Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/070102

Holomorphic Solutions of a Class of 3-D Propagated Wave Dynamical Equations Indicated by a Complex Conformable Calculus

Rabha W. Ibrahim[1,2], *Samir B. Hadid*[3,4] *and Shaher Momani* [4,5,*]

[1] Informetrics Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam
[2] Faculty of Mathematics & Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam
[3] Department of Mathematics and Sciences, College of Humanities and Sciences, Ajman University, Ajman, UAE
[4] Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, UAE
[5] Department of Mathematics, Faculty of Science, University of Jordan, Amman 11942, Jordan

Received: 2 Oct. 2019, Revised: 26 Oct. 2019, Accepted: 6 Nov. 2019
Published online: 1 Jan. 2021

Abstract: In this paper, we present holomorphic assemblies of a class of nonlinear conformable time-fractional wave equations type Khokhlov-Zabolotskaya (KZ) in a complex purview. To achieve this objective, we introduce a characterization of a complex conformable calculus (CCC) of a symmetric differential operator (SDO) and investigate its properties. Moreover, the operator is extended to a complex domain satisfying symmetric illustrations. Employing the proposed operator, we generalize KZ equation symmetrically. The indications imply that the suggested techniques are powerful, reliable and appropriate for employing all styles of differential equations of complex variables.

Keywords: Complex differential equations, Fractional calculus, Fractional differential operator, Holomorphic solution, Majorization, Subordination and superordination, Unit disk.

1 Introduction

Transmission of sound pulses and sound rays in weakly nonlinear data with enumerations of small curvature of wave faces is of significant concern for numerous applications in science, medicine, geology and industry (see [1]). Mathematical model of transmission of nonlinear sound pulses depends on the investigation of the pretended Khokhlov-Zabolotskaya (KZ) equation. If we let the flow velocity of the environment at a point ξ at time t as $w(\xi,t)$, then the KZ equation can be inscribed in the layout

$$\Big(w(\xi,t)_t + \big(\varsigma_0 + \varsigma_1 w(\xi,t)\big)(w(\xi,t))_\xi\Big)_\xi + \frac{\varsigma_0}{2}(w(\xi,t))_{\xi\xi} = 0,$$

where ς_0 and ς_1 are parameters for the linear waves in the suggested environment. If this motion described with a satisfactory accuracy as a plane unidirectional data flow wave, then the conforming wave equation moderates to [2]

$$(w(\xi,t))_t + \varsigma_0(w(\xi,t))_\xi = 0,$$

where $w(\xi,t) = \Omega(\xi - \varsigma_0 t)$. This solution modified in the formal $w(\xi,t) = \Omega\,(\xi - (\varsigma_0 + \varsigma_1 w(\xi,t))t)$ for the Hopf equation

$$(w(\xi,t))_t + (\xi - (\varsigma_0 + \varsigma_1 w(\xi,t))\,(w(\xi,t))_\xi = 0.$$

Rudenko and Soluyan [3] presented the symmetric geometries evolution of waves over weakly nonlinear environment in the following construction

$$(w(\rho,t))_t + (\rho - (\varsigma_0 + \varsigma_1 w(\rho,t))\,(w(\rho,t))_\rho + \frac{\varsigma_2}{t}w(\rho,t) = 0, \quad t \neq 0,$$

* Corresponding author e-mail: shahermm@yahoo.com

where ρ is the radius of the geometric shape (cylindrical and spherical) and ς_2 is the shape's parameter. For example $\varsigma_2 = 0.5$ the shape is cylindrical and $\varsigma_2 = 1$ is a spherical case. Our geometric and symmetric investigation will be in the open unit disk. We shall use the idea of the geometric function theory to study the KZ equation.

In view of the Riemann-Liouville calculus,

$$D_t^\alpha \phi(t) = \frac{d}{dt}\int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}\phi(\tau)\,d\tau, \quad 0 \le \alpha < 1,$$

the 1-D fractional KZ (F-KZ) equation is formulated by [4]

$$D_t^\alpha w(\xi,t) + \varsigma_0 (w(\xi,t))_\xi = 0,$$

where $\alpha \in (0,1)$ characterizes to the fractional instruction variation and ξ signifies the 3-D variable expanse lengthways the stuff line although t is the time in its sizes usage and $w(\xi,t)$ requires a wave amplitude. Ray (see [5] and [6] respectively) introduced different methods to obtain the exact and analytic solutions of F-KZ. Ibrahim [7] introduced a collection of holomorphic outcomes of the 3D fractional complex wave equation utilizing a class of complex convolution formula

$$D_t^\alpha w(z,t) + D_z^\beta w(z,t) = 0, \quad z \in \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\},$$

where

$$D_z^\beta \phi(z) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dz}\int_0^z (z-\eta)^{-\beta}\phi(\eta)\,d\eta, \quad 0 \le \beta < 1$$

is the Srivastava-Owa fractional differential operator for analytic function $\phi(z)$.

Definition 1. *Let $\nu \in [0,1]$. A differential operator $\mathscr{D}^\nu$ is conformable if and only if $\mathscr{D}^0$ is the identity operator and $\mathscr{D}^1$ is the classical differential operator. Specifically, $\mathscr{D}^\nu$ is conformable if and only if for differentiable function $\phi(t)$,*

$$\mathscr{D}^0\phi(t) = \phi(t) \;\; \text{and} \;\; \mathscr{D}^1\phi(t) = \frac{d}{dt}\phi(t) = \phi'(t).$$

Also, they noted that in control theory, a proportional-derivative controller for controlling output P at time t with two tuning parameters has the algorithm

$$P(t) = \kappa_p \Xi(t) + \kappa_d \frac{d}{dt}\Xi(t),$$

where κ_p is the proportional gain, κ_d is the derivative gain, and Ξ is the error between the state-run variable and the progression variable. In this investigation, one can recover the indication of CCC by containing SDO

Definition 2. *Suppose that $\nu \in [0,1]$. An operator $\mathfrak{S}^\mu$ is known as SDO if and only if for any differential function ϕ satisfying*

$$\mathfrak{S}^\mu \phi(t) = \left(\frac{\kappa_1(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right)\phi'(t) - \left(\frac{\kappa_0(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right)\phi'(-t). \tag{1.1}$$

such that $\kappa_1(\mu,t) \neq -\kappa_0(\mu,t)$,

$$\lim_{\mu\to 0}\kappa_1(\mu,t) = 1, \quad \lim_{\mu\to 1}\kappa_1(\mu,t) = 0, \quad \kappa_1(\mu,t) \neq 0, \ \forall t, \ \mu \in (0,1),$$

and

$$\lim_{\mu\to 0}\kappa_0(\mu,t) = 0, \quad \lim_{\mu\to 1}\kappa_0(\mu,t) = 1, \quad \kappa_0(\mu,t) \neq 0, \ \forall t, \mu \in (0,1).$$

It is easy to show the next propositions.

Proposition 1. *For the differential functions ϕ and ψ the following illustrations are held*

1. For all $A, B \in \mathbb{R}$, it indicates $\mathfrak{S}^\mu(A\phi + B\psi) = A\mathfrak{S}^\mu\phi + B\mathfrak{S}^\mu\psi$; ;
2. For all $C \in \mathbb{R}$ it presents $\mathfrak{S}^\mu(C) = 0$;
3. $\mathfrak{S}^\mu(\phi.\psi) = \phi\mathfrak{S}^\mu(\psi) + \psi.\mathfrak{S}^\mu(\phi)$;
4. $\mathfrak{S}^\mu(\phi \diagup \psi) = \frac{\psi\mathfrak{S}^\mu(\phi) - \phi\,\mathfrak{S}^\mu(\psi)}{\psi^2}$; provided $\psi \neq 0$.

Proof. Firstly, we show the following illustration

$$\begin{aligned}
\mathfrak{S}^{\mu}(A\phi+B\psi)(t) &= \left(\frac{\kappa_1(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right)(A\phi+B\psi)'(t)-\left(\frac{\kappa_0(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right)(A\phi+B\psi)'(-t)\\
&= A\left[\left(\frac{\kappa_1(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right)\phi'(t)-\left(\frac{\kappa_0(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right)\phi'(-t)\right]\\
&+ B\left[\left(\frac{\kappa_1(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right)\psi'(t)-\left(\frac{\kappa_0(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right)\psi'(-t)\right]\\
&= A\,\mathfrak{S}^{\mu}\phi(t)+B\,\mathfrak{S}^{\mu}\psi(t).
\end{aligned}$$

For the multiplication, we indicate the following proof

$$\begin{aligned}
\mathfrak{S}^{\mu}(\phi.\psi)(t) &= \left(\frac{\kappa_1(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right)(\phi.\psi)'(t)-\left(\frac{\kappa_0(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right)(\phi\psi)'(-t)\\
&= \left(\frac{\kappa_1(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right)(\phi.\psi'+\psi.\phi')(t)-\left(\frac{\kappa_0(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right)(\phi.\psi'+\psi.\phi')(-t)\\
&= \phi.\left[\left(\frac{\kappa_1(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right)\psi'(t)-\left(\frac{\kappa_0(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right)(\psi')(-t)\right]\\
&+ \psi\left[\left(\frac{\kappa_1(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right)\phi'(t)-\left(\frac{\kappa_0(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right)\phi')(-t)\right]\\
&= \phi.\mathfrak{S}^{\mu}(\psi)+\psi.\mathfrak{S}^{\mu}(\phi).
\end{aligned}$$

Last, we get the next division property

$$\begin{aligned}
&\mathfrak{S}^{\mu}(\phi/\psi)(t)\\
&= \left(\frac{\kappa_1(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right)(\phi/\psi)'(t)-\left(\frac{\kappa_0(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right)(\phi/\psi)'(-t)\\
&= \left(\frac{\kappa_1(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right)(\frac{\phi'\psi-\phi\psi'}{\psi^2})(t)-\left(\frac{\kappa_0(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right)(\frac{\phi'\psi-\phi\psi'}{\psi^2})(-t)\\
&= \frac{\psi.\left[\left(\frac{\kappa_1(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right)\phi'(t)-\left(\frac{\kappa_0(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right)\phi')(-t)\right]}{\psi^2}\\
&- \frac{\phi\left[\left(\frac{\kappa_1(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right)\psi'(t)-\left(\frac{\kappa_0(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right)(\psi')(-t)\right]}{\psi^2}\\
&= \frac{\psi.\mathfrak{S}^{\mu}(\phi)-\phi.\mathfrak{S}^{\mu}(\psi)}{\psi^2}.
\end{aligned}$$

2 Complex conformable calculus

In this investigation, we address a class of normalized analytic functions which is denoted by $\wedge$ and structured by

$$w(z)=z+\sum_{n=2}^{\infty}w_n z^n,\quad z\in\cup, \tag{2.1}$$

where $\cup$ indicates the open unit disk. Using the above series, we present the following definition

Definition 3.*Suppose that $w \in \wedge$ and a parameter $\mu \in [0,1]$. The complex symmetric operator is defined by the construction*

$$\begin{aligned}
\mathfrak{S}^0 w(z) &= w(z)\\
\mathfrak{S}^\mu w(z) &= \left(\frac{\kappa_1(\mu,z)}{\kappa_1(\mu,z)+\kappa_0(\mu,z)}\right)(zw'(z)) - \left(\frac{\kappa_0(\mu,z)}{\kappa_1(\mu,z)+\kappa_0(\mu,z)}\right)(zw'(-z))\\
&= \left(\frac{\kappa_1(\mu,z)}{\kappa_1(\mu,z)+\kappa_0(\mu,z)}\right)\left(z+\sum_{n=2}^{\infty} n w_n z^n\right) - \left(\frac{\kappa_0(\mu,z)}{\kappa_1(\mu,z)+\kappa_0(\mu,z)}\right)\left(-z+\sum_{n=2}^{\infty} n(-1)^n w_n z^n\right)\\
&= z+\sum_{n=2}^{\infty} n\left(\frac{\kappa_1(\mu,z)+(-1)^{n+1}\kappa_0(\mu,z)}{\kappa_1(\mu,z)+\kappa_0(\mu,z)}\right) w_n z^n\\
&:= z+\sum_{n=2}^{\infty} W_n z^n, \quad W_n := n\left(\frac{\kappa_1(\mu,z)+(-1)^{n+1}\kappa_0(\mu,z)}{\kappa_1(\mu,z)+\kappa_0(\mu,z)}\right) w_n,
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
\mathfrak{S}^{2\mu} w(z) &= \mathfrak{S}^\mu[\mathfrak{S}^\mu w(z)] = \mathfrak{S}^\mu[z+\sum_{n=2}^{\infty} W_n z^n]\\
&= \left(\frac{\kappa_1(\mu,z)}{\kappa_1(\mu,z)+\kappa_0(\mu,z)}\right)\left(z+\sum_{n=2}^{\infty} n W_n z^n\right) - \left(\frac{\kappa_0(\mu,z)}{\kappa_1(\mu,z)+\kappa_0(\mu,z)}\right)\left(-z+\sum_{n=2}^{\infty} n(-1)^n W_n z^n\right)\\
&= z+\sum_{n=2}^{\infty} n\left(\frac{\kappa_1(\mu,z)+(-1)^{n+1}\kappa_0(\mu,z)}{\kappa_1(\mu,z)+\kappa_0(\mu,z)}\right) W_n z^n\\
&= z+\sum_{n=2}^{\infty} n^2\left(\frac{\kappa_1(\mu,z)+(-1)^{n+1}\kappa_0(\mu,z)}{\kappa_1(\mu,z)+\kappa_0(\mu,z)}\right)^2 w_n z^n
\end{aligned}$$

$$\vdots$$

$$\mathfrak{S}^{m\mu} w(z) = \mathfrak{S}^\mu[\mathfrak{S}^{(m-1)\mu} w(z)] = z+\sum_{n=2}^{\infty} n^m\left(\frac{\kappa_1(\mu,z)+(-1)^{n+1}\kappa_0(\mu,z)}{\kappa_1(\mu,z)+\kappa_0(\mu,z)}\right)^m w_n z^n.$$

so that $\forall z \in \cup,\ \mu$
$, \in (0,1),\ \kappa_1(\mu,z) \neq -\kappa_0(\mu,z),$

$$\lim_{\mu\to 0}\kappa_1(\mu,z) = 1, \quad \lim_{\mu\to 1}\kappa_1(\mu,z) = 0, \quad \kappa_1(\mu,z) \neq 0,$$

and

$$\lim_{\mu\to 0}\kappa_0(\mu,z) = 0, \quad \lim_{\mu\to 1}\kappa_0(\mu,z) = 1, \quad \kappa_0(\mu,z) \neq 0$$

Obviously, when $m = 0$, we have $w(z)$. In terms of the Hadamard product, we indicate

$$\mathfrak{S}^\mu w(z) = \left(z+\sum_{n=2}^{\infty} n^m\left(\frac{\kappa_1(\mu,z)+(-1)^{n+1}\kappa_0(\mu,z)}{\kappa_1(\mu,z)+\kappa_0(\mu,z)}\right)^m z^n\right) * \left(z+\sum_{n=2}^{\infty} w_n z^n\right).$$

$$\begin{aligned}
\mathfrak{S}^{0.25}\left(\frac{z}{(1-e^{it}z)^2}\right)_z &= z+4e^{(it)}z^2 - 8/3e^{(it)}z^{(5/2)}\\
&+((8e^{(it)})/9+9e^{(2it)})z^3 - 8/27e^{(it)}z^{(7/2)} + ((8e^{(it)})/81+16e^{(3it)})z^4\\
&-8/243(e^{(it)}(1+324e^{(2it)}))z^{(9/2)} + ((8e^{(it)})/729+32/9e^{(3it)}+25e^{(4it)})z^5\\
&-(8(e^{(it)}(1+324e^{(2it)}))z^{(11/2)})/2187+O(z^6)
\end{aligned} \tag{2.3}$$

while in terms of t, we get

$$\mathfrak{S}^{0.25}\left(\frac{z}{(1-e^{it}z)^2}\right)_t = (2ie^{(-it)}(\sqrt(t)+3e^{(2it)})z^2)/(\sqrt(t)+3)+(6ie^{(-2it)}(\sqrt(t)+3e^{(4it)})z^3)/(\sqrt(t)+3) + (12ie^{(-3it)}(\sqrt(t)+3e^{(6it)})z^4)/(\sqrt(t)+3)+(20ie^{(-4it)}(\sqrt(t)+3e^{(8it)})z^5)/(\sqrt(t)+3) + (30ie^{(-5it)}(\sqrt(t)+3e^{(10it)})z^6)/(\sqrt(t)+3)+O(z^7) \tag{2.4}$$

We illustrate the following properties:

Proposition 2. *Let $\mu \in (0,1)$ and the complex operator $\mathfrak{S}^\mu$ in (2.2). Then for $w,u \in \wedge$ and for all $A,BC \in \mathbb{C}$;*

1. $\mathfrak{S}^\mu(Aw+Bu) = A\,\mathfrak{S}^\mu w + B\,\mathfrak{S}^\mu u$;
2. $\mathfrak{S}^\mu(C) = 0$;
3. $\mathfrak{S}^\mu(u.w) = u.\mathfrak{S}^\mu(w) + w.\mathfrak{S}^\mu(u)$;
4. $\mathfrak{S}^\mu(u/w) = \frac{w.\mathfrak{S}^\mu(u) - u.\mathfrak{S}^\mu(w)}{w^2}$; where $w \neq 0$.

Based on the symmetric operators $\mathfrak{S}^\mu$, the 2D complex KZ (CC-KZ) equation is structured by

$$\mathfrak{S}^\mu_t\, w_t(z) + \mathfrak{S}^\mu_z\, w_t(z) = \Phi(w_t(z)), \quad z \in \cup, \tag{2.5}$$

where $w_t(z)$ is a 2D-amplitude during time t and

$$\mathfrak{S}^\mu_t\, w_t(z) = \left(\frac{\kappa_1(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right) w'_t(z) - \left(\frac{\kappa_0(\mu,t)}{\kappa_1(\mu,t)+\kappa_0(\mu,t)}\right) w'_{(-t)}(z)$$

and

$$\mathfrak{S}^\mu_z\, w_t(z) = z + \sum_{n=2}^{\infty} n\left(\frac{\kappa_1(\mu,z)+\kappa_0(\mu,z)(-1)^{n+1}}{\kappa_1(\mu,z)+\kappa_0(\mu,z)}\right) w_n z^n.$$

Moreover, we consider

$$\kappa_0(\mu,t) = \mu t^{1-\mu}, \quad \kappa_0(\mu,z) = \nu z^{1-\mu}, \quad \kappa_1(\mu,t) = (1-\mu)\,t^\mu, \quad \kappa_1(\mu,z) = (1-\mu)\,z^\mu.$$

Our method is indicated by employing the concept of majorization of coefficients problem, which can be defined as follows: let $f(z) = \sum f_n z^n$ and $g(z) = \sum g_n z^n$, $b_n \geq 0$ for all $n \geq 0$, then $f \ll g \Leftrightarrow |f_n| \leq |g_n|$.

It is well known that there is a complete connection between the majorization $(\ll)$ and the subordination $(\prec)$ concepts in the univalent functions theory. We recognize that, the majorization converts the subordination in some settings. In addition, the majorization signifies the maximum bounds of outcomes of differential equations. One can utilize this procedure to estimate the outcome of Eq.(2.5) using well-known functions such as the Koebe rotated function. The estimation can be recognized in various opinions. Initially, for unique objective functions, where the estimation technique indicates how holomorphic functions can be estimated by other types of holomorphic functions which have requisite possessions. Furthermore, the objective function (or cost function) can be used by suggesting convex holomorphic functions [10], subordination and superordination theory [11], or optimization (majorization) utilizing coefficient estimates [12]). Different approaches can be realized in [13–15].

3 Holomorphic outcomes

We proceed to find the holomorphic solution of (2.5). A holomorphic outcome $w_t(z)$ of (2.5) is titled an attractive if and only if the term $\mathfrak{S}^\mu_t\, w_t(z) + \mathfrak{S}^\mu_z\, w_t(z)$ is majorized by the functional $\Phi(w_t(z))$, which means that $|c_n| \leq |\varphi_n|$, where

$$\mathfrak{S}^\mu_t w_t(z) + \mathfrak{S}^\mu_z w_t(z) = \sum c_n z^n$$

and

$$\Phi(w) = \sum \varphi_n z^n, \quad \varphi_n > 0, \forall n.$$

Meanwhile, the left hand of Eq.(2.5) includes a fractional power in relations with $\mu \in (0,1)$. Consequently, the Berkson-Porta functional $\Phi(w_t(z)) = (\varpi - z)(1-\bar{\varpi}z)\,w_t(z)$, $z \in \cup$ does not fulfill the majority condition because of disappearing the fractional power positions. Accordingly, we recommend a functional $\Phi(w_t(z))$ in positions of the common bilinear functional

$$\mathfrak{B}^{\ell}(z) = \left(\frac{(\eta z)+1}{-(\varsigma z)+1}\right)^{\ell}, \quad \eta,\varsigma \in \bar{\cup}, z \in \cup\, \ell \in [0,\infty).$$

In view of $\mathfrak{B}^{\ell}(z)$, we define the following functional (see Fig.3.1)

$$\Phi^{\mu}\left(w_t(z)\right) = \left(\frac{1+\sqrt{z}}{1-z}\right)^{\mu} w_t(z), \tag{3.1}$$

$$\left(z \in \cup,\, \mu \in (0,1),\, \eta = \left(\frac{1}{\sqrt{z}}\right), \varsigma = 1,\right)$$

admitting, for example, the series

$$\begin{aligned}
&\left(\frac{1+\sqrt{z}}{1-z}\right)^{0.25} w_t(z) = z + 0.25z^{3/2} + z^2(0.15625 + 2e^{it}) + z^{5/2}(0.117188 + 0.5e^{(it)}) \\
&+ z^3(0.0952148 + 0.3125e^{(it)} + 3e^{(2it)}) + z^{(7/2)}(0.0809326 + 0.234375e^{(it)} + 0.75e^{(2it)}) \\
&+ z^4(0.070816 + 0.19043e^{(it)} + 0.46875e^{(2it)} + 4e^{(3it)}) + z^{(9/2)}(0.0632286 + 0.161865e^{(it)} + 0.351563e^{(2it)} + e^{(3it)}) \\
&+ z^5(0.0573009 + 0.141632e^{(it)} + 0.285645e^{(2it)} + 0.625e^{(3it)} + 5e^{(4it)}) \\
&+ z^{(11/2)}(0.0525258 + 0.126457e^{(it)} + 0.242798e^{(2it)} + 0.46875e^{(3it)} + 1.25e^{(4it)}) + O(z^6)
\end{aligned} \tag{3.2}$$

Eq.(3.2) can be estimated by consuming the total roots utilizing the parametric connections as follows:

$$\begin{aligned}
\Phi^{0.25}(w_t(z)) \simeq\, & z + 0.25z^{(3/2)} + 9.5z^2 + 9.5z^{(5/2)} + 12.5z^3 + 12.5z^{(7/2)} + 12.5z^4 \\
& + 18.5z^{(9/2)} + 19.0z^5 + 19.0z^{(11/2)} + O(z^6).
\end{aligned} \tag{3.3}$$

Correspondingly, we indicate $\Phi^{0.5}(w_t(z))$ and its estimation respectively,

$$\begin{aligned}
&\left(\frac{1+\sqrt{z}}{1-z}\right)^{0.5} w_t(z) = z + \frac{z^{3/2}}{2} + (2e^{(it)} + 3/8)z^2 + (5/16 + e^{(it)})z^{(5/2)} + ((3e^{(it)})/4 + 3e^{(2it)} + .27)z^3 \\
&+ ((5e^{(it)})/8 + 3/2e^{(2it)} + .24)z^{(7/2)} + ((35e^{(it)})/64 + 9/8e^{(2it)} + 4e^(3it) + .22)z^4 + ((63e^{(it)})/128 \\
&+ 15/16e^{(2it)} + 2e^{(3it)} + 0.2)z^{(9/2)} + ((231e^{(it)})/512 + 105/128e^{(2it)} + 3/2e^{(3it)} \\
&+ 5e^{(4it)} + 0.2)z^5 + ((0.4e^{(it)}) + 0.7e^{(2it)} + 5/4e^{(3it)} + 5/2e^{(4it)} + 0.2)z^{(11/2)} + O(z^6)
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
\Phi^{0.5}(w_t(z)) \simeq\, & z + \frac{z^{3/2}}{2} + 9.5z^2 + 9.5z^{(5/2)} + 9.5z^3 \\
& + 12.5z^{(7/2)} + 17.5z^4 + 17.5z^{(9/2)} + 24z^5 + 24z^{11/2} + O(z^6).
\end{aligned} \tag{3.5}$$

Proposition 3. *Assume the wave equation (2.5)-(3.1). Then $w_t(z) = \frac{z}{(1-e^{it}z)^2}$ (the rotated Koebe function) is a holomorphic attractive outcome for (2.5).*

Proof. In view of $\mathfrak{S}^{\mu}$, we check two cases for the fractional values $\mu = 0.25$ and $\mu = 0.5$.

$$\begin{aligned}
&\mathfrak{S}_t^{0.25}\frac{z}{(1-e^{it}z)^2} + \Delta_z^{0.25}\frac{z}{(1-e^{it}z)^2} \\
&= (2ie^{(-it)}(\sqrt{t} + 3e^{(2it)})z^2)/(\sqrt{t}+3) + (6ie^{(-2it)}(\sqrt{t} + 3e^{(4it)})z^3)/(\sqrt{t}+3) \\
&+ (12ie^{(-3it)}(\sqrt{t} + 3e^{(6it)})z^4)/(\sqrt{t}+3) + (20ie^{(-4it)}(\sqrt{t} + 3e^{(8it)})z^5)/(\sqrt{t}+3) \\
&+ (30ie^{(-5it)}(\sqrt{t} + 3e^{(10it)})z^6)/(\sqrt{t}+3) + O(z^7) \\
&+ z + 4e^{(it)}z^2 - 8/3e^{(it)}z^{(5/2)} + ((8e^{(it)})/9 + 9e^{(2it)})z^3 - 8/27e^{(it)}z^{(7/2)} + ((8e^{(it)})/81 \\
&+ 16e^{(3it)})z^4 - 8/243(e^{(it)}(1 + 324e^{(2it)}))z^{(9/2)} + ((8e^{(it)})/729 + 32/9e^{(3it)} + 25e^{(4it)})z^5 \\
&- (8(e^{(it)}(1 + 324e^{(2it)}))z^{(11/2)})/2187 + O(z^6) \\
&\approx z + 7.5z^2 + 8/3z^{5/2} + 1.34z^3 + 8/27z^{7/2} + 1.14z^4 + 8/234z^{(9/2)} + 1.14z^5 + O(z^6)
\end{aligned} \tag{3.6}$$

The coefficients of the last equation are calculated by using the total roots in terms of t, for example the coefficient of z^2 appears by finding the roots of

$$\frac{(2ie^{(-it)}(\sqrt{t}+3e^{(2it)}))}{(\sqrt{t}+3)+4e^{(it)}}=0.$$

This implies three roots $t_1=-0.567673+1.03527i$, $t_2=2.26348+0.915718i$, $t_3=5.38212+0.782888i$. By taking the total radius $|t_1+t_2+t_3|\approx 7.5$. Comparing Eq.(3.3) and Eq.(3.6), we conclude that $\mathfrak{S}_t^{0.25}\frac{z}{(1-e^{it}z)^2}+\mathfrak{S}_z^{0.25}\frac{z}{(1-e^{it}z)^2}$ is optimized by the function $\Phi^{0.25}(w_t(z))$. Correspondingly, $\mu=0.5$ gives

$$\begin{aligned}
&\mathfrak{S}_t^{0.5}\frac{z}{(1-e^{it}z)^2}+\Delta_z^{0.5}\frac{z}{(1-e^{it}z)^2}\\
&=(ie^{(-it)}+ie^{(it)})z^2+((3i)e^{(-2it)}+(3i)e^{(2it)})z^3+((6i)e^{(-3it)}+(6i)e^{(3it)})z^4+((10i)e^{(-4it)}\\
&+(10i)e^{(4it)})z^5+((15i)e^{(-5it)}+(15i)e^{(5it)})z^6+O(z^7)+z+9e^{(2it)}z^3+25e^{(4it)}z^5+O(z^6)\\
&\approx z+4.7z^2+1.13z^3+11z^4+0.55z^5+O(z^6).
\end{aligned}\tag{3.7}$$

Which means that $\mathfrak{S}_t^{0.5}\frac{z}{(1-e^{it}z)^2}+\mathfrak{S}_z^{0.5}\frac{z}{(1-e^{it}z)^2}$ is optimized by the $\Phi^{0.5}(w_t(z))$. And it holds for all $\mu\in(0,1)$.

Proposition 4. *Consume the wave equation (2.5)-(3.1). Then it indicates a probability of measure $\wp$ on $(\partial\cup)^2$, for $\mu\to 1$ achieving*

$$\int_{(\partial\cup)^2}\Phi(w_t(z))\,d\wp.$$

Proof. Consume that for $\flat,\natural\in\partial\cup$ achieving $\flat=1/\sqrt{z}, |z|<1$ then $|\flat|=1$ and

$$\left(\frac{1+\flat z}{1+\natural z}\right)^{\mu}=\frac{(1+z^{0.5})^{\mu}}{1+\natural z}.\frac{1}{(1+\natural z)^{\mu-1}}\ll\frac{(1+z^{0.5})^{\mu}}{1-z}.\frac{1}{(1-z)^{\mu-1}}=\left(\frac{1+z^{0.5}}{1-z}\right)^{\mu},\quad \mu\to 1.\tag{3.8}$$

According to Theorem 1.11 in [16], the $\left(\frac{1+\flat z}{1+\flat z}\right)^{\mu}$ indicates a probability of measure $\wp$ in $(\partial\cup)^2$ achieving

$$\phi(z)=\int_{(\partial\cup)^2}\left(\frac{1+\flat z}{1+\natural z}\right)^{\mu}d\wp(\flat,\natural),\quad z\in\cup.$$

Then in view of Proposition 3, there is a diffusion constant $\Bbbk$ achieving

$$\int_{(\partial\cup)^2}\left(\frac{1+\flat z}{1+\natural z}\right)^{\mu}d\wp(\flat,\natural)=\Bbbk\int_{(\partial\cup)^2}\left(\frac{1+\flat z}{1-\natural z}\right)^{\mu}w_t(z)d\wp(\flat,\natural),\quad z\in\cup$$

or $\phi(z)=\Bbbk\int_{(\partial\cup)^2}\Phi(w_t(z))\,d\wp(\flat,\natural)$ occurs.

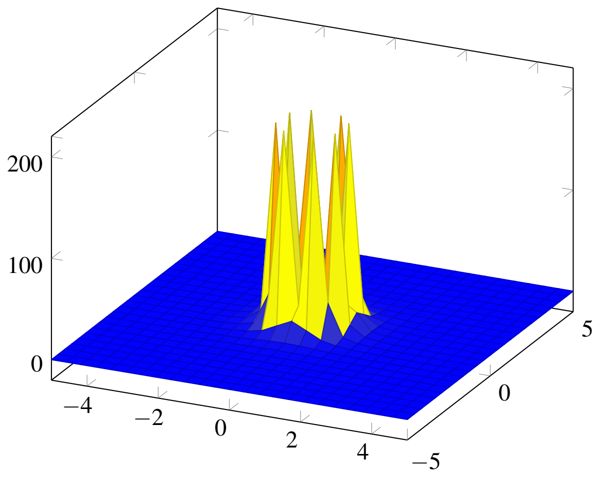

Fig. 3.1: The holomorphic outcome of Eq. (2.5)-(3.1)

Acknowledgment: This effort is sustained by the University Ajman grant: 2019-IRG-HBS-11.

References

[1] M.F. Hamilton and D. T. Blackstock, *Nonlinear acoustic*, Academic Press, San Diego, California, 1998.
[2] A. M. Kamchatnov and M. V. Pavlov, On exact solutions of nonlinear acoustic equations, *Wave motion* **67**, 81–88 (2016).
[3] O. V. Rudenko and S. I. Soluyan, *Theoretical foundations of nonlinear acoustics*, Plenum, New York, 1977.
[4] F. Prieur and S. Holm, Nonlinear acoustic wave equations with fractional loss operators, *J. Acous. Soc. Amer.* **3**, 1125–1132 (2011).
[5] S. Ray Saha, New exact solutions of nonlinear fractional acoustic wave equations in ultrasound, *Comput. Math. Appl.* **71**(3) 859–868 (2016).
[6] S. Ray Saha, New analytical exact solutions of time fractional KdV-KZK equation by Kudryashov methods, *Chinese Phys.* **B 25**(4) 040204 (2016).
[7] R. W. Ibrahim, Fractional complex transforms for fractional differential equations, *Adv. Differ. Equ.* **2012**(1), 192(2012) .
[8] D. R. Anderson and D. J. Ulness, Newly defined conformable derivatives, *Adv. Dyn. Syst. Appl.* **10**(2), 109-137(2015).
[9] L. Broer and P. H. A. Sarluy, On simple waves in non-linear dielectric media, *Physica* **30**(7), 1421-1432 (1964).
[10] R. W. Ibrahim and M. Darus, Analytic study of complex fractional Tsallis' entropy with applications in CNNs, *Entropy* **20**(10), 722 (2018).
[11] R. W. Ibrahim and M. Darus, Subordination and superordination for univalent solutions for fractional differential equations, *J. Math. Anal. Appl.* **345**(2), 871-879 (2008).
[12] R. W. Ibrahim, Existence and uniqueness of holomorphic solutions for fractional Cauchy problem, *J. Math. Anal. Appl.* **380**(1), 232-240 (2011).
[13] M. S. Ghanbari, M. S. Osman and D. Baleanu, Generalized exponential rational function method for extended Zakharov–Kuzetsov equation with conformable derivative, *Mod. Phys. Lett. A*, 1950155 (2019).
[14] J. Shumaila, S. Riaz, K. S. Alimgeer, M. Atif, A. Hanif and D. Baleanu, First integral technique for finding exact solutions of higher dimensional mathematical physics models, *Symmetry* **11**(6), 783 (2019).
[15] M. A. Fahad, D. Baleanu and E. Abdelhaim, *Chin. J. Phys.* **58**, 18-28 (2019).
[16] St. Ruscheweyh, *Convolutions in G geometric function theory*, Presses Univ, Montreal, Montreal, Que, 1982.