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An Accurate Analytical Solution to Strongly Nonlinear Differential Equations

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Abstract: The study presents an alternative analytical method called Newton Harmonic Balance Method (NHBM) to provide an analytical solution for two nonlinear differential equations that appear in specific dynamics. This method is based on combining Newton's method and the harmonic balance method. Because the periodic solution is analytically proved, the relation between the natural frequency and the amplitude is obtained in an analytical form. The study compares the present results with the previous ones obtained by other methods to ensure the quality of the NHBM. Comparisons with Runge-Kutta numerical integration solutions are also made and excellent agreement has been observed. The NHBM enables to linearize the governing equations prior to applying the harmonic balance method. Moreover, it can lead to adequately accurate solutions for nonlinear oscillators.

Keywords: Analytical approximation, Second order Newton harmonic balance method, Nonlinear oscillators, Motion equation, High accuracy, Tapered beam.

1 Introduction

The NHBM is one of the most efficient methods for obtaining analytical approximate solutions for strongly nonlinear differential equations. It has many applications in nonlinear oscillator's problems in fractional differential and difference equations, delay differential equations, heat equations and several other modified versions with applications [1–3].

One of the most interesting areas in many physics and engineering problems is nonlinear vibrations. Mainly nonlinear vibration of oscillation systems is modeled by nonlinear differential equations that are widely used to describe many important phenomena in chemistry, biology, fluid dynamics, plasma, optical fibers and other areas of science. Obtaining an exact solution for these nonlinear problems is difficult and time-consuming. Therefore researchers have tried to find new approaches to overcome this shortcoming [4–6].

Recently, many scientists have used various analytical methods to solve nonlinear equations in mechanical systems. Some of these methods such as homotopy perturbation method [7–9] and variational iteration [10, 11] are powerful and can be used for almost all types of nonlinear equations. There are other methods such as optimal homotopy asymptotic [12], frequency amplitude formulation [13, 14], harmonic balance [15–17], energy balance and global residue harmonic balance [18–20], coupled homotopy variational formulation [21–23], variational approach [24–26], iteration perturbation [27], differential transform [28], Hamiltonian approach [29, 30], Newton harmonic balance [31–35] and other analytical and numerical ones [36–41].

In the present study, the approximate analytical frequency of strongly nonlinear differential equations is achieved by means of second-order Newton harmonic balance method. The accuracy of the present analytical approximate solutions has been illustrated by comparing them with numeric results and results obtained from other methods. The study mainly concludes that the nonlinear frequency can be investigated by simple formulas with respect to amplitude.

This paper is organized as follows: In Section 2, we briefly discuss the description of the Newton harmonic balance method (NHBM). Section 3 is devoted to analytical solutions for nonlinear oscillations with a natural frequency. Two illustrative examples are demonstrated to show the accuracy and reliability of (NHBM). Section 4 is depicted for the analytical and

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numerical calculations of all outcomes; also a comparison is made with some other methods. Finally, concluding remarks are presented in Section 5.

2 The Method of Solution

In order to illustrate the main concept of the method, the study considers the ordinary differential equation governing a conservative nonlinear oscillator with the initial conditions:

$$\ddot{u}+f(u,\dot{u},\ddot{u})=0,\quad u(0)=A,\ \ \dot{u}(0)=0. \tag{1}$$

To construct the analytical approximate solutions for Eq. (1), one new independent variable $\tau=\omega t$ is introduced, in terms of this new variable, Eq. (1) becomes

$$\omega^2u''+f(u,\omega u',\omega^2u'')=0,\quad u(0)=A,\ \ u'(0)=0, \tag{2}$$

where u' and u'' are the first and second differentiation with respect to τ respectively. Applying Newton's approach, the displacement and squared angular frequency can be expressed as Eq. (3) in which Δu_1 and $\Delta\omega_1^2$ are small increments of original displacement u_1 and squared angular frequency ω^2 respectively.

$$u(\tau)=u_1(\tau)+\Delta u_1(\tau),\quad \omega^2=\omega_1^2+\Delta\omega_1^2. \tag{3}$$

Substituting Eq. (3) into Eq. (2) and linearizing it for the first-order analytical approximation, the following equation is set:

$$u_1(\tau)=A\cos(\tau),\quad \Delta u_1=\Delta u_1''=\Delta\omega_1^2=0. \tag{4}$$

For the second analytical approximation, we set Eq. (4) and solve a set of simultaneous equations in terms of C and $\Delta\omega_1^2$.

$$\Delta u_1=C(\cos(\tau)-\cos(3\tau)). \tag{5}$$

The corresponding approximate analytical periodic solutions $u(\tau)$ and the second-order analytical approximate frequency ω_2 are as follows:

$$\omega_2=\omega=\sqrt{\omega_1^2+\Delta\omega_1^2}, \tag{6}$$

$$u(\tau)=A\cos(\tau)+C\left(\cos(\tau)-\cos(3\tau)\right). \tag{7}$$

3 Applications

In this section, the NHBM is applied to solve two particular physical examples of conservative oscillators. The results are compared with those obtained by exact or numerical solutions along with some other analytical techniques.

3.1 Example 1:

For the first example, the nonlinear dynamics of a particle on a rotating parabola is considered. The governing equation of motion and initial conditions are introduced by Nayfeh and Mook [4]:

$$\begin{gathered}(1+4q^2u^2)\frac{d^2u}{dt^2}+4q^2u\left(\frac{du}{dt}\right)^2+\Omega u=0,\\ u(0)=A,\quad \frac{du}{dt}(0)=0.\end{gathered} \tag{8}$$

By introducing a new independent variable $\tau=\omega t$, Eq. (8) is changed to:

$$\begin{gathered}(1+4q^2u^2)\omega^2u''+4\omega^2q^2uu'^2+\Omega u=0,\\ u(0)=A,\quad u'(0)=0.\end{gathered} \tag{9}$$

3.1.1 First-order analytical approximation

For the lowest-order (first-order) analytical approximate solution, substituting Eq. (3) into Eq. (9), and linearization with respect to Δu_1 and $\Delta\omega_1^2$ yields:

$$\begin{gathered}\left(\omega_1^2+4q^2u_1^2\omega_1^2+8q^2u_1\Delta u_1\omega_1^2+4q^2u_1^2\Delta\omega_1^2+\Delta\omega_1^2\right)u_1''\\ +\left(\omega_1^2+4q^2u_1^2\omega_1^2\right)\Delta u_1''+\left(4q^2u_1\omega_1^2+4q^2\Delta u_1\omega_1^2\right.\\ \left.+4q^2u_1\Delta\omega_1^2\right)u_1'^2+\left(8q^2u_1\omega_1^2\right)u_1'\Delta u_1'+\Omega\left(u_1+\Delta u_1\right)=0.\end{gathered} \tag{10}$$

Substituting Eq. (4) into Eq. (10) and avoiding the presence of secular terms, the angular frequency may be written as:

$$\omega_1=\sqrt{\frac{\Omega}{1+2A^2q^2}}. \tag{11}$$

3.1.2 Second-order analytical approximation

For the second-order analytical approximation, by putting $u_1=A\cos(\tau)$ and $\Delta u_1=C(\cos(\tau)-\cos(3\tau))$ into Eq. (10), expanding the obtained expression in a trigonometric series and then equating the coefficients of $cos(\tau)$ and $cos(3\tau)$ equal to zero, results are achieved in a set of simultaneous equations in terms of $\Delta\omega_1^2$ and C, then we get

$$\begin{gathered}-2A^3q^2\Delta\omega_1^2-2A^3q^2\omega_1^2-A\Delta\omega_1^2-A\omega_1^2\\ +A\Omega-C\omega_1^2+C\Omega=0,\end{gathered} \tag{12}$$

$$\begin{gathered}-2A^3q^2\Delta\omega_1^2-2A^3q^2\omega_1^2+14A^2Cq^2\omega_1^2\\ +9C\omega_1^2-C\Omega=0.\end{gathered} \tag{13}$$

Solving Eqs. (12) and (13) for $\Delta\omega_1$ and C gives

$$\Delta\omega_1=\sqrt{\frac{\Omega_1}{\Omega_2}}, \tag{14}$$

and

$$C = \frac{2A^3q^2\Omega}{\Omega_2}. \tag{15}$$

where

$$\Omega_1 = -28A^4q^4\omega_1^4 + 18A^2q^2\omega_1^2\Omega - 34A^2q^2\omega_1^4 + 10\omega_1^2\Omega - 9\omega_1^4 - \Omega^2,$$
$$\Omega_2 = 28A^4q^4\omega_1^2 + 34A^2q^2\omega_1^2 - 4A^2q^2\Omega + 9\omega_1^2 - \Omega.$$

Hence, the second-order analytical approximation of the periodic solution for Eq. (8) is

$$u(\tau) = A\cos(\tau) + C(\cos(\tau) - \cos(3\tau)), \tag{16}$$

where

$$\omega = \sqrt{\frac{\Omega}{1+2A^2q^2} + \frac{\Omega_1}{\Omega_2}}, \tag{17}$$

3.2 Example 2:

Tapered beams are important model for engineering structures that require a variable stiffness along the length, such as moving arms and turbine blades. In dimensionless form, the governing differential equation corresponding to the fundamental vibration mode of a tapered beam is given by [39]:

$$\frac{d^2u}{dt^2} + \alpha u^2\frac{d^2u}{dt^2} + \alpha u\left(\frac{du}{dt}\right)^2 + u + \beta u^3 = 0, \quad u(0) = A, \quad \frac{du}{dt}(0) = 0. \tag{18}$$

Substituting $\tau = \omega t$, Eq. (18) is changed to:

$$\omega^2u'' + \omega^2\alpha u^2u'' + \omega^2\alpha u(u')^2 + u + \beta u^3 = 0, \quad u(0) = A, \quad u'(0) = 0. \tag{19}$$

3.2.1 First-order analytical approximation

For the first-order analytical approximate solution, inserting Eq. (3) into Eq. (19), the following is obtained:

$$\begin{aligned}&\left(2u_1\alpha\Delta u_1\Delta\omega_1^2 + 2u_1\alpha\Delta u_1\omega_1^2 + u_1^2\alpha\Delta\omega_1^2 + u_1^2\alpha\omega_1^2 + \alpha\Delta u_1^2\Delta\omega_1^2 + \alpha\Delta u_1^2\omega_1^2 + \Delta\omega_1^2 + \omega_1^2\right)u_1''\\ &\left(2u_1\alpha\Delta u_1\Delta\omega_1^2 + 2u_1\alpha\Delta u_1\omega_1^2 + u_1^2\alpha\Delta\omega_1^2 + u_1^2\alpha\omega_1^2 + \alpha\Delta u_1^2\Delta\omega_1^2 + \alpha\Delta u_1^2\omega_1^2 + \Delta\omega_1^2 + \omega_1^2\right)\Delta u_1''\\ &+\left(u_1'^2 + 2u_1'\Delta u_1' + \Delta u_1'^2\right)\left(u_1\alpha\Delta\omega_1^2 + u_1\alpha\omega_1^2 \alpha\Delta u_1\Delta\omega_1^2 + \alpha\Delta u_1\omega_1^2\right) + u_1 + \Delta u_1\\ &+3u_1^2\beta\Delta u_1 + 3u_1\beta\Delta u_1^2 + u_1^3\beta + \beta\Delta u_1^3 = 0.\end{aligned} \tag{20}$$

Linearization of Eq. (20) with respect to Δu_1 and $\Delta\omega_1^2$ yields:

$$\begin{aligned}&\left(2u_1\alpha\Delta u_1\omega_1^2 + u_1^2\alpha\Delta\omega_1^2 + u_1^2\alpha\omega_1^2 + \Delta\omega_1^2 + \omega_1^2\right)u_1''\\ &\left(u_1^2\alpha\omega_1^2 + \omega_1^2\right)\Delta u_1'' + u_1^3\beta + \left(u_1\alpha\Delta\omega_1^2 + u_1\alpha\omega_1^2 + \alpha\Delta u_1\omega_1^2\right)u_1'^2 + 2u_1\alpha\omega_1^2u_1'\Delta u_1'\\ &+3u_1^2\beta\Delta u_1 + u_1 + \Delta u_1 = 0.\end{aligned} \tag{21}$$

Substituting Eq. (4) into Eq. (21) for first-order approximation and the presence of secular terms,

$$\tfrac{1}{4}\left(-2A^3\alpha\omega_1^2 + 3A^3\beta - 4A\omega_1^2 + 4A\right)\cos(\tau) + \tfrac{1}{4}\left(A^3\beta - 2A^3\alpha\omega_1^2\right)\cos(3\tau) = 0. \tag{22}$$

From Eq. (22) the angular frequency may be written as

$$\omega_1 = \sqrt{\frac{4+3A^2\beta}{4+2A^2\alpha}}. \tag{23}$$

3.2.2 Second-order analytical approximation

For the second analytical approximation, by putting $u_1 = A\cos(\tau)$ and $\Delta u_1 = C(\cos(\tau) - \cos(3\tau))$ into Eq. (20) then the obtained expression in a trigonometric series is

$$\begin{aligned}&\tfrac{1}{4}\left(-2A^3\alpha\Delta\omega_1^2 - 2A^3\alpha\omega_1^2 + 3A^3\beta + 6A^2C\beta - 4A\Delta\omega_1^2 - 4A\omega_1^2 + 4A - 4C\omega_1^2 + 4C\right)\cos(\tau)\\ &+\tfrac{1}{4}\left(-2A^3\alpha\Delta\omega_1^2 - 2A^3\alpha\omega_1^2 + A^3\beta + 14A^2C\alpha\omega_1^2 - 3A^2C\beta + 36C\omega_1^2 - 4C\right)\cos(3\tau)\\ &+\tfrac{1}{4}\left(18A^2C\alpha\omega_1^2 - 3A^2C\beta\right)\cos(5\tau) = 0,\end{aligned} \tag{24}$$

then by putting the coefficients of $\cos(\tau)$ and $\cos(3\tau)$ equal to zero, and solving the set of equations in terms of $\Delta\omega_1^2$ and C:

$$-\tfrac{1}{2}A^3\alpha\Delta\omega_1^2 - \tfrac{1}{2}A^3\alpha\omega_1^2 + \tfrac{3}{4}A^3\beta + \tfrac{3}{2}A^2C\beta - A\Delta\omega_1^2 - A\omega_1^2 + A - C\omega_1^2 + C = 0. \tag{25}$$

$$-\tfrac{1}{2}A^3\alpha\Delta\omega_1^2 - \tfrac{1}{2}A^3\alpha\omega_1^2 + \tfrac{1}{4}A^3\beta + \tfrac{7}{2}A^2C\alpha\omega_1^2 - \tfrac{3}{4}A^2C\beta + 9C\omega_1^2 - C = 0. \tag{26}$$

Solving Eqs. (25) and (26) simultaneously, it is obtained that:

$$\begin{aligned}\Delta\omega_1 = \Big[&\left(-28A^4\alpha^2\omega_1^4 + 60A^4\alpha\beta\omega_1^2 - 15A^4\beta^2 - 136A^2\alpha\omega_1^4 + 72A^2\alpha\omega_1^2 + 124A^2\beta\omega_1^2 - 28A^2\beta - 144\omega_1^4 + 160\omega_1^2 - 16\right)\\ &/2\left(14A^4\alpha^2\omega_1^2 - 9A^4\alpha\beta + 68A^2\alpha\omega_1^2 - 8A^2\alpha - 6A^2\beta + 72\omega_1^2 - 8\right)\Big]^{1/2},\end{aligned} \tag{27}$$

and

$$C = \left(2A^5\alpha\beta + 4A^3\alpha - 2A^3\beta\right)/\left(14A^4\alpha^2\omega_1^2 - 9A^4\alpha\beta + 68A^2\alpha\omega_1^2 - 8A^2\alpha - 6A^2\beta + 72\omega_1^2 - 8\right). \tag{28}$$

The second-order analytical approximate frequency is written as:

$$\omega = \sqrt{\omega_1^2 + \Delta\omega_1^2}, \tag{29}$$

where $\Delta\omega_1$ and ω are shown by Eqs. (27) and (29), the approximate analytical periodic solution $u(\tau)$, is as follows:

$$u(\tau) = A\cos(\tau) + C(\cos(\tau) - \cos(3\tau)). \tag{30}$$

Table 1: Comparison between the obtained solutions by variational approach, analytical solution and numerical frequency for example 1

Constant parameters			Pakar et al. [26]	Present method		Exact [26]
A	q	Ω	ω_{VA}	ω_1	ω_2	ω_{Exact}
0.5	1	0.5	0.5774	0.5774	0.581774	0.5815
0.5	0.5	2	1.3333	1.3333	1.33439	1.3344
1	0.8	1.5	0.8111	0.8111	0.830357	0.8288
1	0.7	0.5	0.5025	0.5025	0.511536	0.5108
1.5	0.5	2	0.9701	0.9701	0.990343	0.9888
1.5	0.3	2.5	1.3339	1.3339	1.34145	1.3410
2	0.2	4	1.7408	1.7408	1.74762	1.7473
2	0.4	1	0.6623	0.6623	0.677984	0.6767

Table 2: Comparison between the numerical frequency ω, the approximate frequency and other existing frequencies for $\alpha = \beta = 1$ for example 2

	Razzak & Alam [23]	Hamdan & Dado [39]		Wu et al. [40]		Present method		Exact [23]
A	ω_0	ω_0	ω_1	ω_0	ω_1	ω_1	ω_2	ω_{Exact}
5	1.37581	1.20953	1.24841	1.20953	1.32217	1.20953	1.35719	1.34288
10	1.40388	1.22074	1.26285	1.22074	1.35084	1.22074	1.39814	1.38928
15	1.40955	1.22295	1.26570	1.22295	1.35672	1.22295	1.4069	1.40138
20	1.41158	1.22373	1.26671	1.22373	1.35883	1.22373	1.41006	1.40632
25	1.41252	1.22409	1.26718	1.22409	1.35981	1.22409	1.41155	1.40883
30	1.41304	1.22429	1.26743	1.22429	1.36035	1.22429	1.41236	1.41029
50	1.41379	1.22458	1.26781	1.22458	1.36113	1.22458	1.41354	1.41261
100	1.41411	1.22470	1.26797	1.22470	1.36147	1.22470	1.41405	1.41375
200	1.41419	1.22473	1.26801	1.22473	1.36155	1.22473	1.41417	1.41408
500	1.41421	1.22474	1.26802	1.22474	1.36157	1.22474	1.41421	1.41419
1000	1.41421	1.22474	1.26802	1.22474	1.36157	1.22474	1.41421	1.41421

Table 3: Comparison between the numerical frequency ω, the approximate frequency and other existing frequencies for $\alpha = \beta = 2$ for example 2

	Razzak & Alam [23]	Hamdan & Dado [39]		Wu et al. [40]		Present method		Exact [23]
A	ω_0	ω_0	ω_1	ω_0	ω_1	ω_1	ω_2	ω_{Exact}
5	1.39406	1.21687	1.25786	1.21687	1.34073	1.21687	1.38338	1.37132
10	1.40898	1.22272	1.26541	1.22272	1.35613	1.22272	1.40600	1.40006
15	1.41187	1.22384	1.26685	1.22384	1.35913	1.22384	1.41052	1.40707
20	1.41289	1.22424	1.26736	1.22424	1.36020	1.22424	1.41213	1.40986
25	1.41337	1.22442	1.26760	1.22442	1.36069	1.22442	1.41287	1.41127
30	1.41363	1.22452	1.26773	1.22452	1.36096	1.22452	1.41328	1.41207
50	1.41400	1.22466	1.26791	1.22466	1.36135	1.22466	1.41388	1.41335
100	1.41416	1.22472	1.26799	1.22472	1.36152	1.22472	1.41413	1.41397
200	1.41420	1.22474	1.26801	1.22474	1.36156	1.22474	1.41419	1.41414
500	1.41421	1.22474	1.26802	1.22474	1.36157	1.22474	1.41421	1.41420
1000	1.41421	1.22474	1.26802	1.22474	1.36158	1.22474	1.41421	1.41421

4 Results and Discussion

Many researchers have tried to solve Eqs. (8) and (18) using different methods such as variational approach, coupled homotopy variational approach, and harmonic balance method. In the present study, the current equations are solved using the NHBM. Furthermore, the obtained results are compared with those of the mentioned methods.

To check the validity and accuracy of the method used in this study, some comparisons with the published data along with the exact solutions are displayed in Tables (1-3). The approximated results are shown as being in good agreement with numerically-obtained exact

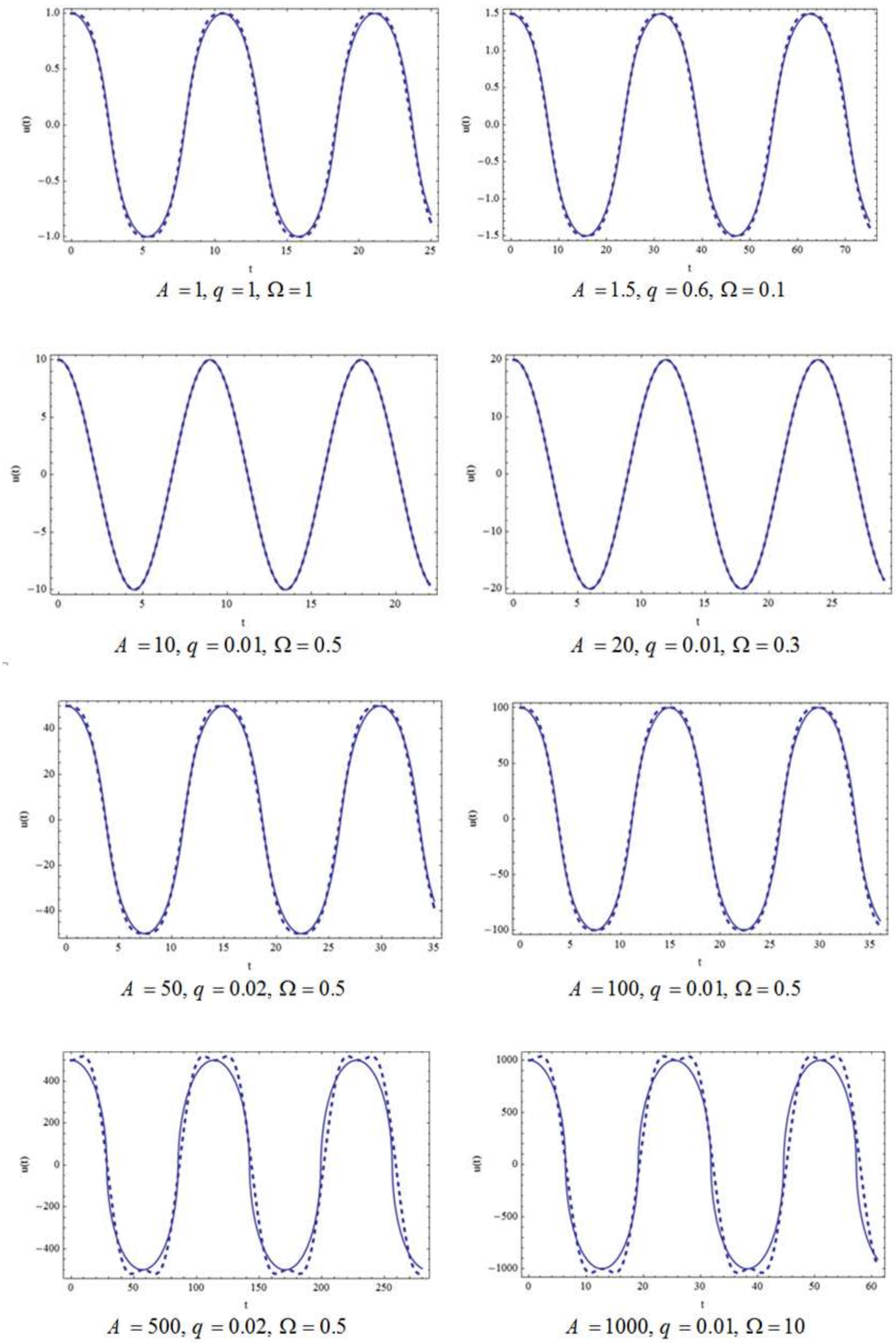

Fig. 1: Comparison between analytical approximate solution (dot line), and numerical solution (solid line) for example 1

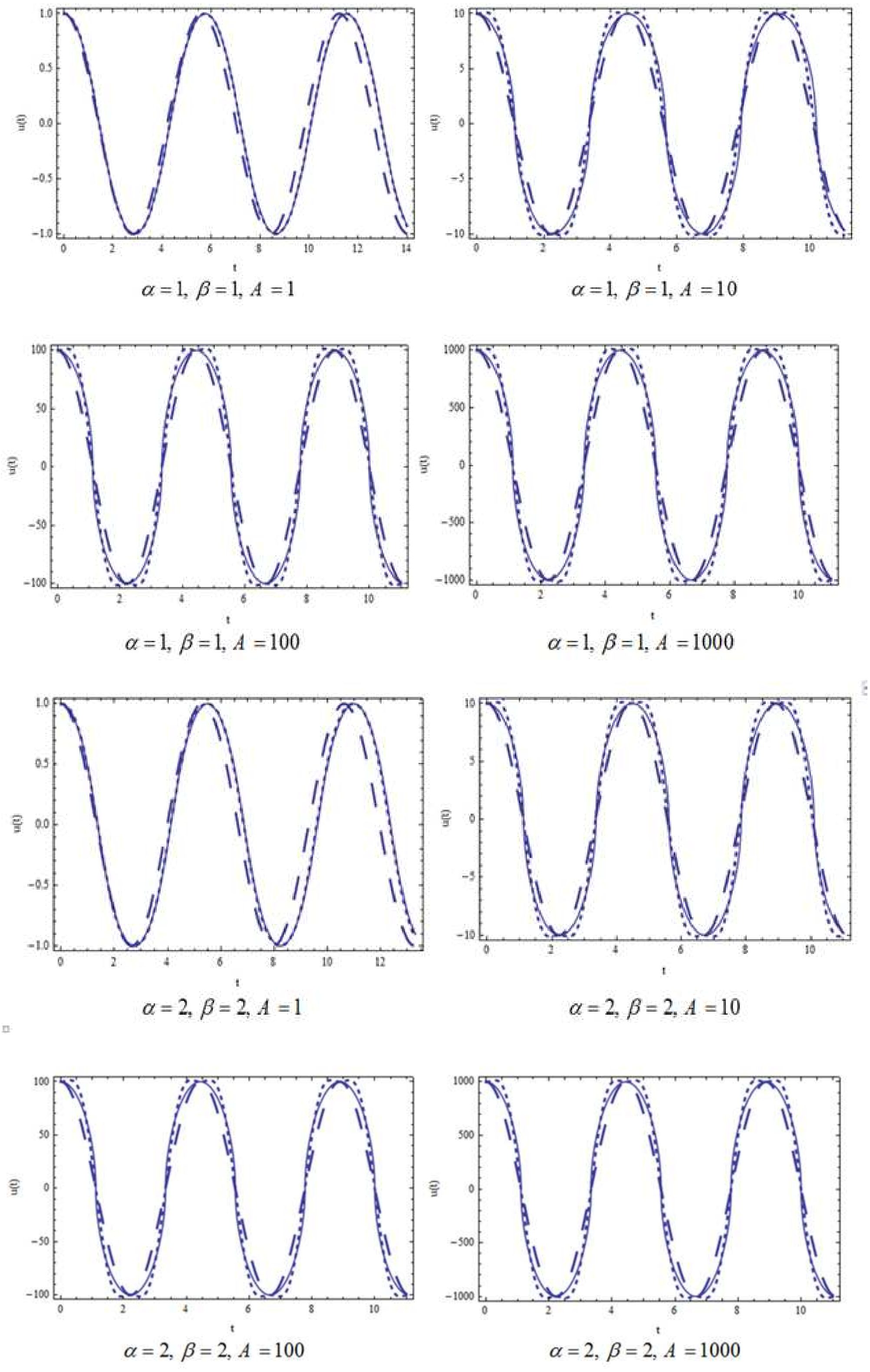

Fig. 2: Comparison between analytical approximate solution (dot line), coupled homotopy variational approach (dashed line) and numerical solution (solid line) for example 2

solutions. The second-order analytical approximate solutions are found to be almost the same as exact solutions. High accuracy results and simple solution procedures are advantages of the proposed method, which could be applied to other nonlinear oscillatory problems arising in nonlinear science and engineering. We observe from the Tables and graphical results that the approximate solution provides relatively good approximation in comparison with the exact periodic solution for small and large amplitude of oscillation.

The results obtained in this paper confirm that the NHBM is a powerful and efficient method for solving nonlinear oscillator differential equations in different fields of sciences and engineering.
From Figures 1 and 2, it can be concluded that Eqs. (8) and (18) can provide excellent approximate frequencies for oscillation amplitude. We plotted the phase curve and the time history response with amplitudes of A=1, 1.5, 10, 20, 50, 100, 500 and 1000 in example 1 and A=1, 10, 100 and 1000 in example 2. The observation shows that the second-order NHBM is in good agreement with the Runge -Kutta method. From the phase curve, it reveals that the nonlinear equations have a stable periodic solution. It also shows that it is effective to use NHBM to approximate the Runge-Kutta numerical solution.

5 Conclusion

The NHBM has been successfully applied to analyze and determine approximate periodic solutions for the governing equations of nonlinear oscillator's differential equations. It can clearly be seen that the second-order approximate solutions are in excellent agreement with the exact solutions. Furthermore, the NHBM gives an idea about the contributions from different harmonics. We conclude that the NHBM is not only a simple but also an elegant way to study a wide class of realistic non-exactly solvable problems, as well. The NHBM is expected to be suitable for other nonlinear ordinary differential equations. Moreover, the proposed method is a direct, concise and effective powerful mathematical tool for obtaining analytical approximate solutions of other nonlinear differential equations. Our results are in perfect agreement with numerical software.

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