parameters are assumed to be $(a_1 = 2, b_1 = 3, a_2 = 3, b_2 = 3, c_1 = 2, d_1 = 2, c_2 = 4, d_2 = 3)$ and non-informative prior for p. All results are based on 1000 replications and the average values and MSE of estimate are given in Tables 1-5.

All results are obtained using Mathematica 10.0.

Table 1: Average estimates and corresponding MSE for parameter λ_1

n	T	R	MLE	Bayes				
				SE	LINEX		GE	
					$q=1.5$	$q=-1.5$	$h=1.5$	$h=-1.5$
30	2	15	1.03969 (0.06210)	0.90071 (0.07053)	0.95257 (0.00841)	1.03281 (0.01984)	0.88275 (0.07495)	0.93974 (0.04835)
		20	1.04140 (0.05661)	0.96086 (0.12990	0.97991 (0.00487)	1.01951 (0.00839)	0.91825 (0.05921)	0.95598 (0.03365)
		25	1.04239 (0.05757)	0.92714 (0.06116)	0.95601 (0.01098)	0.99791 (0.01172)	0.91563 (0.04658)	0.95836 (0.02564)
	3	15	1.03987 (0.06139)	0.90177 (0.08988)	0.98437 (0.01002)	1.00749 (0.01613)	0.89909 (0.05515)	0.94672 (0.04322)
		20	1.04274 (0.05734)	0.95262 (0.07973)	0.94780 (0.00889)	0.99050 (0.01779)	0.91234 (0.03526)	0.94809 (0.02281)
		25	1.06048 (0.05858)	0.91229 (0.06224)	0.96360 (0.00544)	0.97712 (0.00597)	0.89714 (0.02146)	0.96022 (0.01815)
	4	15	1.05763 (0.06866)	0.93029 (0.07247)	0.95801 (0.01128)	0.98064 (0.01434)	0.90799 (0.05947)	0.94241 (0.04395)
		20	1.04749 (0.05436)	0.94130 (0.02774)	0.96335 (0.00503)	1.00455 (0.01307)	0.89761 (0.04095)	0.95374 (0.01900)
		25	1.03578 (0.05137)	0.93392 (0.01178)	0.96897 (0.00736)	1.00436 (0.00559)	0.90562 (0.04010)	0.95410 (0.00557)
50	2	35	1.04584 (0.04577)	0.94121 (0.06147)	0.98229 (0.00540)	0.99743 (0.00731)	0.93045 (0.02814)	0.95784 (0.01742)
		40	1.02468 (0.04306)	0.95665 (0.04158)	0.96122 (0.00577)	0.99646 (0.00348)	0.90696 (0.02269)	0.94860 (0.01700)
		45	1.03805 (0.04332)	0.94171 (0.02652)	0.96631 (0.00687)	0.99366 (0.00691)	0.91177 (0.02922)	0.97588 (0.01893)
	3	35	1.01953 (0.03991)	0.94007 (0.05570)	0.96389 (0.00370)	0.97985 (0.00465)	0.92372 (0.01673)	0.93705 (0.01279)
		40	1.01945 (0.04272)	0.93775 (0.04938)	0.98915 (0.00528)	0.99132 (0.00514)	0.92959 (0.01948)	0.96193 (0.00686)
		45	1.03654 (0.03788)	0.96062 (0.03411)	0.96695 (0.00428)	0.99791 (0.00206)	0.92892 (0.01139)	0.96687 (0.00880)
	4	35	1.01616 (0.04440)	0.92048 (0.03598)	0.97431 (0.00331)	0.97378 (0.00506)	0.93666 (0.02067)	0.95317 (0.01527)
		40	1.02337 (0.04174)	0.94633 (0.00982)	0.94957 (0.00343)	0.97255 (0.00446)	0.93404 (0.00724)	0.96276 (0.00383)
		45	1.02326 (0.04259)	0.96127 (0.00818)	0.95023 (0.00364)	0.98715 (0.00244)	0.97259 (0.00744)	0.97023 (0.00331)

Table 2: Average estimates and corresponding MSE for parameter λ_2

n	T	R	MLE	Bayes				
				SE	LINEX		GE	
					$q=1.5$	$q=-1.5$	$h=1.5$	$h=-1.5$
30	2	15	1.81271 (0.34522)	1.74593 (0.16537)	2.07954 (0.03045)	1.94914 (0.03312)	2.06609 (0.05911)	1.70847 (0.17597)
		20	1.82887 (0.32856)	1.72201 (0.13815)	2.05948 (0.01189)	1.94928 (0.01710)	2.06743 (0.06638)	1.75532 (0.17146)
		25	1.81793 (0.29001)	1.72499 (0.12576)	2.02780 (0.02295)	1.95840 (0.02096)	2.07544 (0.09033)	1.77192 (0.10441)
	3	15	1.81068 (0.44756)	1.77660 (0.12358)	2.10507 (0.02448)	1.90201 (0.02666)	2.06190 (0.10712)	1.7520 (0.13309
		20	1.87805 (0.35188)	1.71455 (0.13441)	2.05876 (0.01288)	1.91318 (0.02041)	2.02841 (0.10758)	1.75115 (0.18424)
		25	1.97798 (0.42061)	1.72380 (0.10861)	2.00593 (0.01652)	1.89351 (0.02000)	2.06312 (0.07892)	1.72778 (0.09323)
	4	15	1.83570 (0.28754)	1.80590 (0.11291)	2.08468 (0.02444)	1.90701 (0.02196)	2.02616 (0.07255)	1.76225 (0.07962)
		20	1.91209 (0.33526)	1.74403 (0.08933)	2.01659 (0.00796)	1.90798 (0.01813)	1.99352 (0.05193)	1.69303 (0.10660)
		25	1.89766 (0.46739)	1.77873 (0.08333)	2.00181 (0.01649)	1.85627 (0.03147)	1.99241 (0.04585)	1.69268 (0.10980)
50	2	35	1.83114 (0.23835)	1.90887 (0.10476)	2.09310 (0.01687)	1.98624 (0.01145)	2.05070 (0.03483)	1.86360 (0.10101)
		40	1.81750 (0.24405)	1.85463 (0.05828)	2.08006 (0.01435)	1.96614 (0.01396)	2.06683 (0.02977)	1.81693 (0.05234)
		45	1.8620 (0.23410)	1.84953 (0.05281)	2.08761 (0.01932)	1.99221 (0.01421)	1.98290 (0.05532)	1.83081 (0.06563)
	3	35	1.88224 (0.24471)	1.71229 (0.12393)	2.07675 (0.01625)	1.89464 (0.01767)	1.98371 (0.04075)	1.78738 (0.05610)
		40	1.85446 (0.29571)	1.68383 (0.12954)	2.02929 (0.00802)	1.93053 (0.01211)	2.02860 (0.04713)	1.74948 (0.07063)
		45	1.89923 (0.31082)	1.68804 (0.11911)	1.97099 (0.01244)	1.98284 (0.01075)	1.95668 (0.03157)	1.72744 (0.09366)
	4	35	1.88155 (0.24528)	1.77152 (0.08436)	2.07918 (0.01476)	1.97503 (0.01097)	1.96412 (0.03564)	1.73214 (0.08983)
		40	1.91041 (0.28410)	1.76949 (0.07419)	1.97784 (0.01148)	1.99758 (0.00844)	2.01236 (0.02966)	1.76844 (0.07354)
		45	1.92711 (0.52086)	1.78491 (0.07138)	2.03823 (0.00543)	1.92338 (0.01303)	2.05341 (0.03356)	1.71782 (0.10081)

Table 3: Average estimates and corresponding MSE for parameter β_1

n	T	R	MLE	Bayes				
				SE	LINEX		GE	
					$q=1.5$	$q=-1.5$	$h=1.5$	$h=-1.5$
30	2	15	1.63993 (0.19913)	1.51231 (0.15373)	1.54805 (0.02685)	1.63205 (0.04443)	1.45266 (0.06100)	1.55340 (0.10939)
		20	1.60356 (0.17453)	1.56303 (0.12914)	1.57802 (0.03265)	1.62777 (0.03901)	1.49026 (0.04675)	1.52405 (0.08112)
		25	1.56541 (0.14804)	1.50262 (0.08251)	1.55641 (0.02549)	1.62126 (0.04479)	1.48654 (0.07132)	1.63760 (0.07593)
	3	15	1.61223 (0.20575)	1.53370 (0.09571)	1.55062 (0.02152)	1.64451 (0.04088)	1.44003 (0.05865)	1.55850 (0.08123)
		20	1.61605 (0.16673)	1.52662 (0.09123)	1.56365 (0.02970)	1.66920 (0.05937)	1.45918 (0.03287)	1.58601 (0.06605)
		25	1.56571 (0.15283)	1.56581 (0.08963)	1.57064 (0.02653)	1.64782 (0.03945)	1.46622 (0.04104)	1.54690 (0.07101)
	4	15	1.61294 (0.20230)	1.4980 (0.12381)	1.54833 (0.02497)	1.65509 (0.05524)	1.47162 (0.08411)	1.56628 (0.06054)
		20	1.58460 (0.14753)	1.57157 (0.09760)	1.55818 (0.01882)	1.66018 (0.04319)	1.49459 (0.04677)	1.55614 (0.04282)
		25	1.59229 (0.12721)	1.53084 (0.04022)	1.57247 (0.01142)	1.67020 (0.04486)	1.47297 (0.03102)	1.56973 (0.03878)
50	2	35	1.56026 (0.12006)	1.56526 (0.06795)	1.56430 (0.02856)	1.67467 (0.04524)	1.56240 (0.03577)	1.60602 (0.07549)
		40	1.55224 (0.10786)	1.56670 (0.07626)	1.56576 (0.03161)	1.64155 (0.03936)	1.54944 (0.01551)	1.57823 (0.06874)
		45	1.55565 (0.11866)	1.52581 (0.02365)	1.55454 (0.03501)	1.65905 (0.04301)	1.53241 (0.02665)	1.58669 (0.03268)
	3	35	1.57465 (0.11558)	1.54558 (0.05544)	1.52708 (0.02469)	1.65487 (0.03426)	1.54384 (0.03671)	1.56441 (0.03434)
		40	1.56948 (0.10375)	1.60514 (0.07413)	1.54565 (0.01278)	1.65358 (0.03483)	1.51359 (0.01164)	1.58415 (0.02086)
		45	1.53018 (0.09545)	1.60177 (0.04129)	1.54202 (0.01341)	1.64968 (0.02989)	1.53144 (0.01418)	1.58187 (0.01931)
	4	35	1.57673 (0.10994)	1.54986 (0.03184)	1.57694 (0.01721)	1.63720 (0.03636)	1.50820 (0.02125)	1.52878 (0.02619)
		40	1.56886 (0.09802)	1.52066 (0.01134)	1.56463 (0.01374)	1.67493 (0.03598)	1.51304 (0.00731)	1.59957 (0.01642)
		45	1.55262 (0.08413)	1.50878 (0.01369)	1.54758 (0.01077)	1.65608 (0.02906)	1.51659 (0.00504)	1.57255 (0.01260)

Table 4: Average estimates and corresponding MSE for parameter β_2

n	T	R	MLE	Bayes				
				SE	LINEX		GE	
					$q = 1.5$	$q = -1.5$	$h = 1.5$	$h = -1.5$
30	2	15	0.88133 (0.15191)	0.82947 (0.08197)	1.10451 (0.01688)	1.02588 (0.03404)	1.03941 (0.07849)	0.90603 (0.04143)
		20	0.89877 (0.16916)	0.82798 (0.06517)	1.07927 (0.01544)	1.01151 (0.02663)	0.96798 (0.06429)	0.93571 (0.03873)
		25	0.88468 (0.14614)	0.85540 (0.06185)	1.12133 (0.02053)	1.02084 (0.03155)	0.99733 (0.08862)	0.89347 (0.03399)
	3	15	0.87789 (0.18917)	0.88642 (0.07186)	1.09953 (0.01899)	0.99980 (0.02232)	1.03286 (0.07241)	0.93535 (0.02346)
		20	0.93611 (0.18945)	0.91436 (0.06690)	1.08175 (0.00974)	1.00531 (0.01964)	1.03673 (0.06911)	0.86529 (0.05717)
		25	1.00858 (0.23910)	0.83993 (0.07255)	1.04076 (0.00767)	0.97546 (0.01405)	0.99949 (0.07096)	0.86673 (0.04264)
	4	15	0.90291 (0.14099)	0.90001 (0.05045)	1.11688 (0.02380)	1.01823 (0.02559)	1.00891 (0.06950)	0.89744 (0.03646)
		20	0.96481 (0.18968)	0.85891 (0.06769)	1.05302 (0.01030)	0.96256 (0.01324)	1.00835 (0.04010)	0.88267 (0.03849)
		25	0.96024 (0.23676)	0.78448 (0.07164)	1.07434 (0.00987)	0.98196 (0.01093)	0.98355 (0.02403)	0.81687 (0.05698)
50	2	35	0.89492 (0.11590)	0.87784 (0.06129)	1.07604 (0.01020)	0.97328 (0.01880)	0.96741 (0.06312)	0.85734 (0.04618)
		40	0.88873 (0.12190)	0.84832 (0.05102)	1.03628 (0.01257)	1.03082 (0.01721)	0.96444 (0.04343)	0.86342 (0.03573)
		45	0.92437 (0.12571)	0.88463 (0.04852)	1.06096 (0.01332)	0.99363 (0.02143)	0.84363 (0.04674)	0.87214 (0.03418)
	3	35	0.94268 (0.13535)	0.87839 (0.08693)	1.02267 (0.00534)	0.97944 (0.01315)	0.97976 (0.06169)	0.79930 (0.06498)
		40	0.91779 (0.15231)	0.76177 (0.09486)	1.03313 (0.00332)	0.97615 (0.01123)	0.94449 (0.05135)	0.81946 (0.04458)
		45	0.95574 (0.17087)	0.80396 (0.05720)	1.06677 (0.00691)	0.97341 (0.00821)	0.90926 (0.03664)	0.81886 (0.06820)
	4	35	0.92702 (0.13080)	0.82833 (0.07938)	1.06107 (0.00890)	0.97137 (0.01252)	0.97699 (0.05583)	0.84230 (0.04478)
		40	0.95050 (0.14916)	0.80393 (0.06731)	1.00097 (0.00565)	0.96074 (0.01047)	0.86970 (0.04276)	0.83384 (0.03543)
		45	0.98160 (0.31234)	0.83304 (0.04264)	0.98386 (0.00467)	0.98699 (0.08870)	0.98953 (0.03455)	0.84509 (0.03497)

Table 5: Average estimates and corresponding MSE for parameter p

n	T	R	MLE	Bayes				
				SE	LINEX		GE	
					$q=1.5$	$q=-1.5$	$h=1.5$	$h=-1.5$
30	2	15	0.45415 (0.01519)	0.42440 (0.02620)	0.46813 (0.00414)	0.46447 (0.00731)	0.41372 (0.01708)	0.43319 (0.00852)
		20	0.45764 (0.01251)	0.45912 (0.01495)	0.47963 (0.00349)	0.48421 (0.00835)	0.43260 (0.00789)	0.45141 (0.00768)
		25	0.45982 (0.01201)	0.46215 (0.01396)	0.46277 (0.00389)	0.47329 (0.00410)	0.44002 (0.00884)	0.45113 (0.01216)
	3	15	0.45567 (0.01552)	0.43504 (0.01384)	0.46513 (0.00284)	0.46651 (0.00932)	0.42526 (0.01066)	0.44555 (0.01328)
		20	0.45630 (0.01135)	0.45071 (0.01212)	0.46793 (0.00139)	0.46481 (0.00610)	0.42818 (0.00607)	0.45430 (0.00150)
		25	0.45838 (0.00939)	0.45773 (0.01268)	0.47722 (0.00193)	0.46710 (0.00525)	0.43312 (0.00273)	0.46481 (0.00197)
	4	15	0.46184 (0.01469)	0.44839 (0.01454)	0.47034 (0.00508)	0.45005 (0.00878)	0.42613 (0.01174)	0.44306 (0.01557)
		20	0.46313 (0.01171)	0.44586 (0.00790)	0.47233 (0.00360)	0.45740 (0.00781)	0.42932 (0.00360)	0.44492 (0.03346)
		25	0.45095 (0.00801)	0.44991 (0.00106)	0.46460 (0.00209)	0.46329 (0.00250)	0.43441 (0.00156)	0.46176 (0.00139)
50	2	35	0.46028 (0.00908)	0.43523 (0.00934)	0.48707 (0.00338)	0.46200 (0.00388)	0.44613 (0.00697)	0.44556 (0.00474)
		40	0.45400 (0.00826)	0.46710 (0.00857)	0.45615 (0.00131)	0.46658 (0.00227)	0.44922 (0.00549)	0.45583 (0.00652)
		45	0.46310 (0.00910)	0.43950 (0.00835)	0.44979 (0.00247)	0.45111 (0.00387)	0.44301 (0.00335)	0.46853 (0.00555)
	3	35	0.45235 (0.00896)	0.44327 (0.00442)	0.47210 (0.00244)	0.46030 (0.00529)	0.43132 (0.00249)	0.45475 (0.00220)
		40	0.45121 (0.00709)	0.43277 (0.00383)	0.45993 (0.00117)	0.45829 (0.00171)	0.44854 (0.00122)	0.45963 (0.00100)
		45	0.45710 (0.00621)	0.43071 (0.00594)	0.48997 (0.00274)	0.44817 (0.00279)	0.45482 (0.00135)	0.45922 (0.00132)
	4	35	0.45407 (0.00799)	0.43735 (0.00365)	0.47638 (0.00214)	0.44849 (0.00253)	0.44368 (0.00241)	0.44831 (0.00526)
		40	0.45145 (0.00623)	0.44340 (0.00231)	0.46293 (0.00130)	0.45339 (0.00190)	0.44697 (0.00089)	0.45389 (0.00075)
		45	0.45436 (0.00545)	0.47328 (0.00082)	0.48187 (0.00141)	0.44814 (0.00162)	0.45398 (0.00163)	0.47011 (0.00113)

5 Conclusion

In this paper, we have considered estimation of unknown parameters for the mixture of the ICICRD under Type-I censored data. For this purpose, the maximum likelihood and Bayesian estimates using Tierney and Kadane approximation method are obtained. Tables 1-5 indicate that for fixed value of T, the MSE of the estimators decreases as n and R increase. Also, in most cases, MSE of the Bayesian estimates are smaller than the maximum likelihood estimates. This indicates that the Bayesian procedure gives accurate estimates of the parameters as expected.

Conflict of Interest

The authors declare that they have no conflict of interest.

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