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A Study on Fractional Taylor Series and Variations

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Abstract: Fractional mechanics has been recently one of the most efficient branches of mechanics, interpreting successfully various experiments. Two of the most powerful tools for solving mechanics problem, i.e Taylor's series and variational approaches are discussed in the context of fractional analysis.Λ-fractional derivative is introduced and fractional Taylor's series is established, along with the fractional calculus of variations. In addition to the aforementioned derivative the according fractional Λ-space is defined in which the derivative satisfies all the conditions of a derivative required by differential topology. In fact the fractional derivatives in the initial space correspond to the Λ-derivatives in the Λ-space, behaving like the conventional ones in that space. Since geometry is feasible only in the Λ-fractional space, the analysis is performed in that space. Then the results are also derived in the initial space. The fractional beam bending problem is addressed as an application of fractional calculus of variations. Further application involving combination of the fractional variational approach and fractional Taylor's series is presented, analyzing the fractional stability and post-stability problem of a rod, under axial compression.

Keywords: $\Lambda-$fractional derivative, fractional $\Lambda-$ space, fractional Taylor's series, fractional calculus of variations, fractional beam bending problem, branching of solutions, post-critical behavior.

1 Introduction, motivation and preliminaries

Fractional calculus was established by Leibniz [1], Liouville [2], and Riemann [3]. It is a very robust and promising field in science, and in mathematics. Leibniz posed the problem of the definition of not only the fractional derivative but also the fractional differential. In continuum mechanics that field is mostly useful in material models based upon fractional time derivatives, in studying their viscoelastic interaction [4,5]. Moreover, Lazopoulos [6] has proposed a fractional model, for long-range (non-local) dependence of phenomena, lifting Noll's locality axiom. That model describes the behavior of those phenomena more accurately. Furthermore, fractional analysis has been recalled, just to mathematically analyze fractals [7]. Non-local problems into the context of fractional calculus [8] have also been proposed to non-local mechanics. Further applications in various physical areas may be also found in various books [9,10,11,12]. Fractional differential geometry has not been established yet, since fractional derivatives do not meet the requirements of differential topology [18],for defining a differential. In that context Adda [17,23] has provided the differential of the fractional order, which is a nonlinear function of the simple differentials of the variables. In addition, Calcani [14] has presented various aspects of the differential geometry of fractional order. Nevertheless, the main problem of the existence of fractional differential, generating fractional differential geometry is still open. In addition, Baleanu et al. [13], Atanackovic et al. [15,16], Agraval [21], Almeida et al. [22], Baleanu et al. [30,31,32] have presented variational problems with fractional derivatives. Thus the fractional derivatives do not possess the requirements of a derivative corresponding to differential as required for the definition of a fractional differential. Three basic prerequisites are required for defining a derivative corresponding to a differential [33]:

Linearity: D(af(x)+bg(x))=aDf(x)+bDg(x),

Leibniz rule: D(f(x)g(x))=Df(x)g(x)+f(x)Dg(x),

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Chain rule: D(g(f))(x)=Dg(f(x))• Df(x).

Those properties are important for the definition of a differential associated with definition of a derivative. The frequent defined fractional derivatives violate one or more of the above-mentioned properties thus their use in mathematics and physics is questionable. The aforementioned issues limit the power of Fractional Calculus drastically. Since a proper fractional differential cannot be defined, solving fractional differential equations is questionable. That has a major influence on physical problems, defined by physical quantities such as, fractional velocities, fractional accelerations, fractional strains etc. The authors managed to tackle those problems through proposing a new fractional derivative [19,20], the $\Lambda-$fractional derivative along with fractional $\Lambda-$space. That derivative possess all the properties of a conventional derivative and behaves in a conventional manner so that the structure of a differential is possible in that space. Therefore, working on the fractional $\Lambda-$space, the results may be pulled back to the initial space.

The present article introduces the $\Lambda-$fractional derivative in fractional Taylor series and fractional variational problems in Section 2. Taylor series is a robust mathematical tool whose value is proven and cherished throughout the scientific community as it is explained in Section 3. Especially in mechanics, Taylor's series have been an indispensable tool for analyzing problems:They were used in non-linear mechanics using successive approximations, (Wang et al. [24], Ogden [25]).Moreover they were used in studying post-critical stability problems (Thompson et al.[26]) and generally in formulating computing methods. Moreover the variational problems,as it is mentioned in Section 4, minimizing the energy for solving physical problems, also enjoy the appreciation of scientists and mathematicians. Furthermore, variational procedures are quite important in field theories, geometry, physics etc. Thus, the establishment of a robust procedure analyzing fractional variational problems is a necessary tool of fractional analysis. As an application of the fractional variational calculus, the beam bending problem under constant distributed loading is formulated and investigated. Since branching problems are closely related to Taylor's series expansions, the branching of the equilibrium solutions of an axially compressed rod is explored to demonstrate how the branching of fractional problems may be worked out with the help of $\Lambda-$derivatives. Conclusions are depicted in Section 5.

2 The $\Lambda-$fractional derivative

The present introductory chapter describes the basic definitions of the fractional calculus. Further information may be found in [9,10,11,12]. The fractional integrals are distinguished in left and right ones. Hence for a real fractional order $0 < \gamma \leq 1$ the left fractional derivative is defined by,

$$ {}_aI_x^\gamma f(x) = \frac{1}{\Gamma(\gamma)}\int_a^x \frac{f(s)}{(x-s)^{1-\gamma}}ds, \tag{1}$$

$$ {}_xI_b^\gamma f(x) = \frac{1}{\Gamma(\gamma)}\int_x^b \frac{f(s)}{(s-x)^{1-\gamma}}ds, \tag{2}$$

where γ is the fractional order of the integrals where Γ(x)=(x-1)! with Γ(x) Euler's Gamma function. Furthermore, the corresponding fractional derivatives are defined by:

$$ {}_a^{RL}D_x^\gamma f(x) = \frac{d}{dx}({}_aI_x^{1-\gamma}(f(x))) = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_a^x \frac{f(s)}{(x-s)^\gamma}ds, \tag{3}$$

where the right one is defined by:

$$ {}_x^{RL}D_b^\gamma f(x) = \frac{d}{dx}({}_xI_b^{1-\gamma}(f(x))) = -\frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_x^b \frac{f(s)}{(s-x)^\gamma}ds. \tag{4}$$

The relation connecting the left fractional derivative and the according function is given by:

$$ {}_a^{RL}D_x^\gamma({}_aI_x^\gamma f(x)) = f(x). \tag{5}$$

Similar relations are valid for the right fractional case. The $\Lambda-$fractional derivative is defined as:

$$ {}_a^\Lambda D_x^\gamma f(x) = \frac{{}_a^{RL}D_x^\gamma f(x)}{{}_a^{RL}D_x^\gamma x}. \tag{6}$$

Considering Eq.(3),

$$ {}_a^\Lambda D_x^\gamma f(x) = \frac{\frac{d_aI_x^{1-\gamma}f(x)}{dx}}{\frac{d_aI_x^{1-\gamma}x}{dx}} = \frac{d_aI_x^{1-\gamma}f(x)}{d_aI_x^{1-\gamma}x}, \tag{7}$$

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where

$$X =_a I_x^{1-\gamma} x, \quad F(X) =_a I_x^{1-\gamma} f(x(X)). \tag{8}$$

The $\Lambda-$ fractional derivative in the $\Lambda-$ fractional space (X, F(X)) corresponds to a conventional derivative, exhibiting local properties. Therefore conventional differential calculus along with conventional differential geometry may be developed in the $\Lambda-$space (X,F(X)). Then the results concerning functions themselves may be transferred to the original space. Let us point out that derivatives may not be transferred into the original space, since fractional derivatives do not exist in the original space.

$$f(x) =_a^{RL} D_x^{1-\gamma} F(X(x)) =_a^{RL} D_x^{1-\gamma} I_x^{1-\gamma} f(x). \tag{9}$$

Applying the aforementioned theory, let us consider:

$$f(x) = x^2. \tag{10}$$

Then the $\Lambda-$ plane (X,F(X)) is defined by

$$X = \frac{x^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)}, \tag{11}$$

$$F(X) =_a I_x^{1-\gamma} f(x(X)) =_a I_x^{\gamma} f(x) = \frac{1}{\Gamma(1-\gamma)} \int_a^x \frac{s^2}{(x-s)^{\gamma}} ds = \frac{2}{(-6+11\gamma-6\gamma^2+\gamma^3)\Gamma(1-\gamma)} x^{(3-\gamma)}. \tag{12}$$

Further considering from Eq.(11),

$$x = ((2-3\gamma+\gamma^2)\Gamma(1-\gamma)X)^{\frac{1}{2-\gamma}}. \tag{13}$$

Eq.(13) yields

$$F(X) = -\frac{2(((2-3\gamma+\gamma^2)\Gamma(1-\gamma)X)^{\frac{1}{2-\gamma}})^{3-\gamma}}{\Gamma(1-\gamma)(-6+11\gamma-6\gamma^2+\gamma^3)}. \tag{14}$$

Hence, the curve in the original plane (x, f(x)) shown in Fig. 1. corresponds to the fractional plane (space) shown in Fig.2,

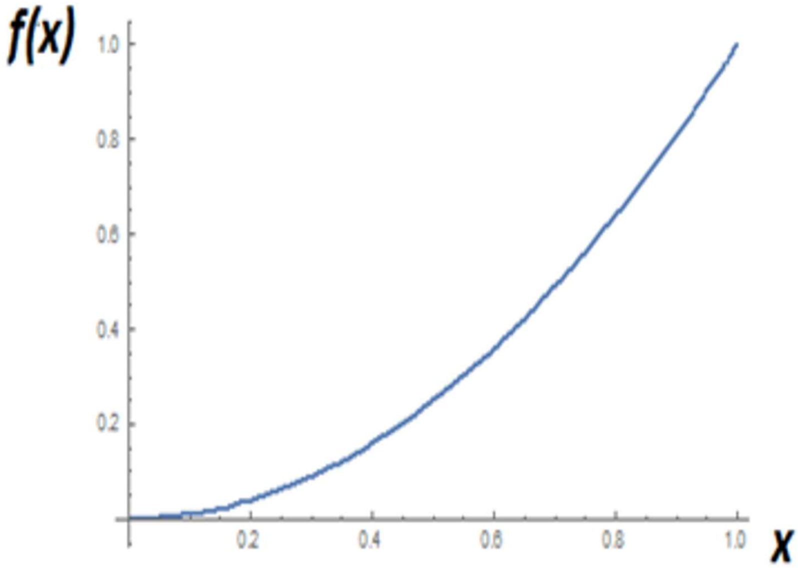

Fig. 1: The curve in the initial space.

for γ=0.6. Further, the derivative,

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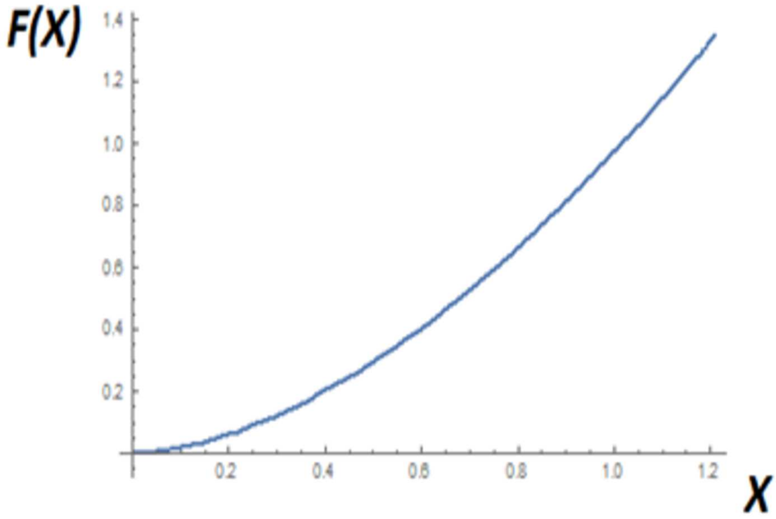

Fig. 2: Corresponding curve in the Λ-plane when γ=0.6.

$$F(X) = \frac{24(5-\gamma)(2-3\gamma+\gamma^2)\Gamma(1-\gamma)(M)^{\frac{3}{2-\gamma}}}{(2-\gamma)\Gamma(6-\gamma)} \quad (15)$$

with $M = (2-3\gamma+\gamma^2)X\Gamma(1-\gamma)$. For X_0=0.6 and γ=0.6, the corresponding derivative in the fractional $\Lambda-$space is equal to $D(F(X_0))$= 1.34. Consequently, the tangent Y(X) of the curve at a point X_0 in the $\Lambda-$space is the line,

$$Y(X) = F(X_0) + \frac{d(F(X_0))}{dX}(X - X_0). \quad (16)$$

Let us point out that geometry may not be transferred in the original space, since formulation of fractional differential is

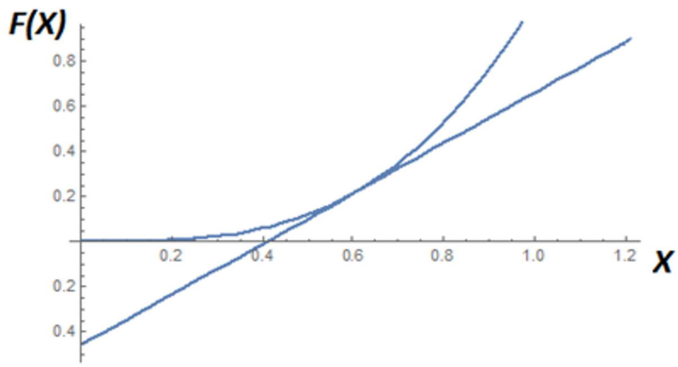

Fig. 3: Figure 3.The corresponding in the fractional $\Lambda-$Plane with its tangent at X_0.

not possible in the initial space (x, f(x)). Functions may be transferred only from the fractional $\Lambda-$space to the original

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space. In addition, for the derivation it is unnecessary to express the various functions with respect to X. Since

$$\frac{d(F(X))}{dX} = \frac{dF(X(x))/dx}{d(X(x))/dx} \tag{17}$$

there is no need to perform the substitution of x for the X variable. The calculus through the x variable is simpler but also effective.

3 Fractional Taylor series

Let us consider a function f(x). Suppose that in a neighbourhood of a point x=a the function y=f(x) has derivatives up to the order (n+1). Then, according to Taylor's formula for the conventional calculus, for all x in that neighbourhood, the following holds:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^n(a)}{2!}(x-a)^n + o(x-a)^n. \tag{18}$$

However, the use of the well-known fractional derivatives for building a formula similar to Taylor's has not been successful, since the well- known fractional derivatives cannot satisfy the conditions demanded by differential topology [17,18,27] just to correspond to differentials. Nevertheless the Λ-FD behaves in the fractional Λ-space according to the differential topology demands. The initial space (x,f(x)) is substituted by the fractional Λ-space (X,F(X)), where,

$$X =_a I_x^{1-\gamma} x, \quad F(X) =_a I_x^{1-\gamma} f(x). \tag{19}$$

Moreover, Eq.(19) yields

$$X = \frac{x^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)}. \tag{20}$$

In addition, Eq.(19) implies that:

$$F(x) =_a I_x^{1-\gamma} f(x) = \frac{1}{\Gamma(1-\gamma)} \int_a^x \frac{f(s)}{(x-s)^\gamma} ds. \tag{21}$$

Inverting Eq.(20) we get

$$x = ((2-3\gamma+\gamma^2)\Gamma(1-\gamma)X)^{\frac{1}{2-\gamma}} = x(X). \tag{22}$$

Inserting Eq.(22) into Eq.(21), we have obtained the function f(x) expressed in the X variable,

$$F(X) = F(x(X)). \tag{23}$$

For better explanation between the original and the fractional $\Lambda-$space, consider the function

$$f(x) = x^5 \tag{24}$$

According to Eq.(21),

$$F(x) = \frac{1}{\Gamma(1-\gamma)} \int_a^x \frac{s^5}{(x-s)^\gamma} ds = \frac{120 x^{6-\gamma}}{\Gamma(7-\gamma)}. \tag{25}$$

Nevertheless, the fractional Λ-space is formulated if we substitute x from Eq.(22). Indeed,

$$F(X) = \frac{120(((2-3\gamma+\gamma^2)\Gamma(1-\gamma)X)^{\frac{1}{2-\gamma}})^{6-\gamma}}{\Gamma(7-\gamma)}. \tag{26}$$

Figures (4,5) clarify the original and the fractional $\Lambda-$space. Let us point out that the Λ-FD is defined as

$${}_a^\Lambda D_x^\gamma f(x) = \frac{d_a I_x^{1-\gamma} f(x)}{d_a I_x^{1-\gamma} x} = \frac{dF(X)}{dX}. \tag{27}$$

Therefore, the Λ-FD in the fractional Λ-space behaves like the conventional derivative. To illustrate the point, the tangent space in the fractional Λ-space is shown in Fig.4 at the point X=0.7. The tangent line of the curve F(X) in the Λ-space at

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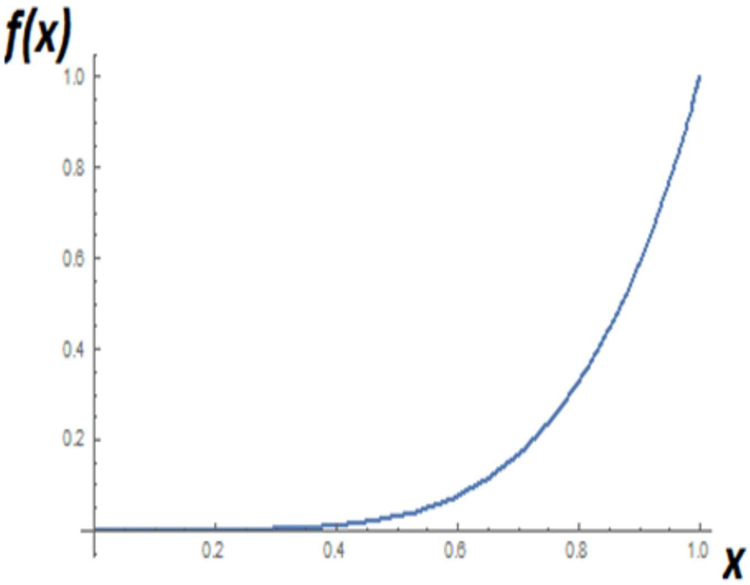

Fig. 4: The function $f(x) = x^5$ in the initial space.

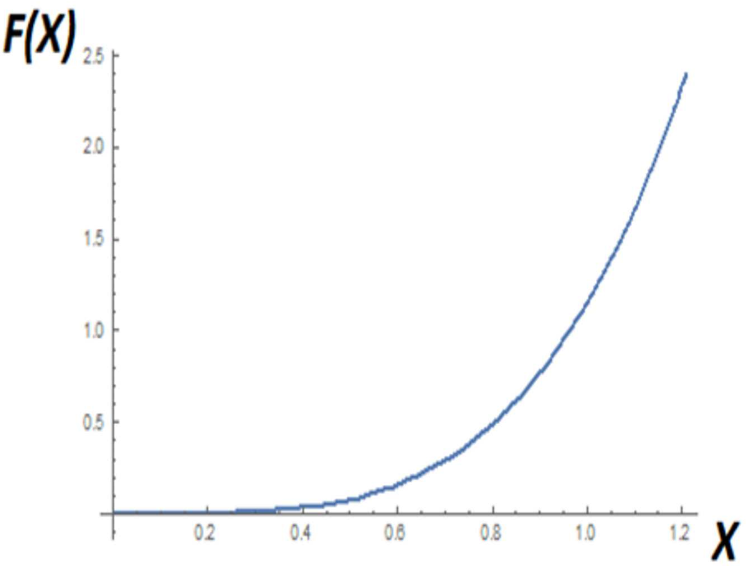

Fig. 5: The curve in the $\Lambda -$ fractional plane.

the point X=0.7 is defined by the equation:

$$G(X) = F(X_0) + \frac{dF(X_0)}{dX}(X - X_0) = (1.15X^{3.86})_{X=0.7} + (4.44X^{3.86})_{X=0.7}(X - 0.7). \tag{28}$$

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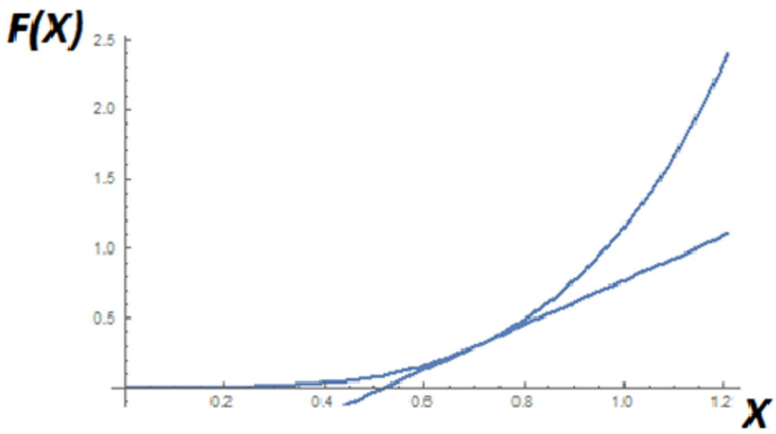

Fig. 6: The curve in the fractional Λ-space with its tangent space at X=0.7.

Taking into consideration the correspondence between x and X, Eq.(22), the point $X_0 = 0.7$ for γ=0.6 corresponds to the point $x_0 = 0.9$. However, configurating the tangent G(X) into the original space we find the curve

$$g(x) = f(x)_{x=0.9} + ({}_a^C D_x^\gamma f(x)(\frac{dF(X_0)}{dX})_{\gamma=0.6})(\frac{x^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)} - 0.7)_{\gamma=0.6}. \quad (29)$$

Performing the necessary calculus with the help of the Mathematica pack, the curve g(x) is defined by the equation

$$g(x) = 0.59 + 2.93(0.81x^{1.4} - 0.7). \quad (30)$$

Fig.5 shows the initial curve with the configuration in the original space of the tangent space in the Λ-space. It may be

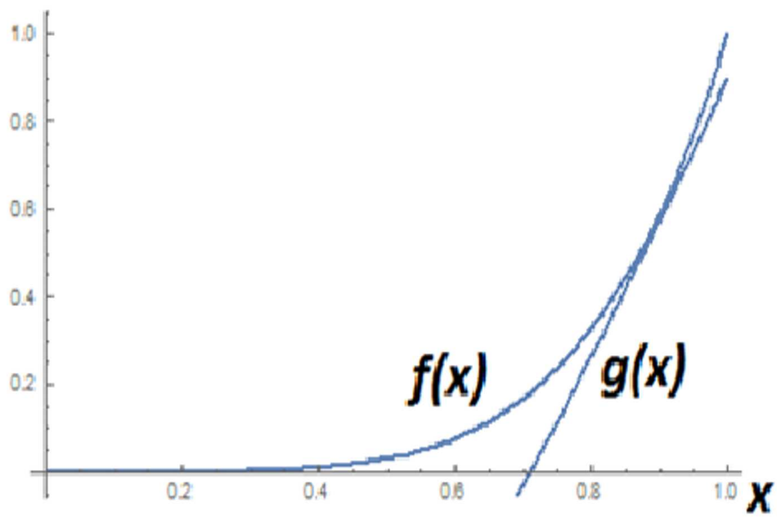

Fig. 7: Figure 7.The original plane with the curve f(x) and the transferring of the fractional tangent.

concluded by Fig.5, that the curve with the fractional tangent space is very close to the conventional tangent space at the contact point. Coming back to the main topic of the fractional Taylor's formula, the function F(X) is considered in the fractional Λ-space. Then F(X) behaves in that space according to conventional mathematical calculus. Hence,

$$F(X) = F(B) + \frac{dF(B)}{dX}(X-B) + \frac{1}{2!}\frac{d^2F(B)}{dX^2}(X-B)^2 + ... + R_n(X). \tag{31}$$

Since, for every Y(X) in the Λ-space there exists a y(x) with,

$$x(X) = ((2-3\gamma+\gamma^2)\Gamma(1-\gamma)X)^{\frac{1}{2-\gamma}} \tag{32}$$

according to Eq.(22), and since,

$$Y(X) =_a I_x^{1-\gamma} y(x) \tag{33}$$

basic property of the fractional derivative yields,

$$y(x) =_a^{RL} D_x^{1-\gamma} Y(X(x)). \tag{34}$$

Therefore, the fractional Taylor's formula in the original space is defined by

$$f(x) = f(b) + \frac{1}{1!}({}_a^C D_x^{1-\gamma}(\frac{dF(X)}{dX}))(\frac{x^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)} - B) + \frac{1}{2!}({}_a^C D_x^{1-\gamma}(\frac{d^2F(B)}{dX^2}))(\frac{x^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)} - B)^2 + ... + R_n(X). \tag{35}$$

4 The fractional variational problem

It is well known that the extremal (minimum) of the integral,

$$I = \int_{X_1}^{X_2} F(X,Y,Y')dX \tag{36}$$

in the Λ-space, is defined by the Euler-Lagrange equation,

$$\frac{\partial F}{\partial Y} - \frac{d}{dX}(\frac{\partial F}{\partial Y'}) = 0. \tag{37}$$

In fractional analysis the integral Eq.(36) along with the Euler-Lagrange equation are valid in the Λ-space, where the derivatives are the Λ-fractional derivatives that behave in that space according to the conventional analysis rules. As far as the boundary conditions are concerned, the natural b.cs are considered in the Λ-space, and then transferred in the initial. The bending problem that is considered as a second application clarifies the procedure. Solving the differential equation in the Λ-space, the solution is transferred to original plane through the Riemann-Liouville derivative, see Eq.(34).

4.1 Applications

As a first application, fractional Taylor's expansion will be performed for the function $f(x) = x^5$ around the point x=0. Indeed, for the function in the original space (x,f(x)), the fractional Λ-plane is defined by (X,F(X)), where X is defined by Eq.(20) and F(X) by Eq.(21). Then in that space, the Λ- fractional derivative,

$${}_a^{\Lambda} D_x^{\gamma} f(x) = \frac{dF(X)}{dX} \tag{38}$$

behaves like a conventional one. Hence,

$${}_a^{\Lambda} D_x^{\gamma}({}_a^{\Lambda} D_x^{\gamma} f(x)) = \frac{d^2F(X)}{dX^2}. \tag{39}$$

The same is valid for the nth Λ-FD. Therefore the conventional Taylor's formula in the Λ- fractional plane is defined by Eq.(35). For the specific case with the fractional order γ=0.6 and X=0.6 and a=0, Taylor's expansion in the fractional Λ-space is given by:

$$F(X) = 0.1603 + 1.0307(X-0.6) + 2.4539(X-0.6)^2 + ... + R_n(X). \tag{40}$$

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Recalling Eq.(22), X=0.6 corresponds to x=0.81. Since F(X) is given by Eq.(26)

$$F(X) = \frac{120(((2-3\gamma+\gamma^2)\Gamma(1-\gamma)X)^{\frac{1}{2-\gamma}})^{6-\gamma}}{\Gamma(7-\gamma)}. \tag{41}$$

Taylor's expansion in the original space is expressed by Eq.(34) and for B=0.6 and b=0.81,

$$f(x) = 0.3487 + 2.0053(0.8050x^{1.4} - 0.6) + 9.7675(0.8050x^{1.4} - 0.6)^2 + ... + R_n(X). \tag{42}$$

Fig.8 shows the coincidence of the initial function and Taylor's expansion in the neighborhood of the point x=0.81.

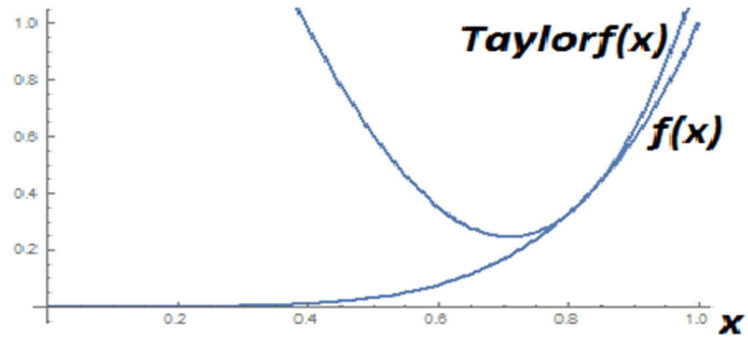

Fig. 8: The function f(x) and its fractional Taylor's expansion at the point x=0.81.

4.2 *The fractional beam bending*

As a second application in the area of fractional calculus of variations, the fractional bending of a beam will be considered. The total potential of the conventional system is defined by,

$$V = \int_0^L [EI(y'')^2 - py]dx. \tag{43}$$

Furthermore the governing equilibrium equation for the deflection y(x) is defined by the minimum of the total potential V. For the fractional bending of the beam the potential

$$V_1 = \int_0^{L_1} [\frac{1}{2}EI(Y''(X))^2 - pY(X))]dX = \int_0^{L_1} F(Y''(X)^2, Y(X))dX \tag{44}$$

is considered, with

$$Y(X) = {}_aI_x^{1-\gamma}y(x) = \frac{1}{\Gamma(1-\gamma)}\int_a^x \frac{y(s)}{(x-s)^\gamma}ds, \tag{45}$$

and,

$$X = \frac{x^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)}. \tag{46}$$

Since in the fractional Λ-space everything behaves exactly in the conventional way, the fractional variation of V_1 is governed by the equation,

$$\frac{d}{dX^2}(\frac{\partial F}{\partial Y''}) + \frac{\partial F}{\partial Y} = 0 \tag{47}$$

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that yields

$$(EIY''(X))'' = p. \tag{48}$$

The Eq.(47) accepts the solution,

$$Y(X) = \frac{p}{EI}\frac{X^4}{4!} + C_1\frac{X^3}{3!} + C_2\frac{X^2}{2!} + C_3X + C_4. \tag{49}$$

According to Eq.(46), the fractional axial length of the beam in theΛ-space equals:

$$L_1 = \frac{L^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)}. \tag{50}$$

Consequently the natural boundary conditions, denoting zero fractional bending moments at the ends X=0, L_1 of the beam demand,

$$Y''(0) = Y''(L_1) = 0, \tag{51}$$

$$C_1 = \frac{p}{EI}\frac{L_1}{2} \tag{52}$$

and again Eq.(46) yields,

$$X = \frac{x^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)}. \tag{53}$$

Consequently, introducing the value of X into Eq.(49) we get for $\gamma = 0.6, L = 1$,

$$y(x) = {}_a^C D_x^{1-\gamma} Y(X) = 1.599C_3x^{0.8} - 0.1269\frac{p}{EI}x^3.6 + 0.0748\frac{p}{EI}x^5 \tag{54}$$

describing the elastic line in the original x,y space. The place boundary conditions in the original space are

$$y(0) = y(1) = 0. \tag{55}$$

Therefore,

$$C_3 = 0.0169\frac{p}{EI} \tag{56}$$

and the fractional elastic line is defined by the curve

$$y^l(x) = \frac{p}{EI}(0.0749x^5 - 0.1269x^{3.6} + 0.0521x^{0.8}). \tag{57}$$

Considering also the effect of the right fractional derivative we get the symmetric elastic curve,

$$y^r(x) = \frac{p}{EI}(0.0749(1-x)^5 - 0.1269(1-x)^{3.6} + 0.0521(1-x)^{0.8}). \tag{58}$$

Therefore, the elastic curve is described by the mean value of the left and right y(x) values

$$y(x) = \frac{1}{2}(y^l(x) + y^r(x)). \tag{59}$$

Let us point out that the conventional ($\gamma = 1$) elastic line of the bending beam is defined by,

$$y(x) = \frac{p}{EI}\frac{(x^4 - 2x^3 + x)}{24}. \tag{60}$$

Fig.9 shows the fractional elastic line of the bending beam and the conventional one.

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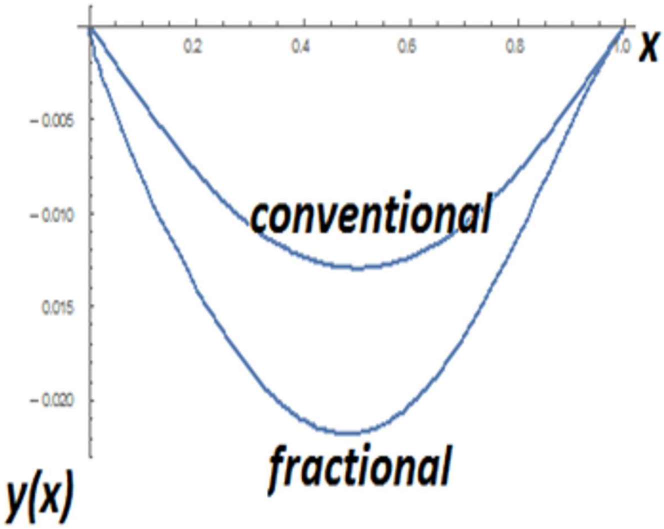

Fig. 9: The fractional and the conventional beam elastic curve.

4.3 The branching problem of a compressed fractional rod

To implement both the variational problem and Taylor expansions, the branching problem of a fractional compressed rod is going to be discussed in the present chapter. Let us consider a simply supported rod in the Λ-space of length L compressed by an axial force P. Then the free energy V of the bar is equal to:

$$V = EI\int_0^L [(\frac{Y''(X)}{\sqrt{1+Y'(X)^2}})^2 - P(1-\frac{1}{\sqrt{1+Y'(X)^2}})]ds. \tag{61}$$

Since for small deformations, $|Y'(X)| << 1$, Eq.(61) becomes:

$$V = \int_0^L [\frac{EI}{2}((Y''(X)(1-Y'(X)^2)))^2 - P(1-(1+\frac{1}{2}Y'(X)^2))]ds. \tag{62}$$

The equilibrium equation for the compressed rod in the Λ-space becomes,

$$Y^{IV}(X) + k^2 Y''(X) = Y^{IV}(X)Y'(X)^2 + Y''(X)^3 + 4Y'(X)Y''(X)Y'''(X) \tag{63}$$

with the b.cs:

$$Y(0) = Y''(0) = Y(L) = Y''(L) = 0. \tag{64}$$

The kernel of the non-linear problem, Eqs.(63,64) is defined by:

$$Y^{IV}(X) + k^2 Y''(X) = 0, \tag{65}$$

with the b.cs. Eq.(63). The solution to the linear problem, Eqs.(65,64) is:

$$Y(X) = \xi Sin\frac{\pi}{L}X, \tag{66}$$

with $k_0 = \frac{\pi}{L}$. Increasing the loading parameter k^2 with $k^2 = k_0^2(1+t^2)$, with $|t| << 1$ and applying the Fredholm alternative theorem, see Trenoging & Vainberg [28] we get:

$$\int_0^L (Y^{IV}(X)Y'(X)^2 + Y'(X)^3 + 4Y'(X)Y''(X)Y'''(X))(Sin\frac{\pi}{L}X)dX = 0, \tag{67}$$

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where, Y(X) is defined by Eq.(66). The branching Eq.(67) yields:

$$\xi = \sqrt{2}\frac{L}{\pi}t. \tag{68}$$

Hence,

$$Y(X) = \sqrt{2}\frac{L}{\pi}tSin\frac{\pi}{L}X. \tag{69}$$

Since the branching problem has been solved in the fractional Λ-space, where differential exists and so geometry could be generated, the solution should be transferred into the initial plane (x,y). Recalling Eqs.(32,34) the branching problem may be transferred from the Λ-fractional space to the initial space. Indeed, with L=1 and $\gamma = 0.6$, Eq.(14) yields,

$$X = 0.805043x^{1.4} \tag{70}$$

and

$$Y(x) = \sqrt{2}\pi Sin(2.52x^{1.4}). \tag{71}$$

Since according to Eq.(34), $L = X = 1$ corresponds to $l = x = 0.97$, Eq.(34) yields,

$$y(x) = {}_a^{RL}D_x^{1-\gamma}Y(X(x)) = \frac{\sqrt{2}}{\Gamma(0.6)}\int_0^{0.97}\frac{Sin(2.529s^{1.4})}{(0.97-s)^{0.4}}ds. \tag{72}$$

With the help of Mathematica, Fig.(10) shows the trace of the Eq.(72) with X=1.

Let us point out that the final elastic curve of the branching problem is the mean value of Eq.(71) and its symmetric. Hence the final branching of the compressed bar is shown in Fig.(11).

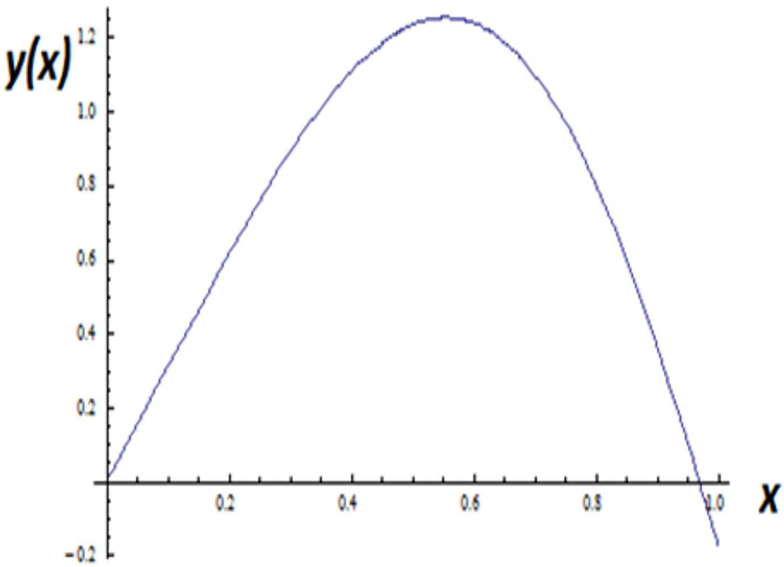

Fig. 10: The figure of Eq.(72) in the initial plane (x,y(x)).

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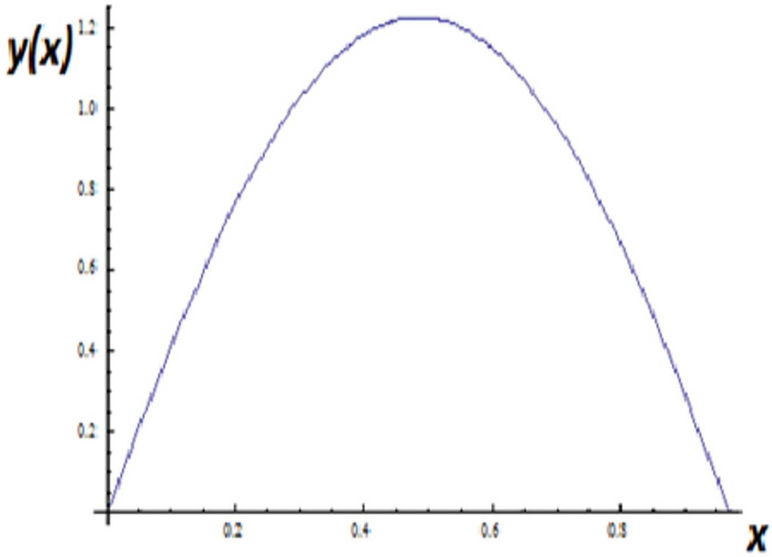

Fig. 11: The branching elastic curve of the rod with L=1.

5 Conclusion

It is proven that the introduced new fractional $\Lambda-$derivative ${}_{a}^{\Lambda}D_{x}^{\gamma}f(x)$ bridges the conventional mathematical analysis with the fractional analysis that is quite important in various fields of physics and mathematics, since it is inherently non-local. After the fractional differential geometry, is presented on the basis of $\Lambda-$fractional derivative, the present paper proves that Taylor's formula and variational problems may be worked out perfectly combining the original with the fractional $\Lambda-$space. The branching problem of the fractional non-linear equations has also been presented through the branching of the equilibrium equations of a simply supported rod under compressing axial force. Applications from beam bending problems are also presented.

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