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Oscillatory Properties of a Certain Class of Mixed Fractional Differential Equations

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Abstract: In this paper, we use the generalized Riccati technique and the integral averaging method to establish new oscillation criteria, for the solutions of a class of mixed fractional order nonlinear differential equations with the conformable fractional derivative and the Riemann-Liouville left-sided fractional derivative. We show the validity and effectiveness of our results by providing various examples.

Keywords: Fractional differential equation, Riccati technique, Oscillation.

1 Introduction

Fractional differential equations have gained importance and popularity during the past three decades or so, mainly due to the feature of these equations to accurately describe nonlinear phenomena. As a result, these equations have several applications in science and engineering. For example, the nonlinear oscillation of an earth quake can be modeled using fractional derivatives. The use of fractional derivatives in the fluid dynamics traffic model eliminates the deficiency arising from the assumption of continuous traffic flow. Fractional differential equations are also used in modeling chemical processes, signal processing, hydraulics of dams, temperature field problems in oil strata, diffusion problems, waves in liquids and gases [1,2,3,4,5,6,7]. Thus, several researchers have been trying to develop new way which is easier to work with definitions of fractional derivatives, like the Riemann-Liouville, Caputo and Grunwald-Letnikov, for details, see [8,9] and references cited therein.

Recently, a well-behaved limit-based derivative called the conformable fractional derivative was suggested in Khalil et al., [10] in this direction have grown rapidly, day by day. Some of these studies are cited in the references [10,11,12,13,14].

Over the years, the development of oscillation theory has played a major role in the physical sciences and engineering. Well-known applications of the theory of oscillations include the oscillations in buildings and machines, self- excited vibrations in synchrotron accelerators, the vibrations in the operation of rocket engines, electro magnetic vibration in radio technology and optical science, the complicated oscillation in chemical reaction and also the work in lossless transmission lines in high speed computer networks [15, 16]. All these different phenomena share a common theoretical foundation in oscillation theory. They all can be described by an oscillatory differential equation. There are many books on oscillation theory. We choose to refer to [17,18].

For studies in the oscillation theory of fractional differential equations with the Liouville right-sided definition see [19,20,21,22,23,24]. While for studies of fractional partial differential equations with Riemann-Liouville fractional left-sided derivative see [25,26,27].

To the best of author's knowledge, it seems that there has been no work done on conformable and Riemann-Liouville left sided fractional derivatives appearing in the mixed fractional order differential equations.

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The authors motivated by this gap have initiated the following oscillation problem of a class of mixed fractional order nonlinear differential equation of the form

$$T_{\alpha_3}\left[\frac{1}{r_2(t)}f_2\left(T_{\alpha_2}\left(\frac{1}{r_1(t)}f_1\left(D_+^{\alpha_1}x(t)\right)\right)\right)\right]$$
$$+F\left(t,\int_0^t(t-s)^{-\alpha_1}x(s)ds\right)=0,$$
$$t\geq t_0>0, 0<\alpha_i<1, i=1,2,3. \quad (1)$$

We assume that the following conditions are satisfied:
(A_1) $r_1(t)\in C^{\alpha_2+\alpha_3}([t_0,\infty),\mathbb{R}_+)), r_2(t)\in C^{\alpha_3}([t_0,\infty),\mathbb{R}_+))$;
(A_2) $f_i\in C^{\alpha_2+\alpha_3}(\mathbb{R},\mathbb{R})$ is an increasing odd function and there exist positive constants δ_i and δ_i' such that $\delta_i'\geq\frac{x}{f_i(x)}\geq\delta_i>0$ for $xf_i(x)\neq 0$, i=1,2 and $\delta=\delta_1\delta_2$;
(A_3) $(a) f_1^{-1}\in C(\mathbb{R},\mathbb{R})$ with $uf_1^{-1}(u)>0$ for $u\neq 0$, and there exist some positive constants λ_1 such that $f_1^{-1}(uv)\leq\lambda_1 f_1^{-1}(u)f_1^{-1}(v)$ for $uv\neq 0$,
$(b) f_2^{-1}\in C(\mathbb{R},\mathbb{R})$ with $uf_2^{-1}(u)>0$ for $u\neq 0$, and there exist some positive constants λ_2 such that $f_2^{-1}(uv)\geq\lambda_1 f_2^{-1}(u)f_2^{-1}(v)$ for $uv\neq 0$, i=1,2 and $\lambda=\lambda_1\lambda_2$;
(A_4) $F(t,K)\in C^1([t_0,\infty)\times\mathbb{R},\mathbb{R}_+))$ and there exist a function $Q(t)\in C^1([t_0,\infty),\mathbb{R}_+))$ such that $\frac{F(t,K)}{f_2(K)}\leq Q(t)$ for $K\neq 0$ whereas $f_2(k)$ and Q(t) are of the same sign.

By a solution of Eq.(1), we mean a function $x(t)$ defined on some ray $[T_x,\infty), T_x\geq t_0$ such that $\int_0^t(t-s)^{-\alpha_1}x(s)ds\in C^1([T_x,\infty),\mathbb{R}), \frac{1}{r_1(t)}f_1\left(D_+^{\alpha_1}x(t)\right)\in C^{\alpha_2+\alpha_3}([T_x,\infty),\mathbb{R})$, which satisfies Eq.(1) for any $t\geq T_x$. From this point onward, we will always assume that solutions of Eq.(1) existing on some half-line $[T_x,\infty), T_x\geq t_0$. We restrict our attention only to the nontrivial solutions of Eq.(1), i.e, the solutions $x(t)$ such that $\sup\{|x(t)|:t\geq T\}$ for all $T>T_x$.

A nontrivial solution of Eq.(1) is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory. Eq.(1) is called oscillatory if all of its solutions are oscillatory.

Our main objective in this paper is to obtain new oscillation criteria for the class of nonlinear mixed fractional differential equations defined by Eq. (1) and provide a detailed discussion of the main results by making use of the generalized Riccati technique and the integral averaging method.

Having these ideas in mind, this paper is organized as follows: In Section 2, we recall the basic definitions of the Riemann-Liouville derivative and the conformable fractional derivatives together with basic lemmas concerning the above set of operators. In Section 3, we present new sufficient conditions for the oscillation of the solutions of Eq.(1). In Section 4, we provide examples to illustrate the main results.

2 Preliminaries

Before starting our analysis of Eq.(1), we have to explain the function of the operators $D_+^{\alpha}x(t)$ and $T_\alpha(f)(t)$. First, we introduce some core concepts and results about the Riemann-Liouville and Khalil's conformable fractional derivative. We start by defining the Riemann-Liouville operator.

Definition: 2.1.[8] The Riemann-Liouville fractional derivative of order α of $x(t)$ is defined by

$$D_+^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t(t-s)^{-\alpha}x(s)ds, t\in\mathbb{R}_+=(0,\infty),$$

here $\Gamma(.)$ is the gamma function defined by $\Gamma(t)=\int_0^\infty e^{-s}s^{t-1}ds, t\in\mathbb{R}_+$.

Lemma: 2.1.[25] Let $x(t)$ be a solution of Eq.(1) and

$$K(t)=\int_0^t(t-s)^{-\alpha}x(s)ds \; for \; \alpha\in(0,1) \quad (2)$$

where $t>0$. Then

$$K'(t)=\Gamma(1-\alpha)D_+^{\alpha}x(t). \quad (3)$$

Next, we provide the definition of the conformable fractional derivative proposed by Khalil et al.[10],

Definition: 2.2. Given a function $f:[0,\infty)\to\mathbb{R}$. Then the conformable fractional derivative of f of order α, is defined by

$$T_\alpha(f)(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}$$

for all $t>0, \alpha\in(0,1)$. If f is α- differentiable in some $(0,a), a>0, and \lim_{t\to 0^+}f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0)=\lim_{t\to 0^+}f^{(\alpha)}(t).$$

We will sometimes write $f^{(\alpha)}(t)$ for $T_\alpha(f)(t)$, to denote the conformable fractional derivatives of f of order α.

Some properties of conformable fractional derivative:
Let $\alpha\in(0,1]$ and f and g be α - differentiable at a point $t>0$. Then
(P_1) $T_\alpha(t^p)=pt^{p-\alpha}$ for all $p\in\mathbb{R}$.
(P_2) $T_\alpha(\lambda)=0$, for all constant functions $f(t)=\lambda$.
(P_3) $T_\alpha(fg)=fT_\alpha(g)+gT_\alpha(f)$.
(P_4) $T_\alpha(\frac{f}{g})=\frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$.
(P_5) If, in addition, f is differentiable, then $T_\alpha(f)(t)=t^{1-\alpha}\frac{df}{dt}(t)$.

Definition: 2.3.[24] Let $\alpha \in (0,1]$ and $0 \le a < b$. A function $f : [a,b] \to \mathbb{R}$ is α-fractional integrable on [a,b] if the integral

$$\int_a^b f(x) d_\alpha x := \int_a^b f(x) x^{\alpha-1} dx$$

exists and is finite.

Remark 2.1

$$I_\alpha^a(f)(t) = I_1^a(t^{\alpha-1} f) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx, \tag{4}$$

where the integral I_1 is the usual Riemann improper integral, and $\alpha \in (0,1]$.

Lemma: 2.2.[11] Let $f : (a,b) \to \mathbb{R}$ be differentiable and $0 < \alpha \le 1$. Then, for all $t > a$ we have

$$I_\alpha^a T_\alpha^a(f)(t) = f(t) - f(a). \tag{5}$$

3 Main Results

We begin this section with the following lemmas that are essential to the proofs of our main theorems.

Lemma: 3.1. Assume that $x(t)$ is an eventually positive solution of Eq.(1) and

$$\int_{t_0}^{\infty} \frac{1}{f_2^{-1}\left(\frac{1}{r_2(s)}\right)} d_{\alpha_2} s = \infty, \tag{6}$$

$$\int_{t_0}^{\infty} r_1(s) ds = \infty, \tag{7}$$

$$\int_{t_0}^{\infty} f_1^{-1}\left[r_1(\tau) \int_\tau^\infty f_2^{-1}\left(r_2(\xi) \int_\xi^\infty Q(s) d_{\alpha_3} s\right) d_{\alpha_2}\xi\right] d\tau = \infty. \tag{8}$$

Then there exists a sufficiently large T such that $T_{\alpha_2}\left(\frac{1}{r_1(t)} f_1\left(D_+^{\alpha_1} x(t)\right)\right) > 0$ on $[T, \infty)$ or $\lim_{t\to\infty} K(t) = 0$.

Proof. Let $t_1 > t_0$ such that $x(t) > 0$ on $[t_1, \infty)$, so that $K(t) > 0$ on $[t_1, \infty)$. From Eq. (1), we get

$$\begin{aligned} T_{\alpha_3}&\left[\frac{1}{r_2(t)} f_2\left(T_{\alpha_2}\left(\frac{1}{r_1(t)} f_1\left(D_+^{\alpha_1} x(t)\right)\right)\right)\right] \\ &= -F\left(t, \int_0^t (t-s)^{-\alpha_1} x(s) ds\right) \\ &\le -Q(t) f_2(K(t)) < 0, \end{aligned} \tag{9}$$

$t \ge t_1$. Then $\frac{1}{r_2(t)} f_2\left(T_{\alpha_2}\left(\frac{1}{r_1(t)} f_1\left(D_+^{\alpha_1} x(t)\right)\right)\right)$ is strictly decreasing on $[t_1, \infty)$. Thus, $T_{\alpha_2}\left(\frac{1}{r_1(t)} f_1\left(D_+^{\alpha_1} x(t)\right)\right)$ is eventually of one sign. For $t_2 > t_1$ sufficiently large, we claim $T_{\alpha_2}\left(\frac{1}{r_1(t)} f_1\left(D_+^{\alpha_1} x(t)\right)\right) > 0$ on $[t_2, \infty)$. Otherwise, assume that there exists a sufficiently large $t_3 > t_2$ such that $T_{\alpha_2}\left(\frac{1}{r_1(t)} f_1\left(D_+^{\alpha_1} x(t)\right)\right) < 0$ on $[t_3, \infty)$. Then $\frac{1}{r_1(t)} f_1\left(D_+^{\alpha_1} x(t)\right)$ is strictly decreasing on $[t_3, \infty)$. Taking the α_3- integral from t_3 to t, we have

$$\begin{aligned} \frac{1}{r_1(t)} f_1\left(D_+^{\alpha_1} x(t)\right) - \frac{1}{r_1(t_3)} f_1\left(D_+^{\alpha_1} x(t_3)\right) &= \\ \int_{t_3}^t T_{\alpha_2}\left(\frac{1}{r_1(s)} f_1\left(D_+^{\alpha_1} x(s)\right)\right) d_{\alpha_2} s & \\ \le f_2^{-1}\left[\frac{1}{r_2(t_3)}\right] T_{\alpha_2}\left(\frac{1}{r_1(t_3)} f_1\left(D_+^{\alpha_1} x(t_3)\right)\right) & \\ \times \int_{t_3}^t \frac{1}{f_2^{-1}\left[\frac{1}{r_2(s)}\right]} d_{\alpha_2} s. & \end{aligned} \tag{10}$$

By Eq.(6), we have $\lim_{t\to\infty} \frac{1}{r_1(t)} f_1\left(D_+^{\alpha_1} x(t)\right) = -\infty$. So there exists a sufficiently large t_4 with $t_4 > t_3$ such that $D_+^{\alpha_1} x(t) < 0, t \in [t_4, \infty)$.

Furthermore $D_+^{\alpha_1} x(t) = \frac{K'(t)}{\Gamma(1-\alpha_1)} < 0$.

$$\begin{aligned} \frac{1}{\Gamma(1-\alpha_1)}[K(t) - K(t_4)] &= \frac{1}{\Gamma(1-\alpha_1)} \int_{t_4}^t K'(s) ds \\ &\le \frac{1}{\Gamma(1-\alpha_1)} \frac{K'(t_4)}{r_1(t_4)} \int_{t_4}^t r_1(s) ds. \end{aligned}$$

By Eq.(7), we deduce that $\lim_{t\to\infty} K(t) = -\infty$, which contradicts the fact that $K(t)$ is an eventually positive solution of Eq.(7). So $T_{\alpha_2}\left(\frac{1}{r_1(t)} f_1\left(D_+^{\alpha_1} x(t)\right)\right) > 0$ on $[t_2, \infty)$. Thus, eventually, $D_+^{\alpha_1} x(t)$ becomes of one sign. Now, we assume that $D_+^{\alpha_1} x(t) < 0, t \in [t_5, \infty)$, for some sufficiently large $t_5 > t_4$. Since furthermore, $K'(t) > 0$, we have $\lim_{t\to\infty} K(t) = \beta \ge 0$. We claim that $\beta = 0$. Otherwise, assume that $\beta > 0$. That $K(t) \ge \beta$ on $[t_5, \infty)$ and for $t \in [t_5, \infty)$ by Eq.(1) we have,

$$\begin{aligned} T_{\alpha_3}&\left[\frac{1}{r_2(t)} f_2\left(T_{\alpha_2}\left(\frac{1}{r_1(t)} f_1\left(D_+^{\alpha_1} x(t)\right)\right)\right)\right] \\ &\le -Q(t) f_2(\beta) < 0. \end{aligned} \tag{11}$$

α_3- integrating from t to ∞ and using (A_3) yields,

$$\int_t^\infty T_{\alpha_3}\left[\frac{1}{r_2(s)}f_2\left(T_{\alpha_2}\left(\frac{1}{r_1(s)}f_1\left(D_+^{\alpha_1}x(s)\right)\right)\right)\right]d_{\alpha_3}s$$
$$\leq -\int_t^\infty Q(s)f_2(\beta)d_{\alpha_3}s,$$
$$-\left[\frac{1}{r_2(t)}f_2\left(T_{\alpha_2}\left(\frac{1}{r_1(t)}f_1\left(D_+^{\alpha_1}x(t)\right)\right)\right)\right]$$
$$\leq -f_2(\beta)\int_t^\infty Q(s)d_{\alpha_3}s,$$
$$T_{\alpha_2}\left(\frac{1}{r_1(t)}f_1\left(D_+^{\alpha_1}x(t)\right)\right)$$
$$> \lambda_2\beta f_2^{-1}\left[r_2(t)\int_t^\infty Q(s)d_{\alpha_3}s\right]. \quad (12)$$

α_2 - integrating both sides of Eq.(12) from t to ∞ and using (A_3) we get

$$\int_t^\infty T_{\alpha_2}\left(\frac{1}{r_1(s)}f_1\left(D_+^{\alpha_1}x(s)\right)\right)d_{\alpha_2}s$$
$$> \lambda_2\beta\int_t^\infty f_2^{-1}\left[r_2(\xi)\int_\xi^\infty Q(s)d_{\alpha_3}s\right]d_{\alpha_2}\xi,$$
$$D_+^{\alpha_1}x(t)$$
$$< -f_1^{-1}\left(r_1(t)\lambda_2\beta\int_t^\infty f_2^{-1}\left[r_2(\xi)\int_\xi^\infty Q(s)d_{\alpha_3}s\right]d_{\alpha_2}\xi\right),$$
$$\frac{K'(t)}{\Gamma(1-\alpha_1)} < -\lambda_2\lambda_1 f_1^{-1}(\beta)$$
$$\times f_1^{-1}\left(r_1(t)\int_t^\infty f_2^{-1}\left[r_2(\xi)\int_\xi^\infty Q(s)d_{\alpha_3}s\right]d_{\alpha_2}\xi\right). \quad (13)$$

Again integrating Eq.(13) from t_5 to t, yields

$$K(t) \leq K(t_5) - \lambda\Gamma(1-\alpha_1)f_1^{-1}(\beta)$$
$$\times\int_{t_5}^t f_1^{-1}\left(r_1(\tau)\int_t^\infty f_2^{-1}\left[r_2(\xi)\int_\xi^\infty Q(s)d_{\alpha_3}s\right]d_{\alpha_2}\xi\right)d\tau. \quad (14)$$

Letting $t\to\infty$, from Eq.(8), we get $lim_{t\to\infty}K(t) = -\infty$, which is a contradiction with $D_+^{\alpha_1}x(t) > 0$. This completes the proof.

Lemma: 3.2. Assume that $x(t)$ is an eventually positive solution of Eq.(1) such that

$$T_{\alpha_2}\left(\frac{1}{r_1(t)}f_1\left(D_+^{\alpha_1}x(t)\right)\right) > 0, D_+^{\alpha_1}x(t) > 0 \quad (15)$$

on $[t_1,\infty)$, where $t_1 > t_0$ is sufficiently large. Then we have

$$K'(t) \geq \frac{\delta\Gamma(1-\alpha_1)R_1(t_1,t)r_1(t)f_2\left(T_{\alpha_2}\left(\frac{1}{r_1(t)}f_1\left(D_+^{\alpha_1}x(t)\right)\right)\right)}{r_2(t)}, \quad (16)$$

$$K(t) \geq \frac{\delta\Gamma(1-\alpha_1)R_2(t_1,t)f_2\left(T_{\alpha_2}\left(\frac{1}{r_1(t)}f_1\left(D_+^{\alpha_1}x(t)\right)\right)\right)}{r_2(t)}, \quad (17)$$

where $R_1(t_1,t) = \int_{t_1}^t r_2(s)d_{\alpha_2}s, R_2(t_1,t) = \int_{t_1}^t r_1(s)R_1(t_1,s)ds$ for $t_1 > t_0$.

Proof. By Eq.(9), $\frac{1}{r_2(t)}f_2\left(T_{\alpha_2}\left(\frac{1}{r_1(t)}f_1\left(D_+^{\alpha_1}x(t)\right)\right)\right)$ is a strictly decreasing function on $[t_1,\infty)$. So ,

$$\frac{1}{r_1(t)}f_1\left(D_+^{\alpha_1}x(t)\right) \geq \int_{t_1}^t \frac{r_2(s)T_{\alpha_2}\left(\frac{1}{r_1(s)}f_1\left(D_+^{\alpha_1}x(s)\right)\right)}{r_2(s)}d_{\alpha_2}(s),$$
$$f_1\left(D_+^{\alpha_1}x(t)\right) \geq \frac{r_1(t)}{r_2(t)}T_{\alpha_2}\left(\frac{1}{r_1(t)}f_1\left(D_+^{\alpha_1}x(t)\right)\right)R_1(t_1,t).$$

From (A_2) we obtain,

$$D_+^{\alpha_1}x(t) \geq \delta R_1(t_1,t)\frac{r_1(t)}{r_2(t)}f_2\left(T_{\alpha_2}\left(\frac{1}{r_1(t)}f_1\left(D_+^{\alpha_1}x(t)\right)\right)\right).$$

Therefore,

$$K'(t) \geq \Gamma(1-\alpha_1)\delta R_1(t_1,t)\frac{r_1(t)}{r_2(t)}$$
$$\times f_2\left(T_{\alpha_2}\left(\frac{1}{r_1(t)}f_1\left(D_+^{\alpha_1}x(t)\right)\right)\right)$$

and thus

$$K(t) \geq \frac{\delta\Gamma(1-\alpha_1)R_2(t_1,t)f_2\left(T_{\alpha_2}\left(\frac{1}{r_1(t)}f_1\left(D_+^{\alpha_1}x(t)\right)\right)\right)}{r_2(t)}.$$

Hence the proof is complete.

Next, we will give some new oscillation criteria, through the following theorems.

Theorm: 3.1. Assume that (6)-(8) hold and suppose that $f_2'(v)$ exists such that $f_2'(v) \geq \mu$ for some $\mu > 0$ and for all $v \neq 0$. If there exist two functions $\phi(t) \in C^{\alpha_3}([t_0,\infty),\mathbb{R}_+), \eta(t) \in C^{\alpha_3}([t_0,\infty),[0,\infty))$ such that

$$\int_T^\infty\Bigg(\phi(s)Q(s)s^{\alpha_3-1} - \phi(s)\eta'(s)$$
$$+\phi(s)r_1(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)\eta^2(s)$$
$$-\frac{\left[2\eta(s)\phi(s)r_1(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)+\phi'(s)\right]^2}{4r_1(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)}\Bigg)ds$$
$$=\infty, \quad (18)$$

for sufficiently large T, where $R_1(T,s)$ is defined by Lemma 3.2, then every solution of Eq.(1) is oscillatory or satisfies $lim_{t\to\infty}K(t) = 0$.

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of Eq.(1). Then without loss of generality, we can assume that there is a solution $x(t)$ of Eq.(1) such that $x(t) > 0$ on $[t_1,\infty)$, where t_1 is sufficiently large. By Lemma 3.1, we have $D_+^{\alpha_1}x(t) > 0$, and $T_{\alpha_2}\left(\frac{1}{r_1(t)}f_1\left(D_+^{\alpha_1}x(t)\right)\right)>0$ on $[t_2,\infty)$, where t_2 is sufficiently large. We define the following generalized Riccati function

$$w(t)=\phi(t)\left(\frac{\frac{1}{r_2(t)}f_2\left(T_{\alpha_2}\left(\frac{1}{r_1(t)}f_1\left(D_+^{\alpha_1}x(t)\right)\right)\right)}{f_2(K(t))}+\eta(t)\right). \quad (19)$$

Thus, $w(t)>0$ on $[t_2,\infty)$. α_3 - differentiating, we have,

$$\begin{aligned}T_{\alpha_3}w(t)&=T_{\alpha_3}\phi(t)\\&\times\left(\frac{\frac{1}{r_2(t)}f_2\left(T_{\alpha_2}\left(\frac{1}{r_1(t)}f_1\left(D_+^{\alpha_1}x(t)\right)\right)\right)}{f_2(K(t))}+\eta(t)\right)\\&+\phi(t)T_{\alpha_3}\left(\frac{\frac{1}{r_2(t)}f_2\left(T_{\alpha_2}\left(\frac{1}{r_1(t)}f_1\left(D_+^{\alpha_1}x(t)\right)\right)\right)}{f_2(K(t))}\right)\\&+\phi(t)T_{\alpha_3}\eta(t).\end{aligned}$$

Then, making use of Eq.(9) and Eq.(16), we get

$$\begin{aligned}w'(t)&\le\frac{\phi'(t)}{\phi(t)}w(t)-\phi(t)Q(t)t^{\alpha_3-1}+\phi(t)\eta'(t)\\&-\phi(t)\frac{1}{r_2(t)}f_2\left(T_{\alpha_2}\left(\frac{1}{r_1(t)}f_1\left(D_+^{\alpha_1}x(t)\right)\right)\right)t^{1-\alpha_3}\mu\\&\times\frac{\delta\Gamma(1-\alpha_1)R_1(t_1,t)\frac{r_1(t)}{r_2(t)}f_2\left(T_{\alpha_2}\left(\frac{1}{r_1(t)}f_1\left(D_+^{\alpha_1}x(t)\right)\right)\right)}{f_2(K(t))^2}\\w'(t)&\le\frac{\phi'(t)}{\phi(t)}w(t)-\phi(t)Q(t)t^{\alpha_3-1}+\phi(t)\eta'(t)\\&-\phi(t)\mu t^{1-\alpha_3}\delta\Gamma(1-\alpha_1)R_1(t_1,t)r_1(t)\left(\frac{w(t)}{\phi(t)}-\eta(t)\right)^2,\end{aligned}$$

$$\begin{aligned}w'(t)&\le-\phi(t)Q(t)t^{\alpha_3-1}\\&+\phi(t)\eta'(t)-\phi(t)r_1(t)t^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_1,t)\eta^2(t)\\&+\frac{\left[2\eta(t)\phi(t)r_1(t)t^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_1,t)+\phi'(t)\right]^2}{4r_1(t)\phi(t)t^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_1,t)}.\end{aligned} \quad (20)$$

Integrating the above inequality from t_2 to t, we obtain,

$$\begin{aligned}&\int_{t_2}^{t}\Big(\phi(s)Q(s)s^{\alpha_3-1}\\&-\phi(s)\eta'(s)+\phi(s)r_1(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)\eta^2(s)\\&-\frac{\left[2\eta(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)r_1(s)+\phi'(s)\right]^2}{4r_1(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)}\Big)ds\\&\le w(t_2).\end{aligned}$$

By letting $t\to\infty$, we get a contradiction to Eq.(18). The proof of the theorem is complete.

The following theorem, even though it is similar to the previous one, nevertheless, involves a slight variation in the definition of the Riccati Eq.(22).

Theorm: 3.2. Assume that (6)-(8) hold. If there exist two functions $\phi(t)\in C^{\alpha_3}([t_0,\infty),\mathbb{R}_+),\eta(t)\in C^{\alpha_3}([t_0,\infty),[0,\infty))$ such that

$$\begin{aligned}&\int_T^{\infty}\Big(\frac{\phi(s)Q(s)s^{\alpha_3-1}}{\delta_2'}-\phi(s)\eta'(s)\\&+\phi(s)r_1(s)\delta\Gamma(1-\alpha_1)R_1(T,s)\eta^2(s)\\&-\frac{\left[2\eta(s)\phi(s)\delta\Gamma(1-\alpha_1)R_1(T,s)r_1(s)+\phi'(s)\right]^2}{4r_1(s)\phi(s)\delta\Gamma(1-\alpha_1)R_1(T,s)}\Big)ds\\&=\infty,\end{aligned} \quad (21)$$

for sufficiently large T, where $R_1(T,s)$ is defined by Lemma 3.2, then every solution of (1) is oscillatory or satisfies $lim_{t\to\infty}K(t)=0$.

Proof. Suppose to the contrary that $x(t)$ is nonoscillatory solution of Eq.(1). Then without loss of generality, we may assume that there is a solution $x(t)$ of Eq.(1) such that $x(t)>0$ on $[t_1,\infty)$, where t_1 is sufficiently large. By Lemma 3.1, we have $D_+^{\alpha_1}x(t)>0$, and $T_{\alpha_2}\left(\frac{1}{r_1(t)}f_1\left(D_+^{\alpha_1}x(t)\right)\right)>0$ on $[t_2,\infty)$, where t_2 is sufficiently large. We define the following generalized Riccati function

$$w(t)=\phi(t)\left(\frac{\frac{1}{r_2(t)}f_2\left(T_{\alpha_2}\left(\frac{1}{r_1(t)}f_1\left(D_+^{\alpha_1}x(t)\right)\right)\right)}{K(t)}+\eta(t)\right). \quad (22)$$

Then $w(t)>0$ on $[t_2,\infty)$. The rest of the proof is similar to the proof of Theorem 3.1 and hence the details are omitted.

Next, we establish new oscillation criteria for Eq.(1) using the integral average method.

Let $\mathbb{D}_0=\{(t,s):t>s\ge t_0\}$ and $\mathbb{D}=\{(t,s):t\ge s\ge t_0\}$. There exists a function $H\in C'(\mathbb{D};\mathbb{R})$ which is said to belong to the class $\mathbb{P}$ if
(T_1) $H(t,t)=0$ for $t\ge t_0,H(t,s)>0$ on $\mathbb{D}_0$,
(T_2) H has a continuous and nonpositive partial derivative on $\mathbb{D}_0$ with respect to the second variable.

Theorm: 3.3. Assume $(6)-(8)$ hold.

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\Big[\phi(s)Q(s)s^{\alpha_3-1}-\phi(s)\eta'(s)\\&+\phi(s)r_1(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)\eta^2(s)\\&-\frac{\left[2\eta(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)r_1(s)+\phi'(s)\right]^2}{4r_1(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)}\Big]ds\\&=\infty,\end{aligned} \quad (23)$$

for all sufficiently large T, where ϕ and η are defined as in Theorem 3.1, then every solution of (1) is oscillatory or satisfies $lim_{t\to\infty}K(t)=0$.

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of Eq.(1). Then without loss of generality, we can assume that there is a solution $x(t)$ of Eq.(1) such that $x(t)>0$ on $[t_1,\infty)$, where t_1 is sufficiently large. By Lemma 3.1, we have $D_+^{\alpha_1}x(t)>0$, and $T_{\alpha_2}\left(\frac{1}{r_1(t)}f_1\left(D_+^{\alpha_1}x(t)\right)\right)>0$ on $[t_2,\infty)$ where t_2 is sufficiently large. Let $w(t)$ be defined as in Theorem 3.1. By Eq.(20), we have

$$\begin{aligned}&\phi(t)Q(t)t^{\alpha_3-1}-\phi(t)\eta'(t)\\&+\phi(t)r_1(t)t^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,t)\eta^2(t)\\&-\frac{\left[2\eta(t)\phi(t)t^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,t)r_1(t)+\phi'(t)\right]^2}{4r_1(t)\phi(t)t^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,t)}\\&\le -w'(t).\end{aligned}\tag{24}$$

Multiplying both sides by $H(t,s)$ and then integrating from t_2 to t yields

$$\begin{aligned}&\int_{t_2}^t H(t,s)\Big[\phi(s)Q(s)s^{\alpha_3-1}-\phi(s)\eta'(s)\\&+\phi(s)r_1(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,s)\eta^2(s)\\&-\frac{\left[2\eta(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,s)r_1(s)+\phi'(s)\right]^2}{4r_1(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,s)}\Big]ds\\&\le-\int_{t_2}^t H(t,s)w'(s)ds\le H(t,t_2)w(t_2)\le H(t,t_0)w(t_2).\end{aligned}$$

Then,

$$\begin{aligned}&\int_{t_0}^t H(t,s)\Big[\phi(s)Q(s)s^{\alpha_3-1}-\phi(s)\eta'(s)\\&+\phi(s)r_1(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,s)\eta^2(s)\\&-\frac{\left[2\eta(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,s)r_1(s)+\phi'(s)\right]^2}{4r_1(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,s)}\Big]ds\\&\le H(t,t_0)\int_{t_0}^{t_2}\Big|\phi(s)Q(s)s^{\alpha_3-1}-\phi(s)\eta'(s)\\&+\phi(s)r_1(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,s)\eta^2(s)\\&-\frac{\left[2\eta(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,s)r_1(s)+\phi'(s)\right]^2}{4r_1(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,s)}\Big|ds\\&+H(t,t_0)w(t_2).\end{aligned}$$

Taking limit supremum on both sides, we have

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t H(t,s)\Big[\phi(s)Q(s)s^{\alpha_3-1}-\phi(s)\eta'(s)\\&+\phi(s)r_1(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,s)\eta^2(s)\\&-\frac{\left[2\eta(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,s)r_1(s)+\phi'(s)\right]^2}{4r_1(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(t_2,s)}\Big]ds\\&<\infty,\end{aligned}$$

which contradicts (23). The proof of the theorem is complete.

In Theorem 3.3, if we take $H(t,s)$ to have the form of some special functions such as $(t-s)^m$ or $log\left(\frac{t}{s}\right)$, then we obtain the following corollaries.

Corollary: 3.1. Assume that (6)-(8) hold. Furthermore, suppose that

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{(t-t_0)^m}\int_{t_0}^t (t-s)^m\Big[\phi(s)Q(s)s^{\alpha_3-1}-\phi(s)\eta'(s)\\&+\phi(s)r_1(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)\eta^2(s)\\&-\frac{\left[2\eta(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)r_1(s)+\phi'(s)\right]^2}{4r_1(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)}\Big]ds\\&=\infty,\end{aligned}$$

for sufficiently large T. Then every solution of Eq.(1) is oscillatory or satisfies $lim_{t\to\infty}K(t)=0$.

Corollary: 3.2. Assume that (6)-(8) hold. Furthermore, suppose that

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{logt-logt_0}\\&\times\int_{t_0}^t (logt-logs)\Big[\phi(s)Q(s)s^{\alpha_3-1}-\phi(s)\eta'(s)\\&+\phi(s)r_1(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)\eta^2(s)\\&-\frac{\left[2\eta(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)r_1(s)+\phi'(s)\right]^2}{4r_1(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)}\Big]ds\\&=\infty,\end{aligned}$$

for sufficiently large T. Then every solution of Eq.(1) is oscillatory or satisfies $lim_{t\to\infty}K(t)=0$.

4 Examples

We conclude this paper with three examples to illustrate our main results.

Example: 4.1. Consider the mixed fractional differential equation

$$\begin{aligned}&T_{\frac{1}{6}}\left[\frac{1}{t^{\frac{2}{3}}}\left(T_{\frac{1}{3}}\left(D_+^{\frac{1}{2}}x(t)\right)\right)\right]+\frac{t^{\frac{5}{6}}\sin(t+\frac{\pi}{4})}{\sqrt{2\pi}[\sin tC(x)-\cos tS(x)]}\\&\times\int_0^t (t-s)^{-\frac{1}{2}}x(s)ds=0\end{aligned}\tag{25}$$

for $t\ge 1$. This corresponds to Eq.(1) with $r_2(t)=t^{\frac{2}{3}}, r_1(t)=1, \alpha_1=\frac{1}{2}, \alpha_2=\frac{1}{3}, \alpha_3=\frac{1}{6}, f_1(x)=f_2(x)=x, Q(t)=\frac{t^{\frac{5}{6}}\sin(t+\frac{\pi}{4})}{\sqrt{2\pi}[\sin tC(x)-\cos tS(x)]}, R_1(T,s)=(s-t)$ and $C(x),S(x)$ are the Fresnel integrals namely

$$C(x)=\int_0^x cos(\frac{1}{2}\pi t^2)dt,\quad S(x)=\int_0^x sin(\frac{1}{2}\pi t^2)dt.$$

Take $|C(x)| \leq \pi$ and $|S(x)| \leq \pi$. Now, consider

$$\int_1^\infty \frac{1}{f_2^{-1}\left(\frac{1}{r_2(s)}\right)} d_{\alpha_2} s = \int_1^\infty \frac{1}{f_2^{-1}\left(\frac{1}{r_2(s)}\right)} s^{\alpha_2 - 1} ds = \int_1^\infty ds = \infty.$$

$$\int_1^\infty f_1^{-1}\left[r_1(\tau)\int_\tau^\infty f_2^{-1}\left(r_2(\xi)\int_\xi^\infty Q(s) d_{\alpha_3} s\right) d_{\alpha_2}\xi\right] d\tau = \int_1^\infty f_1^{-1}\left[r_1(\tau)\int_\tau^\infty f_2^{-1} \times \left(r_2(\xi)\int_\xi^\infty Q(s) s^{\alpha_3 - 1} ds\right)\xi^{\alpha_2 - 1} d\xi\right] d\tau$$
$$= \int_1^\infty \left[\int_\tau^\infty \left(\int_\xi^\infty \frac{s^{\frac{5}{6}}(\sin s + \cos s) s^{\frac{-5}{6}}}{2\pi\sqrt{\pi}(\sin s - \cos s)} ds\right) d\xi\right] d\tau = \infty.$$

Let $\eta = 2, \phi(s) = 1, \phi'(s) = 0, \mu = \delta = 1$. For any $T \geq 1$,

$$\int_T^\infty \phi(s) Q(s) s^{\alpha_3 - 1} - \phi(s)\eta'(s) + \phi(s) r_1(s) s^{1-\alpha_3} \mu\delta\Gamma(1-\alpha_1) R_1(T,s)\eta^2(s) - \frac{[2\eta(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)r_1(s) + \phi'(s)]^2}{4r_1(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)} ds$$
$$= \int_T^\infty \frac{s^{\frac{5}{6}}(\sin s + \cos s) s^{\frac{-5}{6}}}{2\pi\sqrt{\pi}[\sin s - \cos s]} + s^{\frac{5}{6}}\sqrt{\pi}(s-1)4 - \frac{[4s^{\frac{5}{6}}\sqrt{\pi}(s-1)]^2}{4s^{\frac{5}{6}}\sqrt{\pi}(s-1)} ds \to \infty.$$

Thus all the conditions of Theorem 3.1 are satisfied. Therefore, every solution of Eq.(25) is oscillatory. In fact, $x(t) = \sin t$ is one such solution of Eq.(25).

Example: 4.2. Consider the fractional differential equation

$$T_{\frac{1}{7}}\left[\frac{1}{t^{\frac{12}{5}}}\left(T_{\frac{3}{5}}\left(\frac{1}{t^{-2}} D_+^{\frac{1}{2}} x(t)\right)\right)\right] + \frac{\sqrt{\pi}}{t^{\frac{8}{7}}\sqrt{2\pi}[\sin t C(x) - \cos t S(x)]} \times \int_0^t (t-s)^{-\frac{1}{2}} x(s) ds = 0, \tag{26}$$

for $t \geq 1$ which has the form of Eq.(1) with $r_2(t) = t^{\frac{12}{5}}, r_1(t) = t^{-2}, \alpha_1 = \frac{1}{2}, \alpha_2 = \frac{3}{5}, \alpha_3 = \frac{1}{7}, f_i(x) = x, i = 1,2, Q(t) = \frac{\sqrt{\pi}}{t^{\frac{8}{7}}\sqrt{2\pi}[\sin t C(x) - \cos t S(x)]}$ while $C(x)$ and $S(x)$ are defined as in Example 4.1.

We have

$$\int_1^\infty r_1(s) ds = \int_1^\infty s^{-2} ds < \infty.$$

$$\int_1^\infty f_1^{-1}\left[r_1(\tau)\int_\tau^\infty f_2^{-1}\left(r_2(\xi)\int_\xi^\infty Q(s) d_{\alpha_3} s\right) d_{\alpha_2}\xi\right] d\tau \geq \int_1^\infty \tau^2\left[\int_\tau^\infty \xi^2\left(\int_\xi^\infty \frac{s^{\frac{-6}{7}}}{s^{\frac{8}{7}}(\sin s - \cos s)\pi\sqrt{2}} ds\right) d\xi\right] d\tau = \infty.$$

Let $\eta = 0, \phi(s) = 1, \phi'(s) = 0, \mu = \delta = 1$. For any $T \geq 1$,

$$\int_T^\infty \phi(s) Q(s) s^{\alpha_3 - 1} - \phi(s)\eta'(s) + \phi(s) r_1(s) s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1) R_1(T,s)\eta^2(s) - \frac{[2\eta(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)r_1(s) + \phi'(s)]^2}{4r_1(s)\phi(s)s^{1-\alpha_3}\mu\delta\Gamma(1-\alpha_1)R_1(T,s)} ds$$
$$\geq \int_T^\infty \left[\frac{s^{\frac{1}{7}-1}}{\pi\sqrt{2}s^{\frac{8}{7}}[\sin s - \cos s]}\right] ds \geq \frac{1}{\pi 2\sqrt{2}}.$$

We observe that some of the conditions of Theorem 3.1 are not satisfied. In fact, Eq.(7) and Eq.(18) are not satisfied. Hence, Eq.(26) has a nonoscillatory solution $x(t) = (t-s)^{\frac{1}{2}}$.

Example: 4.3. Consider the fractional differential equation

$$T_{\frac{1}{7}}\left[\frac{1}{t^{\frac{4}{5}}}\left(T_{\frac{1}{5}}\left(D_+^{\frac{1}{2}} x(t)\right)\right)\right] + \frac{t^{\frac{6}{7}}\sin(t + \frac{\pi}{4})}{\sqrt{2\pi}[\sin t C(x) - \cos t S(x)]}\int_0^t (t-s)^{-\frac{1}{2}} x(s) ds = 0 \tag{27}$$

for $t \geq 1$. This corresponds to Eq.(1) with $r_2(t) = t^{\frac{4}{5}}, r_1(t) = 1, \alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{5}, \alpha_3 = \frac{1}{7}, f_1(x) = f_2(x) = x, Q(t) = \frac{t^{\frac{6}{7}}\sin(t+\frac{\pi}{4})}{\sqrt{2\pi}[\sin t C(x) - \cos t S(x)]}$ while $C(x)$ and $S(x)$ are defined as in Example 4.1. Now, consider

$$\int_1^\infty r_1(s) ds = \int_1^\infty ds = \infty.$$

$$\int_1^\infty f_1^{-1}\left[r_1(\tau)\int_\tau^\infty f_2^{-1}\left(r_2(\xi)\int_\xi^\infty Q(s) d_{\alpha_3} s\right) d_{\alpha_2}\xi\right] d\tau = \int_1^\infty \left[\int_\tau^\infty \xi^{\frac{-8}{5}}\left(\int_\xi^\infty \frac{s^{\frac{6}{7}}(\sin s + \cos s) s^{\frac{-6}{7}}}{2\pi\sqrt{\pi}(\sin s - \cos s)} ds\right) d\xi\right] d\tau = \infty.$$

Let $\eta = 1, \phi(s) = 1, \phi'(s) = 0, \mu = \delta = 1$. For any $T \geq 1$,

$$\int_T^\infty \left(\frac{\phi(s)Q(s)s^{\alpha_3 - 1}}{\delta_2'} - \phi(s)\eta'(s) \right.$$
$$+ \phi(s)r_1(s)\delta\Gamma(1-\alpha_1)R_1(T,s)\eta^2(s)$$
$$\left. - \frac{[2\eta(s)\phi(s)\delta\Gamma(1-\alpha_1)R_1(T,s)r_1(s) + \phi'(s)]^2}{4r_1(s)\phi(s)\delta\Gamma(1-\alpha_1)R_1(T,s)} \right) ds$$
$$= \int_T^\infty \left[\frac{s^{\frac{6}{7}}(\sin s + \cos s)s^{\frac{-6}{7}}}{2\pi\sqrt{\pi}[\sin s - \cos s]} \right] ds \to \infty.$$

Thus all the conditions of Theorem 3.2 are satisfied. Therefore, every solution of Eq.(27) is oscillatory. In fact, $x(t) = \sin t$ is one such solution of Eq.(27).

5 Conclusion

In this present paper, we have derived some new oscillation results for a certain class of nonlinear mixed fractional order differential equations with the Conformable fractional derivative and the Riemann-Liouville left-sided fractional derivative, by using the generalized Riccati technique and the integral averaging method. Our newly-derived oscillation results extend and improve numerous findings in the recent publications on classical literature to mixed fractional differential equations. We believe that this research work would lead to further work on the mixed fractional differential equations.

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