Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/100210

Estimation of Stress-Strength Reliability for Marshall-Olkin Extended Weibull Family Based on Type-II Progressive Censoring

N. A. Mokhlis[1], *S. K. khames*[1,*] *and Sh. W. Sadk*[2]

[1]Department of Mathematics, Faculty of Science, Ain Shams University, Egypt
[2]Department of Mathematics, Faculty of Science, Zigzag University, Egypt

Received: 3 Jan. 2020, Revised: 4 Apr. 2020, Accepted: 6 Apr. 2020
Published online: 1 Jul. 2021

Abstract: In this article, an explicit form of the stress-strength reliability $\mathscr{R} = P(X < Y)$ is introduced when X and Y are independent random variables belonging to Marshall-Olkin extended Weibull family. Also a characterization of the parent distributions associated with $\mathscr{R}$ is presented. Based on Type-II progressive censoring with fixed and random number of removals, maximum likelihood and Bayesian estimators of the parameter $\mathscr{R}$ are obtained. Two distributions for the random number of removals are considered, namely discrete uniform and binomial distributions. Using informative and non-informative priors, the Bayesian estimation is discussed under two different loss functions: the squared error loss function (SELF) and linear exponential loss functions (LINEX). A numerical illustration is performed to highlight the theoretical results that are obtained. Also a real data example is provided.

Keywords: Marshall-Olkin extended distribution, Stress-strength reliability, Type-II progressive censoring, Random removals, Bayesian estimation, Lindley approximation.

1 Introduction

Marshall and Olkin [1] introduced a general method for adding a new parameter α to any family of distributions. Starting with parent survival function (SF), $\overline{F}(x)$, they constructed a new survival function, $\overline{G}(x,\alpha)$, as

$$\overline{G}(x,\alpha) = \frac{\alpha\,\overline{F}(x)}{1-\overline{\alpha}\overline{F}(x)}, \qquad \alpha > 0, \tag{1}$$

where $\overline{\alpha} = 1-\alpha$. In the last decade, several authors used Marshall and Olkin extension for adding a new parameter to the classical distributions to obtain more flexible distributions. Jose [2] displayed the related work of Marshall-Olkin family and its applications in different fields as reliability theory, time series, and stress-strength analysis. Also, Ahmad et al. [3] introduced a brief survey on Marshall-Olkin extended distribution and some other families.
Gurvich et al. [4] introduced a class of extended Weibull distribution with SF

$$\overline{F}(x,\theta,\eta) = e^{-\theta\ \psi(x,\eta)}, \ \ x \in D \subseteq \mathfrak{R}_{+}, \theta > 0, \tag{2}$$

where η could be a vector parameter and $\psi(x,\eta)$ is non-negative, monotonically increasing function of x, not depending on θ and differentiable with respect to x and η. It is noticeable that $\psi(x,\eta) \to 0$ as $x \to a^{+}$ and that $\psi(x,\eta) \to \infty$ as $x \to b^{-}$ when (a,b) is the support of x.
Santos et al. [5] considered the class of extended Weibull given in (2) as a parent distribution of Marshall-Olkin extended distribution and they called it Marshall-Olkin extended Weibull (MOEW) family of distributions.

* Corresponding author e-mail: sohair_smz@women.asu.edu.eg

Let X be a random variable with a probability distribution belonging to MOEW family with parameters $\alpha,\ \theta$ and η, i.e. X having MOEW (α,θ,η), then the SF, $\overline{G}_x(x,\alpha,\theta,\eta)$, and the probability density function (PDF), $g_x(x,\alpha,\theta,\eta)$, of X are given respectively by

$$\overline{G}_x(x,\alpha,\theta,\eta) = \frac{\alpha\ e^{-\theta\psi(x,\eta)}}{1-\overline{\alpha}e^{-\theta\psi(x,\eta)}}, \qquad \alpha>0, \tag{3}$$

and

$$g_x(x,\alpha,\theta,\eta) = \frac{\alpha\ \theta\ \psi^{(1)}(x,\eta)\, e^{-\theta\psi(x,\eta)}}{\left[1-\overline{\alpha}e^{-\theta\psi(x,\eta)}\right]^2}, \qquad \alpha>0, \tag{4}$$

where $\psi^{(1)}(x,\eta)$ is the first partial derivative of $\psi(x,\eta)$ with respect to x.
Notice that for $\alpha=1$, the SF of X is reduced to the parent SF. Clearly, the SF and the PDF in (3) and (4) can be presented as linear combinations of the parent distribution in (2) according to the value of α. First: If $\alpha<1$, then we have $\left|\overline{\alpha}e^{-\theta\psi(x,\eta)}\right|<1$, and using the binomial expansion for the denominator, we get

$$\overline{G}_x(x,\alpha,\theta,\eta) = \alpha\sum_{k=0}^{\infty}(\overline{\alpha})^k e^{-\theta(k+1)\psi(x,\eta)}, \tag{5}$$

and

$$g_x(x,\alpha,\theta,\eta) = \alpha\ \theta\psi^{(1)}(x,\eta)\sum_{k=0}^{\infty}(k+1)\overline{\alpha}^k e^{-\theta(k+1)\psi(x,\eta)}. \tag{6}$$

Second: If $\alpha>1$, we can rewrite the denominator of (3) as $\alpha\left[1-\left(1-\frac{1}{\alpha}\right)\left(1-e^{-\theta\psi(x,\eta)}\right)\right]$. In this case, we have $\left|\left(1-\frac{1}{\alpha}\right)\left(1-e^{-\theta\psi(x,\eta)}\right)\right|<1$. Applying the binomial expansion, we get

$$\overline{G}_x(x,\alpha,\theta,\eta) = e^{-\theta\psi(x,\eta)}\sum_{j=0}^{\infty}\left(1-\frac{1}{\alpha}\right)^j\left(1-e^{-\theta\psi(x,\eta)}\right)^j, \tag{7}$$

Similarly, we get the PDF as

$$g_x(x,\alpha,\theta,\eta) = \frac{\theta\psi^{(1)}(x,\eta)\,e^{-\theta\psi(x,\eta)}}{\alpha}\sum_{j=0}^{\infty}(j+1)\left(1-\frac{1}{\alpha}\right)^j\left(1-e^{-\theta\psi(x,\eta)}\right)^j. \tag{8}$$

Many distributions in the literature could be considered as special cases of MOEW$(\alpha,\ \theta,\eta)$ by choosing the appropriate form of $\psi\ (x,\eta)$. For example, Marshall-Olkin extended exponential ($\psi\ (x,\eta)=x$), Marshall-Olkin extended Rayleigh ($\psi\ (x,\eta)=\frac{x^2}{2}$), Marshall-Olkin extended Pareto ($\psi\ (x,\eta)=\ln\frac{x}{\eta}$), and many other distributions. For more details, see [2].
Stress-strength models have attracted the attention of a large number of researchers due to its importance in many aspects of life. If Y is the strength of a system and X is the stress imposed on this system, then $\mathscr{R}=P(X<Y)$ is the system stress-strength reliability. However, X and Y may represent other variables of interest, such as the life time of two equipments or the effect of two medical treatments. A valuable review of stress-strength models and their applications is presented in [6]. In subsequent years, many researchers studied $\mathscr{R}$ when X and Y have specified distributions, for example [7,8,9,10,11] and others. Mokhlis et al. [12,13] discussed the reliability $\mathscr{R}$ when X and Y are independent random variables having SF given in (2). Gupta et al. [9] proposed a general form for $\mathscr{R}$ when X and Y are independent random variables having Marshall-Olkin extended given in (1) with equal parent SFs, and different α's.
In the present paper, we consider the estimation of the stress-strength reliability parameter $\mathscr{R}=P(X<Y)$ when the stress X and the strength Y are independent random variables having MOEW$(\alpha,\ \theta_1,\eta)$ and MOEW(α,θ_2,η), respectively. In many situations, it is difficult to observe the failures of all units under the study. This could be due to the lack of time, money or other considerations. Thus, censored schemes are often used. A Type-II progressive censored scheme is one of the most popular censoring schemes. Thus, in this article, we consider the estimation of $\mathscr{R}$ based on Type-II progressive censored samples. Type-II progressive censored data is defined, as follows: Suppose that N units are under experimentation and only $n<N$ prefixed number of units is observed until failure. When the first failure $x_{1:n:N}$ occurs, R_1units are randomly eliminated from the $(N-1)$ remaining units. Similarly, when the second failure $x_{2:n:N}$ occurs, R_2 units are eliminated from the remaining $(N-\ R_1-2)$ units. When the n^{th}failure $x_{n:n:N}$ occurs, the experiment terminates. For more details, see [14]. In some practical situations, the assumption of eliminating a fixed number of units at each

failure is impossible. Thus, we also consider both situations when the removals $R_1,\ R_2,\ldots,R_n$ are fixed or random. Two cases of random removals are considered, namely; discrete uniform and binomial random removals. For more details on random removals, see [15,16].
The remaining part of the paper is organized as follows:
In Section 2, the stress-strength reliability $\mathscr{R} = P\ (X < Y\ )$, when X and Y are independent random variables following MOEW$(\alpha,\ \theta_1,\eta)$ and MOEW(α,θ_2,η) respectively, is obtained in an explicit form showing that $\mathscr{R}$ does not depend on η. Also, a characterization of the parent distributions, associated with $\mathscr{R}$, is presented in this section. The estimation of $\mathscr{R}$ based on Type-II progressively censored samples, for fixed as well as random removals with discrete uniform and binomial distributions, is discussed in Sections 3 and 4. In Section 3, the maximum likelihood estimator (MLE) of $\mathscr{R}$ is obtained. In Section 4, Bayes estimators of $\mathscr{R}$ using Lindley's approximation for informative and non-informative priors under two different loss functions (SELF and LINEX) are presented. A simulation study illustrating the theoretical results is performed in Section 5. A real data example is discussed in Section 6. Conclusion is presented in Section 7.

2 Stress-Strength Reliability

In this section, we obtain the stress-strength reliability, $\mathscr{R}$, when the stress X and the strength Y are independent random variables following MOEW$(\alpha,\ \theta_1,\eta)$ and MOEW(α,θ_2,η), respectively. Theorem 1 presents a characterization of the parent distributions of X and Y associated with the stress-strength reliability $\mathscr{R}$.
Theorem 1 Let X and Y be two independent, non-negative, continuous random variables following Marshall-Olkin extended distribution given in (1) with parent SFs $\overline{F}_1(.)$ and $\overline{F}_2(.)$ respectively. Then, the stress-strength reliability $\mathscr{R}$ is

$$\mathscr{R} = \frac{\theta_1\alpha^2}{\theta_1+\theta_2}\,\varphi\left(\alpha,\frac{\theta_1}{\theta_1+\theta_2}\right), \tag{9}$$

where $\varphi(\alpha,\omega) = \int_0^1 [1-\overline{\alpha}u^{\omega}]^{-2}\left[1-\overline{\alpha}u^{1-\omega}\right]du$ is a function of α and ω.
if and only if, the parent SFs are

$$\overline{F}_1(x) = e^{-\theta_1\ \psi(x,\eta)}, \tag{10}$$

and

$$\overline{F}_2(y) = e^{-\theta_2\ \psi(y,\eta)}, \tag{11}$$

Where, $\psi(z,\ \eta)$ is a non-decreasing differential function in z such that $\psi(z,\eta)\to 0$ as $z\to 0$ and $\psi(z,\ \eta)\to\infty$ as $z\to\infty$.
Proof Suppose that X and $Y$are random variables having MOEW$(\alpha,\ \theta_1,\eta)$ and MOEW(α,θ_2,η) with parent SFs given by (10) and (11), respectively. Then, the stress strength reliability is given by

$$\mathscr{R} = \int_0^{\infty} g_X(z,\alpha,\theta_1,\eta)\,\overline{G}_Y(z,\alpha,\theta_2,\eta)\,dz. \tag{12}$$

Using (3) and (4), we get

$$\mathscr{R} = \int_0^{\infty}\frac{\theta_1\alpha^2\psi^{(1)}(z,\eta)\,e^{-[\theta_1+\theta_2]\psi(z,\eta)}}{\left[1-\overline{\alpha}e^{-\theta_1\psi(z,\eta)}\right]^2\left[1-\overline{\alpha}e^{-\theta_2\psi(z,\eta)}\right]}dz. \tag{13}$$

Setting $u = e^{-[\theta_1+\theta_2]\psi(z,\eta)}$ in (13), and after simplification, we get (9).
Conversely, suppose that X and Y are independent random variables following Marshall Olkin extended distribution in (1), and that (9) holds. When $\alpha = 1$, the distributions of X and Y are reduced to the parent distributions and $\varphi\left(1,\frac{\theta_1}{\theta_1+\theta_2}\right) = 1$. Hence, $\mathscr{R} = \frac{\theta_1}{\theta_1+\theta_2}$. Then, according to Theorem 1 in Mokhlis et al. [12], if $\mathscr{R} = \frac{\theta_1}{\theta_1+\theta_2}$ then, X and Y must have the independent distributions given by (10) and (11). Hence, the proof is complete.
Notice that:

1. Using (5) and (6) in (12) for $\alpha < 1$ and (7) and (8) in (12) for $\alpha > 1$, the stress strength reliability $\mathscr{R}$ can be written as

$$\mathscr{R} = \begin{cases} \sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\frac{\alpha^2\theta_1\overline{\alpha}^{k+s}(k+1)}{(k+1)\theta_1+(s+1)\theta_2}, & \alpha < 1, \\ \frac{\theta_1}{\theta_1+\theta_2}, & \alpha = 1, \\ \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{t=0}^{j}\frac{(-1)^{r+t}\binom{k}{r}\binom{j}{t}\left(1-\frac{1}{\alpha}\right)^{k+j}(k+1)\theta_1}{\alpha(r+1)\theta_1+\alpha(t+1)\theta_2} & \alpha > 1. \end{cases} \tag{14}$$

2. The formula of the stress strength function $\mathscr{R}$ obtained in (9) and (14) does not involve η. This means that if $\alpha,\ \theta_1$ and θ_2 are known, the exact value of $\mathscr{R}$ could be determined without knowing η. However, if the parameters are unknown, the estimated value of $\mathscr{R}$ must depend on the value of η or the estimated value of η.

3 Maximum Likelihood Estimation of $\mathscr{R}$

In this section, the MLE of $\mathscr{R}$ is obtained based on Type-II progressively censored data. Fixed as well as random removals with discrete uniform and binomial distributions are discussed. The MLE of $\mathscr{R}$ is

$$\widehat{\mathscr{R}} = \frac{\widehat{\theta}_1 \alpha^2}{\widehat{\theta}_1 + \widehat{\theta}_2}\, \varphi\left(\widehat{\alpha}, \frac{\widehat{\theta}_1}{\widehat{\theta}_1 + \widehat{\theta}_2}\right), \tag{15}$$

where $\widehat{\theta}_1$, $\widehat{\theta}_2$, and $\widehat{\alpha}$ are the maximum likelihood estimators of θ_1, θ_2, and α respectively, based on Type II progressive censoring schemes. Although the expression of $\mathscr{R}$ obtained in (15) does not involve $\widehat{\eta}$, we will obtain the MLE $\widehat{\eta}$ of η since the estimator of the parameters θ_1, θ_2, and α in some cases may depend on $\psi(.,\widehat{\eta})$.
Suppose that $(X_{1:n:N}, X_{2:n:N}, \ldots, X_{n:n:N})$ and $(Y_{1:m:M}, Y_{2:m:M}, \ldots, Y_{m:m:M})$ are Type-II progressive censored samples from MOEW(α, θ_1, η) and MOEW(α, θ_2, η) with censoring schemes$(R_1, R_2, \ldots, R_n)$ and $(\grave{R}_1, \grave{R}_2, \ldots, \grave{R}_m)$, respectively. Three cases for $(R_1, R_2, \ldots, R_n)$ and $(\grave{R}_1, \grave{R}_2, \ldots, \grave{R}_m)$ are considered, as follows:

Case1: Fixed removals

Assume that the censoring schemes $(R_1, R_2, \ldots, R_n)$ and $(\grave{R}_1, \grave{R}_2, \ldots, \grave{R}_m)$ are predetermined fixed numbers. Then the likelihood function can be written as

$$L_1 = c_1 c_2 \prod_{i=1}^{n} g_x(x_i, \alpha, \theta_1, \eta)\, \overline{G}_x(x_i, \alpha, \theta_1, \eta)^{R_i} \prod_{j=1}^{m} g_y(y_j, \alpha, \theta_2, \eta)\, \overline{G}_y(y_j, \alpha, \theta_2, \eta)^{\grave{R}_j}, \tag{16}$$

where
$c_1 = \prod_{k=1}^{n-1} N\left(N - \sum_{i=1}^{k}(R_i - 1)\right)$ and $c_2 = \prod_{s=1}^{m-1} M(M - \sum_{j=1}^{s}(\grave{R}_j - 1))$.
Using (3) and (4) in (16) we get
$L_1 \propto \theta_1^n \theta_2^m \alpha^{n+m+\sum_{i=1}^{n} R_i + \sum_{j=1}^{m} R'_j}$

$$\times \frac{\prod_{j=1}^{m} \psi^{(1)}(y_j, \eta)\, e^{-\theta_1 \sum_{i=1}^{n} (R_i+1)\psi(x_i,\eta) - \theta_2 \sum_{j=1}^{m} (\grave{R}_j+1)\psi(y_j,\eta)}}{\prod_{i=1}^{n}\left[1 - \overline{\alpha} e^{-\theta_1 \psi(x_i,\eta)}\right]^{R_i+2} \prod_{j=1}^{m}\left[1 - \overline{\alpha} e^{-\theta_2 \psi(y_j,\eta)}\right]^{\grave{R}_j+2}}. \tag{17}$$

Then, the likelihood equations are given by

$$\frac{n}{\theta_1} - \sum_{i=1}^{n}(R_i+1)\,\psi(x_i,\eta) - \sum_{i=1}^{n} \frac{(R_i+2)\,\psi(x_i,\eta)\,\overline{\alpha} e^{-\theta_1 \psi(x_i,\eta)}}{\left[1 - \overline{\alpha} e^{-\theta_1 \psi(x_i,\eta)}\right]} = 0, \tag{18}$$

$$\frac{m}{\theta_2} - \sum_{j=1}^{m}\left(\grave{R}_j+1\right)\psi(y_j,\eta) - \sum_{j=1}^{m} \frac{\left(\grave{R}_j+2\right)\psi(y_j,\eta)\,\overline{\alpha} e^{-\theta_2 \psi(y_j,\eta)}}{\left[1 - \overline{\alpha} e^{-\theta_2 \psi(y_j,\eta)}\right]} = 0, \tag{19}$$

$$\frac{n+m+\sum_{i=1}^{n} R_i + \sum_{j=1}^{m} \grave{R}_j}{\alpha} - \sum_{i=1}^{n} \frac{(2+R_i)\, e^{-\theta_1 \psi(x_i,\eta)}}{\left[1 - \overline{\alpha} e^{-\theta_1 \psi(x_i,\eta)}\right]} - \sum_{j=1}^{m} \frac{\left(\grave{R}_j+2\right) e^{-\theta_2 \psi(y_j,\eta)}}{\left[1 - \overline{\alpha} e^{-\theta_2 \psi(y_j,\eta)}\right]} = 0, \tag{20}$$

$$\begin{aligned} &\sum_{i=1}^{n} \frac{\frac{\delta \psi^{(1)}(x_i,\eta_k)}{\partial \eta_k}}{\psi^{(1)}(x_i,\eta_k)} - \sum_{i=1}^{n} \theta_1 (R_i+1) \frac{\delta \psi(x_i,\eta_k)}{\partial \eta_k} - \sum_{i=1}^{n} \frac{(R_i+2)\theta_1 \frac{\delta \psi(x_i,\eta_k)}{\partial \eta_k} \overline{\alpha} e^{-\theta_1 \psi(x_i,\eta_k)}}{\left[1 - \overline{\alpha} e^{-\theta_1 \psi(x_i,\eta_k)}\right]} + \\ &\sum_{j=1}^{m} \frac{\frac{\delta \psi^{(1)}(y_j,\eta_k)}{\partial \eta_k}}{\psi^{(1)}(y_j,\eta_k)} - \sum_{j=1}^{m} \theta_2 \left(\grave{R}_j+1\right) \frac{\delta \psi(y_j,\eta_k)}{\partial \eta_k} - \sum_{j=1}^{m} \frac{\left(\grave{R}_j+2\right)\theta_2 \frac{\delta \psi(y_j,\eta_k)}{\partial \eta_k} \overline{\alpha} e^{-\theta_2 \psi(y_j,\eta_k)}}{\left[1 - \overline{\alpha} e^{-\theta_2 \psi(y_j,\eta_k)}\right]} = 0. \end{aligned} \tag{21}$$

where, $k = 1, .., T$ and $\eta = (\eta_1, \eta_2, \ldots, \eta_T)$. The MLEs $\widehat{\theta}_1, \widehat{\theta}_2$, $\widehat{\alpha}$ and $\widehat{\eta}$ can be obtained by solving equations (18–21) numerically.

Case 2: Removals with discrete uniform distribution

In this case, R_i, $i = 1, \ldots, n$ and $\grave{R}_j$, $j = 1, \ldots, m$ are assumed to be independent random variables following discrete uniform distributions. Thus, $P(R_1 = r_1) = \frac{1}{N-n+1}$, and

$$P(R_i = r_i \mid R_{i-1} = r_{i-1} \ldots R_1 = r_1) = \frac{1}{N - n - \sum_{k=1}^{i-1} r_k + 1},$$

where, $0 \le r_1 \le N-n$, $0 \le r_i \le N-n-\sum_{k=1}^{i-1} r_k$, $i=2,\ldots,n-1$ and $R_n = N-n-\sum_{k=1}^{n-1} r_k$.
Similarly, $P(\grave{R}_1 = \grave{r}_1) = \frac{1}{M-m+1}$, and

$$P\left(\grave{R}_j = \grave{r}_j \mid R\grave{}_{j-1} = \grave{r}_{j-1} \ldots \grave{R}_1 = \grave{r}_1\right) = \frac{1}{M-m-\sum_{s=1}^{j-1} \grave{r}_s + 1},$$

where, $0 \le \grave{r}_1 \le M-m$, $0 \le r'_j \le M-m-\sum_{s=1}^{j-1} \grave{r}_s$,$j=2,\ldots,m-1$ and $\grave{R}_m = M-m-\sum_{s=1}^{j-1} \grave{r}_s$.
Hence, the likelihood function is given by

$$L_2 = L_1 \times P(R_1 = r_1,\, R_2 = r_2,\, \ldots,\, R_n = r_n) \times P(R'_1 = r'_1,\, R'_2 = r'_2,\, \ldots,\, R'_m = r'_m)$$

$$= L_1 \left(\frac{1}{N-n+1} \prod_{i=1}^{n-1} \frac{1}{N-n-\sum_{k=1}^{i-1} r_k + 1} \right) \left(\frac{1}{M-m+1} \prod_{j=1}^{m-1} \frac{1}{M-m-\sum_{s=1}^{j-1} \grave{r}_s + 1} \right), \tag{22}$$

where L_1 is given in (17). Clearly, the joint PDFs of R_i's, $i=1,\ldots,n-1$ and $\grave{R}_j$'s, $j=1,\ldots,m-1$, are free of parameters. Then, the MLEs $\widehat{\theta}_1, \widehat{\theta}_2$, $\widehat{\alpha}$ and $\widehat{\eta}$ can be obtained by solving simultaneously the system of equations (18- 21). The difference here is that r_i and $\grave{r}_j$ are observations of the random removals R_i, $i=1,\ldots,n$ and $\grave{R}_j$, $j=1,\ldots,m$, respectively.

Case3: Removals with binomial distribution

Here, we assume that $R_i, i=1,..,n$ and $\grave{R}_j$, $j=1,..,m$ are independent random variables following binomial distributions. Thus, $P(R_1 = r_1) = \binom{N-n}{r_1} P_1^{r_1} (1-P_1)^{N-n-r_1}$, and

$$P(R_i = r_i \mid R_{i-1} = r_{i-1},\ldots,R_1 = r_1) = \binom{N-n-\sum_{k=1}^{i-1} r_k}{r_i} {P_1}^{r_i} (1-P_1)^{N-n-\sum_{k=1}^{i-1} r_k},$$

where, $0 \le r_1 \le N-n$, $0 \le r_i \le N-n-\sum_{k=1}^{i-1} r_k$, $i=2,\ldots,n-1$ and $R_n = N-n-\sum_{k=1}^{n-1} r_k$.
Similarly, $P(\grave{R}_1 = \grave{r}_1) = \binom{M-m}{\grave{r}_1} {P_2}^{\grave{r}_1} (1-P_2)^{M-m-\grave{r}_1}$, and

$$P\left(\grave{R}_j = \grave{r}_j \mid \grave{R}_{j-1} = r_{j}\grave{}_{-1},\ldots,\grave{R}_1 = \grave{r}_1\right) = \binom{M-m-\sum_{s=1}^{j-1} \grave{r}_s}{\grave{r}_j} {P_2}^{\grave{r}_j} (1-P_2)^{M-m-\sum_{s=1}^{j-1} \grave{r}_s},$$

where, $0 \le \grave{r}_1 \le M-m$, $0 \le r'_j \le M-m-\sum_{s=1}^{j-1} \grave{r}_s$, $j=2,\ldots,m-1$ and $\grave{R}_m = M-m-\sum_{s=1}^{j-1} \grave{r}_s$.
Thus, the likelihood function is given by

$$L_3 = L_1 \times P(R_1 = r_1,\, \ldots,\, R_n = r_n)\, P(R'_1 = r'_1,,\, \ldots,\, R'_m = r'_m)$$

$$= L_1 \times \left(\frac{N-n}{\prod_{i=1}^{n-1} r_i! (N-n-\sum_{i=1}^{n-1} r_i)!} P_1^{\sum_{i=1}^{n-1} r_i} (1-P_1)^{(n-1)(N-n)-\sum_{i=1}^{n-1} (n-i) r_i} \right)$$

$$\times \left(\frac{M-m}{\prod_{j=1}^{m-1} \grave{r}_j! \left(M-m-\sum_{j=1}^{m-1} \grave{r}_j\right)!} P_2^{\sum_{j=1}^{m-1} \grave{r}_j} (1-P_2)^{(m-1)(M-m)-\sum_{j=1}^{m-1} (m-j) \grave{r}_j} \right), \tag{23}$$

where L_1 is given in (17).
Notice that the joint PDFs of R_i's, $i=1,\ldots,n-1$ and $\grave{R}_j$'s, $j=1,\ldots,m-1$ depend only on $P_1$and P_2 respectively, then the MLEs of parameters θ_1, θ_2, α and η can also be obtained by simultaneously solving the same system of equations, (18-21). The MLEs of parameters P_1 and P_2 are obtained by maximizing L_3, so
$\hat{p}_1 = \frac{\sum_{i=1}^{n-1} r_i}{(n-1)(N-n)-\sum_{i=1}^{n-1} (n-i-1) r_i}$, and $\hat{p}_2 = \frac{\sum_{j=1}^{m-1} \grave{r}_j}{(m-1)(M-m)-\sum_{j=1}^{m-1} (m-j-1) \grave{r}_j}$.

4 Bayesian Estimation

In this section we discuss the Bayesian estimation of $\mathscr{R}$ based on Type-II progressive censoring with fixed and random removals, under the SELF and LINEX loss functions, using informative and non-informative priors.

Informative priors:

We assume that the prior PDFs of θ_1, θ_2, η and α for the three cases (Case 1, Case 2 and Case 3) are given respectively by

$$\pi_j(\theta_j) = \frac{1}{a_j} e^{-a_j \theta_j}, \qquad \theta_j > 0, \quad j = 1,2, \tag{24}$$

$$\pi_3(\eta) = \frac{1}{a_3} e^{-a_3 \eta}, \qquad \eta > 0, \tag{25}$$

and

$$\pi_4(\alpha) = \frac{1}{a_4} e^{-a_4 \alpha}, \qquad \alpha > 0. \tag{26}$$

Case 1: The joint posterior PDF of $\theta_1, \theta_2, \alpha$ and η is given by

$$\pi_1^*(\theta_1, \theta_2, \alpha, \eta) = \frac{L_1 e^{-(a_1\theta_1 + a_2\theta_2 + a_3\eta + a_4\alpha)}}{\int L_1 e^{-(a_1\theta_1 + a_2\theta_2 + a_3\eta + a_4\alpha)} \, d(\theta_1, \theta_2, \alpha, \eta)}, \tag{27}$$

where, L_1 is given in (17). **Case 2:** The joint posterior PDF of $\theta_1, \theta_2, \alpha$ and η is given by

$$\pi_2^*(\theta_1, \theta_2, \alpha, \eta) = \frac{L_2 e^{-(a_1\theta_1 + a_2\theta_2 + a_3\eta + a_4\alpha)}}{\int L_2 e^{-(a_1\theta_1 + a_2\theta_2 + a_3\eta + a_4\alpha)} \, d(\theta_1, \theta_2, \alpha, \eta)},$$

where, L_2 is given in (22). Substituting L_2 and after simplifications, we find that

$$\pi_2^*(\theta_1, \theta_2, \alpha, \eta) = \pi_1^*(\theta_1, \theta_2, \alpha, \eta). \tag{28}$$

Case 3:

Since $0 < P_K < 1$, $K = 1,2$, we consider the following prior PDFs for P_K , $K = 1,2$

$$\pi_j(P_k) = \frac{1}{B(b_k, c_k)} P_k^{b_k - 1} (1 - \mathrm{P_k})^{c_k - 1}, \qquad j = 5,6, k = 1,2, \tag{29}$$

where, $B(b_k, c_k)$ is the beta function.

The joint posterior PDF of $\theta_1, \theta_2, \alpha$ and η is given by

$$\pi_3^*(\theta_1, \theta_2, \alpha, \eta, P_1, P_2) = \frac{L_3 e^{-(a_1\theta_1 + a_2\theta_2 + a_3\eta + a_4\alpha)} \pi_5(P_1)\, \pi_6(P_2)}{\int L_3 e^{-(a_1\theta_1 + a_2\theta_2 + a_3\eta + a_4\alpha)} \pi_5(P_1)\, \pi_6(P_2)\, d(\theta_1, \theta_2, \alpha, \eta, P_1, P_2)}.$$

Using (23), we get

$$\pi_3^*(\theta_1, \theta_2, \alpha, \eta, P_1, P_2) = D_1 \pi_1^*(\theta_1, \theta_2, \alpha, \eta), \tag{30}$$

where

$$D_1 = \frac{P_1^{\sum_{i=1}^{n-1} r_i + b_1 - 1} (1 - P_1)^{(n-1)(N-n) - \sum_{i=1}^{n-1} (n-i) r_i + c_1 - 1}}{B(\sum_{i=1}^{n-1} r_i + b_1, (n-1)(N-n) - \sum_{i=1}^{n-1} (n-i) r_i + c_1)} \frac{P_2^{\sum_{j=1}^{m-1} \grave{r}_j + b_2 - 1} (1 - P_2)^{(m-1)(M-m) - \sum_{j=1}^{m-1} (m-j) \grave{r}_j + c_2 - 1}}{B(\sum_{j=1}^{m-1} \grave{r}_j + b_2, (m-1)(M-m) - \sum_{j=1}^{m-1} (m-j) \grave{r}_j + c_2)}.$$

Non-informative priors:

Here, we assume that all the prior PDFs of the parameters $\theta_1, \theta_2, \alpha$ and η for all cases are equal to 1.

Case 1: The joint posterior PDF of θ_1, θ_2 , α and η is given by

$$\pi'^*_1(\theta_1, \theta_2, \alpha, \eta) = \frac{L_1}{\int L_1 \; d(\theta_1, \theta_2, \alpha, \eta)}. \tag{31}$$

Case 2: The joint posterior PDF of θ_1, θ_2 , α and η is given by

$$\pi'^*_2(\theta_1, \theta_2, \alpha, \eta) = \frac{L_2}{\int L_2 \; d(\theta_1, \theta_2, \alpha, \eta)}.$$

Using (22), we find that

$$\pi'^{*}_{2}(\theta_1,\theta_2,\ \alpha,\eta)=\ \pi'^{*}_{1}(\theta_1,\theta_2,\ \alpha,\eta). \tag{32}$$

Case 3: The joint posterior PDF of θ_1, θ_2 , α and η is given by

$$\pi'^{*}_{3}(\theta_1,\theta_2,\ \alpha,\eta,P_1,P_2)=\frac{L_3}{\int L_3\ \ d(\theta_1,\theta_2,\ \alpha,\eta,P_1,P_2)}=D_2\pi_1^{'*}(\theta_1,\theta_2,\ \alpha,\eta). \tag{33}$$

Where $D_2=\frac{P_1^{\sum_{i=1}^{n-1} r_i}(1-P_1)^{(n-1)(N-n)-\sum_{i=1}^{n-1}(n-i)r_i}}{B(\sum_{i=1}^{n-1} r_i+1,(n-1)(N-n)-\sum_{i=1}^{n-1}(n-i)r_i+1)}\frac{P_2^{\sum_{j=1}^{m-1}\grave{r}_j}(1-P_2)^{(m-1)(M-m)-\sum_{j=1}^{m-1}(m-j)\grave{r}_j}}{B(\sum_{j=1}^{m-1}\grave{r}_j+1,(m-1)(M-m)-\sum_{j=1}^{m-1}(m-j)\grave{r}_j+1)}$.

The Bayesian estimator of $\mathscr{R}$ which minimizes the SELF, $L_{SEL}\left(\mathscr{R},\widetilde{\mathscr{R}}_{SEL}\right)$, and LINEX loss function, $L_{LX}(\mathscr{R},\widetilde{\mathscr{R}}_{LX})$, are defined as $\widetilde{\mathscr{R}}_{SEL}$ *and* $\widetilde{\mathscr{R}}_{LX}$ respectively, where

$$L_{SEL}\left(\mathscr{R},\widetilde{\mathscr{R}}_{SEL}\right)=\left(\mathscr{R}-\widetilde{\mathscr{R}}_{SEL}\right)^2,$$

$$L_{LX}\left(\mathscr{R},\widetilde{\mathscr{R}}_{LX}\right)=e^{\beta\left(\mathscr{R}-\widetilde{\mathscr{R}}_{LX}\right)}-\beta\left(\mathscr{R}-\widetilde{\mathscr{R}}_{LX}\right)-1,$$

$$\widetilde{\mathscr{R}}_{SEL}=E(\mathscr{R}),$$

and

$$\widetilde{\mathscr{R}}_{LX}=\frac{-1}{\beta}\ln\left(E\left(e^{-\beta\,\mathscr{R}}\right)\right),$$

where β is a scale parameter of L_{LX}. For more details, see [17]. The expectation is taken with respect to the joint posterior PDF of the parameters.

Notice that: For the three cases considered above, from (28), (30), (32), and (33), we find that the joint posterior PDFs of the informative and non informative priors of the parameters are similar for Cases 1 and 2, while in Case 3, the joint posterior PDFs of the informative and non informative priors are directly proportional to the corresponding joint posteriors of Case 1. Hence, $\widetilde{\mathscr{R}}_{SEL}$and $\widetilde{\mathscr{R}}_{LX}$ will have the same expressions for all cases, with the exception that in Case 1, the r_i's and $\grave{r}_j$'s are fixed, while in Cases 2 and 3, they are observations of discrete random variables that are uniformly and binomially distributed, respectively.

However, the joint posterior PDFs in (27) and (31) are intractable, so the Bayes estimators $\widetilde{\mathscr{R}}_{SEL}$and $\widetilde{\mathscr{R}}_{LX}$ couldn't be obtained in explicit forms. Instead, we apply Lindley's approximation method for obtaining these estimators using the following technique

$$\mathrm{E}(u(\vartheta))=\frac{\int u(\vartheta)\,e^{L^*(\vartheta)+\rho(\vartheta)}d(\vartheta)}{\int e^{L^*(\vartheta)+\rho(\vartheta)}d(\vartheta)},$$

where, $\vartheta=(\vartheta_1,\vartheta_2,\dots\vartheta_r)$, $u(\vartheta)$ is any function of ϑ, $L^*(\vartheta)$ is the log likelihood function of ϑ and $\rho(\vartheta)$ is the log of joint prior of ϑ. Then, the Lindley's approximation of this integral is

$$\mathrm{E}(u(\vartheta))=\ u+\frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}(u_{i,j}+2u_i\rho_j)\,\sigma_{i,j}+\frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{t=1}^{r}L^{*}_{i,j,k}u_t\sigma_{i,j}\sigma_{k,t},$$

where,$v_i=\frac{\delta v}{\partial\vartheta_i}$,$v_{i,j}=\frac{\delta^2 v}{\partial\vartheta_i\partial\vartheta_j}$,$v_{i,j,k}$= $\frac{\delta^3 v}{\partial\vartheta_i\partial\vartheta_j\partial\vartheta_k}$ and the matrix of elements $\sigma_{i,j}$ is equal to the inverse matrix of elements $-L^{*}_{i,j}$. All the partial derivatives are evaluated at the MLEs of ϑ. For more details see [18].

5 Simulation Study

In this section, a Monte Carlo simulation is preformed to compare the performance of the different estimators of $\mathscr{R}$ proposed in Sections 3 and 4. As an application, Marshall-Olkin extended Pareto is chosen as a sub-model of MOEW by assuming that X and Y having Marshal Olkin extended Pareto distribution with parameter(α,θ_1,η) and (α,θ_2,η) respectively with SFs

$$\overline{G}_x(x,\alpha,\theta_1,\eta)=\frac{\alpha\ e^{-\theta_1\ln(\frac{x}{\eta})}}{1-\overline{\alpha}e^{-\theta_1\ln(\frac{x}{\eta})}},\qquad\qquad x>\eta,$$

$$\overline{G}_y\left(y,\alpha,\theta_2,\eta\right)=\frac{\alpha\ e^{-\theta_2\ln\left(\frac{y}{\eta}\right)}}{1-\overline{\alpha}e^{-\theta_2\ln\left(\frac{y}{\eta}\right)}}\ , \qquad\qquad y>\eta.$$

In Marshall-Olkin extended Pareto, since $x>\eta$ and $y>\eta$, so we take $\widehat{\eta}=\min(x_{\text{i}:n:N},y_{\text{j}\ :m:M})$, while the MLEs of the remaining parameters are obtained by solving the system of equations (18-20) after substituting η with $\widehat{\eta}$.
1000 samples of size 30 are generated for both X and Y. The MLE $\widehat{\mathscr{R}}_{MLE}$ is compared with the Bayesian estimators of $\mathscr{R}$ under the squared error loss function $\widetilde{\mathscr{R}}_{\boldsymbol{SEL}}$ and the LINEX loss function $\widetilde{\mathscr{R}}_{LX}$. Furthermore, the Bayesian estimation is considered for informative and non-informative priors. For the informative priors, the hyper-parameters are chosen based on the method applied by Ahn et al. [19] and used by Ali and Aslam [20]. The comparison of the estimates is presented in terms of bias and mean squared error (MSE). For the simulation purpose different sets of parameters are chosen for different values of $\mathscr{R}\ =\ 0.5,\ 0.7$ and 0.9 to see the sensitivity of the estimators with respect to the values of $\mathscr{R}$ (small, moderate and large) as shown in Tables (2-4). The LINEX loss function are considered for three different values of parameter β as represented in Tables (2-4) as $\widetilde{\mathscr{R}}_{LX(\beta=-1)}$, $\widetilde{\mathscr{R}}_{LX(\beta=1)}$ and $\widetilde{\mathscr{R}}_{LX(\beta=0.5)}$. All the calculations are evaluated by Maple (18) program. All the estimators are evaluated for fixed, discrete uniform and binomial random removals. The censoring schemes considered are 20 percent and 50 percent elimination from the sample size. For fixed removal, the elimination is either at the beginning, in the middle or at the end of the sample. Table 1 summarizes the censoring schemes considered for fixed elimination with 0^q, indicating that no elimination occurred during q failures.

Tables 2-4 summarize the results of different estimation methods based on progressive censoring schemes with discrete uniform, binomial random removals (with $P_1=P_2=0.5$) and fixed removals, respectively.

Table 1: Censoring schemes for fixed removals.

Sample size	Elimination	Schemes	
30	20%	Scheme1	$(6, 0^{23})$
		Scheme2	$(0^{12}, 6, 0^{11})$
		Scheme3	$(0^{23}, 6)$
	50%	Scheme1	$(15, 0^{14})$
		Scheme2	$(0^{7}, 15, 0^{7})$
		Scheme3	$(0^{14}, 15)$

Table 2:Bias and MSE of $\left(\mathscr{R}_{\boldsymbol{MLE}},\widetilde{\mathscr{R}}_{\boldsymbol{SEL}},\widetilde{\mathscr{R}}_{\boldsymbol{LX}(\beta=c)}\right)$, $c=-1,1$ *and* 0.5 *for discrete uniform random removal.*

$(\theta_1,\theta_2,\alpha,\eta)\mathscr{R}$		**bias** **MSE**	$\mathscr{R}_{MLE}$	**Non-Informative-prior**				**Informative-prior**			
				$\mathscr{R}_{SEL}$	$\mathscr{R}_{LX(\beta=-1)}$	$\mathscr{R}_{LX(\beta=1)}$	$\mathscr{R}_{LX(\beta=0.5)}$	$\mathscr{R}_{SEL}$	$\mathscr{R}_{LX(\beta=-1)}$	$\mathscr{R}_{LX(\beta=1)}$	$\mathscr{R}_{LX(\beta=0.5)}$
(6,5,3,8) **$\mathscr{R}$=0.5596**	**20%**	**bias**	-0.0054	-0.0061	-0.0085	-0.0037	-0.0049	-0.0027	-0.0051	-0.0003	-0.0015
		MSE	0.0059	0.0060	0.0060	0.0061	0.0060	0.0051	0.0051	0.0052	0.0052
	50%	**bias**	0.0085	0.0072	0.0033	0.0110	0.0091	0.0123	0.0085	0.0161	0.0142
		MSE	0.0061	0.0064	0.0064	0.0065	0.0065	0.0049	0.0048	0.0050	0.0050
(8,4,3,2) **$\mathscr{R}$=0.7116**	**20%**	**bias**	0.0197	0.0168	0.0149	0.0187	0.0177	0.0275	0.0256	0.0293	0.0284
		MSE	0.0030	0.0029	0.0028	0.0030	0.0030	0.0031	0.0029	0.0032	0.0031
	50%	**bias**	0.0047	0.0014	-0.0015	0.0043	0.0029	0.0162	0.0134	0.0190	0.0176
		MSE	0.0051	0.0052	0.0051	0.0053	0.0052	0.0046	0.0044	0.0048	0.0047
(9,1,5,7) **$\mathscr{R}$=0.9509**	**20%**	**bias**	-0.0076	-0.0024	-0.0017	-0.0028	-0.0026	-0.0025	-0.0018	-0.0029	-0.0028
		MSE	0.0001	0.0011	0.0018	0.0008	0.00090	0.00104	0.0018	0.0008	0.0009
	50%	**bias**	-0.0096	-0.0087	-0.0079	-0.0092	-0.0089	-0.0063	-0.0056	-0.0069	-0.0066
		MSE	0.0001	0.0012	0.0021	0.0014	0.0012	0.0012	0.0019	0.0014	0.0012

Table 3: Bias and MSE of $\left(\mathscr{R}_{MLE}, \widetilde{\mathscr{R}}_{sEL}, \widetilde{\mathscr{R}}_{LX(\beta=\complement)}\right), \complement = -1, 1\ and 0.5$ for binomial random removal.

$(\theta_1,\theta_2,\alpha,\eta)\mathscr{R}$		bias MSE	$\mathscr{R}_{MLE}$	**Non-Informative-prior**				**Informative-prior**			
				$\widetilde{\mathscr{R}}_{SEL}$	$\widetilde{\mathscr{R}}_{LX(\beta=-1)}$	$\widetilde{\mathscr{R}}_{LX(\beta=1)}$	$\widetilde{\mathscr{R}}_{LX(\beta=0.5)}$	$\widetilde{\mathscr{R}}_{SEL}$	$\widetilde{\mathscr{R}}_{LX(\beta=-1)}$	$\widetilde{\mathscr{R}}_{LX(\beta=1)}$	$\widetilde{\mathscr{R}}_{LX(\beta=0.5)}$
(6,5,3,8) **$\mathscr{R}$=0.5596**	**20%**	**bias**	0.0083	0.0072	0.0048	0.0097	0.0084	0.0108	0.0083	0.0132	0.0120
		MSE	0.0033	0.00334	0.0033	0.0034	0.0034	0.0028	0.0028	0.0029	0.0029
	50%	**bias**	0.0243	0.0226	0.0186	0.0267	0.0247	0.0262	0.0222	0.0303	0.0282
		MSE	0.0034	0.0034	0.0033	0.0036	0.0035	0.0028	0.0026	0.0030	0.0029
(8,4,3,2) **$\mathscr{R}$=0.7116**	**20%**	**bias**	-0.0107	-0.0123	-0.0140	-0.0106	-0.0115	-0.0030	-0.0047	-0.0013	-0.0021
		MSE	0.0036	0.0037	0.0037	0.0037	0.0037	0.0032	0.0032	0.0032	0.0032
	50%	**bias**	0.0557	0.0479	0.0445	0.0514	0.0496	0.0703	0.0669	0.0735	0.0719
		MSE	0.0053	0.0045	0.0042	0.0049	0.0047	0.0067	0.0062	0.0072	0.0069
(9,1,5,7) **$\mathscr{R}$=0.9509**	**20%**	**bias**	-0.0098	-0.0035	0.0036	0.0067	-0.0050	-0.0014	0.0060	-0.0048	-0.003
		MSE	0.0001	0.0067	0.0229	0.0073	0.0060	0.0067	0.0251	0.0078	0.0061
	50%	**bias**	-0.0076	-0.0044	-0.0043	-0.0045	-0.0044	0.0037	-0.0036	-0.0037	-0.0037
		MSE	0.0001	0.0003	0.0003	0.0003	0.00031	0.0003	0.0003	0.0003	0.00027

Tables 2-4 demonstrate that the informative Bayesian estimation gives better results than the non-informative Bayesian estimation in terms of MSE. Bayesian estimation under LINEX loss function gives a very close result to that under SELF loss function concerning both bias and MSE. 50% elimination gives higher MSE than 20% elimination. Concerning the sensitivity of the estimates with respect to $\mathscr{R}$, we find that the MSE decreases as the value of $\mathscr{R}$ increases. This indicates that if $\mathscr{R}$ is unknown but it is known in advance that $\mathscr{R}$ is large, the presented estimators will be highly recommended. However, if $\mathscr{R}$ is small or moderate, the performance of the estimators is still very good.

6 Real data example

In this section, a real data example from Proschan [21] is presented. Proschan studied the successive failures times of the air conditioning system of different types of airplanes and showed that the failure distribution for each airplane separately was exponentially distributed, but with different failure rate. We present the failure times of two airplanes as shown in Table 5.

Table 5: failure intervals of the air conditioning system of two jet Planes given (in hours).

Plane (7913)	[23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95]
Plane (8044)	[487, 18, 100, 7, 98, 5, 85, 91, 43, 230, 3, 130]

Suppose X represents the failure interval times of air conditioning system of Plane (7913) and Yrepresents the failure interval times of air conditioning system of Plane (8044). Using the (K-S) goodness of fit test, the two data sets are tested for fitting either exponential or Marshall-Olkin extended exponential (MOEE). The results are shown in Table 6.

Table 6: MLEs of the parameters and the corresponding (K-S) values exponential distribution and MOEE

Var.	Distribution	Parameter estimate	K-S	Tabulated value
X	exponential	$\widehat{\theta}_1 = 0.01678$	0.2131	0.2417
	MOEE	$\widehat{\theta}_1$ =0.0101 , $\widehat{\alpha}_1$ =0.38	0.1262	
Y	exponential	$\widehat{\theta}_2$=0. 0093	0.1872	0.3754
	MOEE	$\widehat{\theta}_2$ =0.0055, $\widehat{\alpha}_2$ =0.37	0.1544	

Table 4: Bias and MSE of $\left(\mathscr{R}_{\boldsymbol{MLE}},\widetilde{\mathscr{R}}_{\boldsymbol{sEL}},\widetilde{\mathscr{R}}_{\boldsymbol{LX(\beta=C)}}\right)$, $C=-1,1$ *and* 0.5 for fixed

$(\theta_1,\theta_2,\alpha,\eta)$ $\mathcal{R}$	Sch. & Elim.	bias MSE	$\mathcal{R}_{MLE}$	Non-Informative-prior				Informative-prior			
				$\tilde{\mathcal{R}}_{SEL}$	$\tilde{\mathcal{R}}_{LX(\beta=-1}$	$\tilde{\mathcal{R}}_{LX(\beta=1)}$	$\tilde{\mathcal{R}}_{LX(\beta=0.5)}$	$\tilde{\mathcal{R}}_{SEL}$	$\tilde{\mathcal{R}}_{LX(\beta=-1)}$	$\tilde{\mathcal{R}}_{LX(\beta=1)}$	$\tilde{\mathcal{R}}_{LX(\beta=0.5}$
(6,5,3,8) $\mathcal{R}$ =0.55 96	1 20%	bias	0.0002	-0.0005	-0.0029	0.0019	0.0007	0.0022	-0.0002	0.0046	0.0034
		MSE	0.0060	0.0061	0.0062	0.0062	0.0062	0.0054	0.0053	0.0053	0.0053
	2 20%	bias	0.0005	-0.0009	-0.0033	0.0016	0.0004	0.0034	0.0009	0.0058	0.0046
		MSE	0.0046	0.0049	0.0048	0.0049	0.0049	0.0040	0.0040	0.0041	0.0040
	3 20%	bias	0.0130	0.0116	0.0092	0.0140	0.0128	0.0154	0.0130	0.0178	0.0166
		MSE	0.0037	0.0039	0.0038	0.0040	0.0039	0.0032	0.0031	0.0033	0.0032
	1 50%	bias	-0.0051	-0.0061	-0.0099	-0.0023	-0.0042	-0.0010	-0.0047	0.0027	0.0008
		MSE	0.0090	0.0093	0.0093	0.0093	0.0093	0.0074	0.0073	0.0074	0.0074
	2 50%	bias	-0.0092	-0.0111	-0.0151	-0.0070	-0.0091	-0.0046	-0.0086	-0.0006	-0.0026
		MSE	0.0111	0.0119	0.0119	0.0119	0.0119	0.0089	0.0089	0.0090	0.0090
	3 50%	bias	0.0750	0.0807	0.0768	0.0845	0.0826	0.0739	0.0700	0.0778	0.0759
		MSE	0.0056	0.0065	0.0059	0.0072	0.0068	0.0055	0.0049	0.0061	0.0058
(8,4,3,2) $\mathcal{R}$=0.7116	1 20%	bias	0.0045	0.0019	0.0001	0.0037	0.0028	0.0123	0.0105	0.0140	0.0132
		MSE	0.0028	0.0029	0.0028	0.0029	0.0029	0.0027	0.0026	0.0028	0.0027
	2 20%	bias	-0.0338	-0.0343	-0.0359	-0.0327	-0.0335	-0.0258	-0.0274	-0.0242	-0.0250
		MSE	0.0050	0.0050	0.0051	0.0050	0.0050	0.0042	0.0043	0.0042	0.0042
	3 20%	bias	-0.0397	-0.0396	-0.0411	-0.0382	-0.0389	-0.0313	-0.0328	-0.0299	-0.0306
		MSE	0.0053	0.0053	0.0054	0.0052	0.0053	0.0044	0.0045	0.0044	0.0044
	1 50%	bias	-0.0102	-0.0127	-0.0156	-0.0102	-0.0115	0.0012	-0.0015	0.0038	0.0025
		MSE	0.0052	0.0054	0.0053	0.0054	0.0054	0.0044	0.0043	0.00452	0.0045
	2 50%	bias	-0.0255	-0.0306	-0.0333	-0.0278	-0.0292	-0.0129	-0.0156	-0.0103	-0.0116
		MSE	0.0069	0.0075	0.0075	0.0074	0.0074	0.0056	0.0056	0.0057	0.0057
	3 50%	bias	-0.0360	-0.0385	-0.0407	-0.0362	-0.0373	-0.0233	-0.0255	-0.0211	-0.0222
		MSE	0.0071	0.0074	0.0074	0.0073	0.0073	0.0056	0.0056	0.0056	0.0056
(9,1,5,7) $\mathcal{R}$=0.9509	1 20%	bias	0.0185	0.0169	0.0167	0.0170	0.0169	0.0235	0.0233	0.0236	0.0235
		MSE	0.0004	0.0004	0.0004	0.0004	0.0004	0.0006	0.0006	0.0006	0.0006
	2 20%	bias	0.0183	0.0129	0.0132	0.0122	0.0126	0.0173	0.0176	0.0167	0.0171
		MSE	0.0003	0.0012	0.0009	0.0021	0.0015	0.0013	0.0010	0.0021	0.0016
	3 20%	bias	0.0116	0.0089	0.0125	0.0046	0.0074	0.0122	0.0159	0.0081	0.0108
		MSE	0.0002	0.0054	0.0104	0.0129	0.0063	0.0054	0.0114	0.0120	0.0062
	1 50%	bias	-0.0217	-0.0198	-0.0197	-0.0199	-0.0199	-0.0212	-0.0211	-0.0212	-0.0212
		MSE	0.0005	0.0006	0.0006	0.0006	0.0006	0.0006	0.0006	0.0006	0.0006
	2 50%	bias	0.0099	0.0084	0.0083	0.0085	0.0084	0.0153	0.0152	0.0154	0.0153
		MSE	0.0001	0.0001	0.0001	0.0001	0.0001	0.0003	0.0003	0.0003	0.0003
	3 50%	bias	-0.0161	-0.0163	-0.0163	-0.0162	-0.0163	-0.0122	-0.0122	-0.0121	-0.0121
		MSE	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002	0.0002	0.0002	0.0002

It is clear that the MOEE provides better fit than exponential distribution at level of significance 0.05. Since $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$ are approximately the same, we may consider X and Y as random variables with MOEE($\theta_1,\ \alpha$) and MOEE($\theta_2,\ \alpha$) respectively.

The stress-strength reliability function $\mathscr{R} = P(X < Y)$ is an appropriate measure of the effectiveness of the two air conditioning systems. Using the results in Sections (3) and (4), we consider the reliability estimation for the complete data case (i.e. $R_i = R'_j = 0$), binomial and discrete uniform random removals. The computed results are shown in Table 7.

Table7 shows that the estimated value of $\mathscr{R}$ for the complete data is 0.6238, which means that the failure interval times of air conditioning system of Plane (8044) is less than that of Plane (7913) with probability estimated by 0.6238. Moreover, estimation with 20% elimination gives almost the same results as that with complete data, while the estimation with 50% elimination is far from that with complete data.

Table 7: MLE and Bayes estimators of $\mathscr{R}$

Type of elimination	**Elim.**	$\widehat{\mathscr{R}}_{MLE}$	**Non-Informative-prior**				**Informative-prior**			
			$\widetilde{\mathscr{R}}_{sEL}$	$\widetilde{\mathscr{R}}_{LX(\beta=-1)}$	$\widetilde{\mathscr{R}}_{LX(\beta=1)}$	$\widetilde{\mathscr{R}}_{LX(\beta=0.5)}$	$\widetilde{\mathscr{R}}_{sEL}$	$\widetilde{\mathscr{R}}_{LX(\beta=-1)}$	$\widetilde{\mathscr{R}}_{LX(\beta=1)}$	$\widetilde{\mathscr{R}}_{LX(\beta=0.5)}$
Complete		**0.6238**	**0.6217**	**0.6250**	**0.6183**	**0.6246**	**0.6208**	**0.6242**	**0.6175**	**0.6247**
Binomial	20%	**0.6282**	**0.6152**	**0.6195**	**0.6109**	**0.6312**	**0.6111**	**0.6154**	**0.6070**	**0.6319**
	50%	**0.7147**	**0.7239**	**0.7309**	**0.7168**	**0.7138**	**0.7057**	**0.7129**	**0.6988**	**0.7169**
Discrete uniform	20%	**0.6411**	**0.6496**	**0.6539**	**0.6452**	**0.6399**	**0.6311**	**0.6354**	**0.6268**	**0.6434**
	50%	**0.7053**	**0.7072**	**0.7124**	**0.7020**	**0.7054**	**0.6931**	**0.6982**	**0.6880**	**0.7079**

7 Conclusion

In this paper, an explicit form of the stress-strength reliability $\mathscr{R}$, is obtained when the stress X and the strength Y are independent random variables belonging to MOEW family of distributions. A characterization of the parent distributions associated with the stress-strength reliability $\mathscr{R}$ was also presented. The problem of estimation $\mathscr{R}$ under Type-II progressive censored scheme using classical and Bayesian technique was studied. The Bayesian estimation of $\mathscr{R}$ was considered under two loss function (symmetric and asymmetric loss functions) for informative and non-informative priors. Furthermore, fixed, binomial and discrete uniform removals with 20% and 50% elimination from the sample size were addressed. A Monte Carlo simulation was preformed to compare the performance of the different estimators of $\mathscr{R}$ showing that the results of the informative Bayesian estimates were better than those of the non-informative Bayesian estimates in terms of MSE. However, the results of the Bayesian estimates under LINEX loss function were very close to those under SELF in terms of both bais and MSE. Moreover 50% elimination gave higher MSE than 20% elimination. Concerning the sensitivity of the estimates with respect to $\mathscr{R}$, the MSE decreased as the value of $\mathscr{R}$ increased. However, estimates showed very satisfactory results.

Conflict of Interest

The authors declare that they have no conflict of interest.

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