A Novel Approach to Find Optimal Parameter in the Homotopy-Regularization Method for Solving Integral Equations

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Abstract: The regularization method is one of the important schemes to solve the ill-posed problems. In this work, by combining the Wazwaz's regularization method and the homotopy analysis method, a new and robust approach is presented to solve integral equations which is called the homotopy-regularization method. The solution which is produced by the homotopy-regularization method depends on the regularization parameter. In order to find the optimal value of this parameter, the Controle et Estimation Stochastique des Arrondis de Calculs method is applied which is based on the stochastic arithmetic. A theorem is presented to show the accuracy of the proposed approach. Also, in order to implement the algorithm, the Control of Accuracy and Debugging for Numerical Applications library is applied to perform the Controle et Estimation Stochastique des Arrondis de Calculs method in the stochastic arithmetic automatically. Some examples of the singular and ill-posed integral equations are illustrated. The numerical results show the abilities of the Controle et Estimation Stochastique des Arrondis de Calculs method to find the optimal regularization parameter and the optimal approximation of the homotopy-regularization method.

Keywords: Regularization method, Homotopy analysis method, Stochastic arithmetic, CESTAC method, CADNA library, Integral equation

1 Introduction

The integral equation (IEs) of the first kind are appeared in many fields of engineering such as image processing and electromagnetic which can be considered in the class of ill-posed problems [1]-[7]. In last decade, the authors used different methods to solve the IEs of the first kind such as Galerkin method [3,8,9,10], collocation method [11,12], homotopy analysis method [13,14] and others [15]-[21].

The regularization method (RM) is one of powerful and important methods to solve the IEs of the first kind. Tikhonov [22,23] and Phillips [24] applied the RM to solve incorrectly-posed problems. Also, the RM is generalized by Wazwaz in [20,25] and others [11,17,19, 26,27,28,29]. Many methods such as the homotopy analysis method (HAM) [13,14,30,31,32,33,34,35,36, 37,38], collocation method [11,12,39], integral mean value method [19,40,41] can be combined by the RM to

solve the first kind IEs. The results of the mentioned studies are usually based on the common computer arithmetic by using packages like Matlab, Maple or Mathematica. We apply the Wazwaz's RM and the HAM to solve the singular and ill-posed IEs by means of the stochastic arithmetic (SA) in place of the floating-point arithmetic (FPA). The solution of the homotopy-regularization method (HRM) depends on the regularization parameter. When this parameter leads to zero, the approximate solution can be obtained [20,25].

Now, an important question can be made. What is the optimal value of the regularization parameter and how can we find this value? In this paper, the Controle et Estimation Stochastique des Arrondis de Calculs (CESTAC) method [42]-[46] is applied to find the optimal value of regularization parameter. This method is based on the SA [47]-[52]. In order to implement the CESTAC method the Control of Accuracy and Debugging for Numerical Applications

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Numerical Applications (CADNA) library is applied [53]-[66]. It can convert the common FPA to the SA on a code automatically. For this purpose, the program must be written by C, C++, FORTRAN or ADA and run in Linux operating system [67,68]. Finding the best step of numerical methods, determining the validation of results and neglecting the unnecessary iterations are some abilities of the SA in comparison with the FPA.

The present study is formed by the following organizations: at first the HRM is presented to solve the first kind IEs in section 2. The CADNA library and the CESTAC method are explained briefly in the section 3. Also, a theorem is proved which allows us to apply the CESTAC method and show the accuracy of the proposed approach. Three examples of the first kind IEs are presented in section 4 by means of the proposed algorithm to find the optimal value of the regularization parameter. Finally, the last section is conclusion.

2 The homotopy-regularization method

Consider the following non-linear IE of the first kind

$$g(v) = \lambda \int_{z_1(v)}^{z_2(v)} K(v,s) F(Y(s)) ds, \qquad (1)$$

where $g(v), K(v,s)$ and λ are given, $Y(v)$ is the unknown function and F is a nonlinear operator. When $z_1(v)$ and $z_2(v)$ are constant, Eq. (1) is called the Fredholm IE and when $z_1(v)$ is constant and $z_2(v) = v \leq b$, Eq. (1) is called the Volterra IE. The first kind IE (1) has appeared in many physical and engineering models such as radiography, image and signal processing and so on. Also, we note that IE (1) can be an ill-posed problem which is either without solution or with alternative solutions.

By using transformation $\psi(v) = F(Y(s))$, and applying the RM proposed by Wazwaz the following second kind IE is obtained as

$$\alpha \psi^\alpha(v) = g(v) - \lambda \int_{z_1(v)}^{z_2(v)} K(v,s) \psi^\alpha(s) ds, \qquad (2)$$

where $\alpha > 0$ is the regularization parameter. From Eq. (2), we get

$$\psi^\alpha(v) = \frac{1}{\alpha} g(v) - \frac{1}{\alpha} \lambda \int_{z_1(v)}^{z_2(v)} K(v,s) \psi^\alpha(s) ds. \qquad (3)$$

Let

$$\widehat{\mathcal{N}}\big[\Psi^\alpha(v;\ell)\big]$$

$$= \Psi^\alpha(v;\ell) - \frac{1}{\alpha} g(v) + \frac{1}{\alpha} \lambda \int_{z_1(v)}^{z_2(v)} K(v,s) \Psi^\alpha(s;\ell) ds,$$

is an operator of non-linear form where the homotopy parameter is shown by $\ell \in [0,1]$ and $\Psi^\alpha(v;\ell)$ is an unknown function.

The following zero-order deformation equation can be produced by the prevalent homotopy method [32,33] as

$$(1-\ell)\widehat{\mathcal{L}}[\Psi^\alpha(v;\ell) - \psi_0^\alpha(v)] = \ell\hbar\widehat{H}(v)\widehat{\mathcal{N}}[\Psi^\alpha(v;\ell)], \qquad (4)$$

where the non-zero auxiliary parameter which is called convergence-control parameter is demonstrated by $\hbar$, the auxiliary function is displayed by $\widehat{H}(v) \neq 0$, the auxiliary operator of linear form is shown by $\widehat{\mathcal{L}}$, the primary conjecture of $\psi^\alpha(v)$ is shown by $\psi_0^\alpha(v)$ and $\Psi^\alpha(v;\ell)$ is an unknown function. By putting $\ell = 0,1$ into Eq. (4), since $\hbar \neq 0$ and $\widehat{H}(v) \neq 0$ we get $\Psi^\alpha(v;0) = \psi_0^\alpha(v)$ and $\Psi^\alpha(v;1) = \psi^\alpha(v)$ respectively. The following relation is obtained from Taylor's theorem as

$$\Psi^\alpha(v;\ell) = \psi_0^\alpha(v) + \sum_{d=1}^{\infty} \psi_d^\alpha(v)\ell^d, \qquad (5)$$

where

$$\psi_d^\alpha(v) = \frac{1}{d!} \frac{\partial^d \Psi^\alpha(v;\ell)}{\partial v^d}\bigg|_{\ell=0}.$$

In order to study the convergence of series (5) at $\ell = 1$, we assume that $\hbar, \widehat{H}(v), \psi_0^\alpha(v)$ and $\widehat{\mathcal{L}}$ are so correctly selected. Therefore, the following series solution can be obtained:

$$\psi_d^\alpha(v) = \Psi^\alpha(v;1) = \psi_0^\alpha(v) + \sum_{d=1}^{\infty} \psi_d^\alpha(v). \qquad (6)$$

According to [13,14,32,33], the d-th order deformation equation is defined as follows

$$\widehat{\mathcal{L}}\big[\psi_d^\alpha(v) - \upsilon_d\psi_{d-1}^\alpha(v)\big] = \hbar\widehat{H}(v)\big[\mathfrak{R}_d(\psi_{d-1}^\alpha)\big], \qquad (7)$$

where

$$\mathfrak{R}_d(\psi_{d-1}^\alpha)$$

$$= \psi_{d-1}^\alpha(v) + \frac{1}{\alpha}\lambda \int_{z_1(v)}^{z_2(v)} K(v,s)\psi_{d-1}^\alpha(s)ds - (1-\chi_d)\frac{1}{\alpha}g(v),$$

and

$$\chi_d = \begin{cases} 0, & d \leq 1, \\ 1, & d > 1. \end{cases}$$

By choosing $\widehat{\mathcal{L}}\psi^\alpha = \psi^\alpha$, $\widehat{H}(v) = 1$ and $\hbar = -1$, the following successive relations are produced

$$\begin{cases} \psi_0^\alpha(v) = \dfrac{1}{\alpha}g(v), \\[2mm] \psi_d^\alpha(v) = -\dfrac{1}{\alpha}\lambda \displaystyle\int_{z_1(v)}^{z_2(v)} K(v,s)\psi_{d-1}^\alpha(s)ds, \quad d \geq 1. \end{cases} \qquad (8)$$

The j-th order of regularized approximate solution can be estimated by

$$\psi_j^\alpha(v) = \sum_{d=0}^{j} \psi_d^\alpha(v). \qquad (9)$$

Tikhonov [22,23] and Phillips [24] proved that the solution of IE (3) can be obtained when α leads to zero. Finally, the transformation $Y = F^{-1}(\psi)$ is applied to find the exact solution of Eq. (1).

3 CESTAC method-CADNA library

Let F be a set of real values which are reproduced by computer and the arbitrary value θ is demonstrated as $\Theta \in F$. In the personal computer (PC), Θ with the binary FPA, ρ mantissa bits and the rounding error term is given by:

$$\Theta = \theta - \varepsilon 2^{E-\eta}\rho, \qquad (10)$$

where $2^{-\eta}\rho$ is the missing segment of mantissa which is obtained from round-off error, ε is the sign of θ, E is the binary power of the outcome and ρ is a random parameter on $[-1,1]$ which is distributed uniformly and is applied to perturb on the last mantissa bit of θ. Therefore, the random result of Θ can be calculated where mean (μ) and standard deviation (σ) are applied to guarantee the precision of results. The details of the SA can be found in [47-52].

In PC, for $\eta = 23, 52$, the results can be obtained by single or double accuracies respectively. By n times performing the process for $\Theta_i, i = 1, ..., n$, their distribution is presented in the quasi Gaussian form. Therefore, the mean of them is equal with the exact value of θ and the values of μ and σ can be estimated by these n samples. In the following algorithm which is based on the CESTAC method the value of T distribution with $n-1$ freedom's degree is shown by τ_δ for the confidence interval $1 - \delta$.

Algorithm 1:

Step 1- Apply the perturbation technique to find n samples of Θ as $\Phi = \{\Theta_1, \Theta_2, ..., \Theta_n\}$

Step 2- Compute $\Theta_{ave} = \frac{\sum_{i=1}^{n} \Theta_i}{n}$.

Step 3- Calculate $\sigma^2 = \frac{\sum_{i=1}^{n}(\Theta_i - \Theta_{ave})^2}{n-1}$.

Step 4- Compute $C_{\Theta_{ave}, \Theta} = \log_{10} \frac{\sqrt{n}|\Theta_{ave}|}{\tau_\delta \sigma}$, as estimation of the common significant digits between Θ and Θ_{ave}.

Step 5- If $C_{\Theta_{ave}, \Theta} \leq 0$ or $\Theta_{ave} = 0$, then write $\Theta = @.0$.

The CADNA library can create a new environment to calculate the more precisely approximations by using the random arithmetic. CADNA can be used to check numerical instabilities and find the number of unstable operations.

CADNA enables us to create new numerical types of data with other operators such as ADA, C/C++ or FORTRAN. The codes of CADNA library are similar to programs which are produced by these operators and by minor variations. Hence, we can apply and run CADNA programs. When the informatical zero's sign @.0 is shown, the process of CADNA's algorithm will be stopped. This sign shows that the number of common significant digits between two successive approximations of iterative methods is almost equal with the number of common significant digits between exact and numerical solutions.

The advantages of using the CADNA library based on the SA and the CESTAC method are:
1) The CADNA library is able to identify the optimal iteration and optimal approximation.
2) The accuracy of any numerical result is estimated, during the performance of a code.
3) The numerical instabilities are detected and the branching in an algorithm are checked.
4) Unnecessary iterations are eliminated through which the FPA is not able to distinguish them. In some cases, the termination criterion of iterative methods is not suitable so that the implementation of the algorithm is continued without improvement in the accuracy of the result. In the SA, instead of the termination criterion, a criterion that directly reflects the mathematical condition, is replaced, that must be satisfied by the solution.
5) It is an effective and powerful tool that helps us to achieve the validation of scientific programs and gives them a reliability.

Further information to study are presented in www.cadna.lib6.fr.

Definition 1.[53,55] *The number of common significant digits between two real numbers ω_1 and ω_2 is defined in the following form*

$$\begin{cases} C_{\omega_1, \omega_2} = \log_{10} \left| \frac{\omega_1 + \omega_2}{2(\omega_1 - \omega_2)} \right| \\ \qquad = \log_{10} \left| \frac{\omega_1}{\omega_1 - \omega_2} - \frac{1}{2} \right|, \quad \omega_1 \neq \omega_2, \quad (11) \\ C_{\omega_1, \omega_1} = +\infty. \end{cases}$$

Theorem 1. *Let* $\psi_d^{\alpha_k}(v) = \sum_{i=0}^{d} \psi_i^{\alpha_k}(v)$ *and* $\psi_d^{\alpha_{k+1}}(v) = \sum_{i=0}^{d} \psi_i^{\alpha_{k+1}}(v)$ *be the d-th order approximate solution of Eq. (3) and α_k, α_{k+1} for $k = 0, 1, 2, ...$ are two successive regularization parameters, then*

$$C_{\psi_d^{\alpha_k}, \psi_d^{\alpha_{k+1}}} = C_{\psi_d^{\alpha_k}, \psi} + \log_{10} \left| 1 + O(\frac{1}{d!}) \right|. \qquad (12)$$

Proof: Applying Definition 1 as

$$C_{\psi_d^{\alpha_k},\psi_d^{\alpha_{k+1}}} - C_{\psi_d^{\alpha_k},\psi}$$

$$= \log_{10}\left|\frac{\psi_d^{\alpha_k} + \psi_d^{\alpha_{k+1}}}{2(\psi_d^{\alpha_k} - \psi_d^{\alpha_{k+1}})}\right| - \log_{10}\left|\frac{\psi_d^{\alpha_k} + \psi}{2(\psi_d^{\alpha_k} - \psi)}\right|$$

$$= \log_{10}\left|\frac{\psi_d^{\alpha_k} + \psi_d^{\alpha_{k+1}}}{\psi_d^{\alpha_k} + \psi}\right| + \log_{10}\left|\frac{\psi_d^{\alpha_k} - \psi}{\psi_d^{\alpha_k} - \psi_d^{\alpha_{k+1}}}\right|$$

$$= \log_{10}\left|\frac{\psi_d^{\alpha_k} + \psi_d^{\alpha_{k+1}}}{\psi_d^{\alpha_k} + \psi}\right| + \log_{10}\left|1 + \frac{\psi_d^{\alpha_{k+1}} - \psi}{\psi_d^{\alpha_k} - \psi_d^{\alpha_{k+1}}}\right|. \tag{13}$$

According to the HAM, we can write:

$$\psi_d^{\alpha_k} - \psi_d^{\alpha_{k+1}}$$

$$= \sum_{i=0}^{d}\psi_i^{\alpha_k} - \sum_{i=0}^{d}\psi_i^{\alpha_{k+1}} = \sum_{i=0}^{d}\hbar\Re_i(\psi_i^{\alpha_k}) - \sum_{i=0}^{d}\hbar\Re_i(\psi_i^{\alpha_{k+1}})$$

$$= \hbar\sum_{i=0}^{d}\left[\frac{1}{(i)!}\frac{\partial^i N[\Psi^{\alpha_k}(v;\ell)]}{\partial\ell^i}\bigg|_{\ell=0} - \frac{1}{(i)!}\frac{\partial^i N[\Psi^{\alpha_{k+1}}(v;\ell)]}{\partial\ell^i}\bigg|_{\ell=0}\right]$$

$$= O(1). \tag{14}$$

Also

$$\psi_d^{\alpha_{k+1}} - \psi$$

$$= \sum_{i=0}^{d}\psi_i^{\alpha_{k+1}} - \sum_{i=0}^{\infty}\psi_i^{\alpha_{k+1}} = -\sum_{i=d+1}^{\infty}\psi_i^{\alpha_{k+1}}$$

$$= -\sum_{i=d+1}^{\infty}\hbar\widehat{\mathscr{L}}^{-1}\left[\widehat{H}(v)\frac{1}{(i-1)!}\frac{\partial^{i-1}N[\Psi^{\alpha_{k+1}}(v;\ell)]}{\partial\ell^{i-1}}\bigg|_{\ell=0}\right]$$

$$= O\left(\frac{1}{d!}\right). \tag{15}$$

First term of Eq. (13) leads to zero because for proper value of d the approximate solution $\psi_d^{\alpha_k}$ and $\psi_d^{\alpha_{k+1}}$ are close to the exact solution ψ. By substituting Eqs. (14) and (15) into second term of Eq. (13) the following relation is obtained as

$$C_{\psi_d^{\alpha_k},\psi_d^{\alpha_{k+1}}} - C_{\psi_d^{\alpha_k},\psi}$$

$$= \log_{10}\left|1 + \frac{\psi_d^{\alpha_{k+1}} - \psi}{\psi_d^{\alpha_k} - \psi_d^{\alpha_{k+1}}}\right| \tag{16}$$

$$= \log_{10}\left|1 + \frac{O(\frac{1}{d!})}{O(1)}\right| = \log_{10}\left|1 + O\left(\frac{1}{d!}\right)\right|,$$

and as d increases, the right hand side of Eq. (16) leads to zero and we conclude by

$$C_{\psi_d^{\alpha_k},\psi_d^{\alpha_{k+1}}} \approx C_{\psi_d^{\alpha_k},\psi}.\square \tag{17}$$

Eq. (17) shows that the number of common significant digits between $\psi_d^{\alpha_k}, \psi$ is almost equal with $\psi_d^{\alpha_k}, \psi_d^{\alpha_{k+1}}$. This theorem was presented for fixed value of iteration d. Also, by using two successive regularization parameters α_k and α_{k+1} we can find the optimal value of α.

4 Numerical examples

In many iterative methods based on the FPA applied to solve the linear and non-linear problems, the general absolute error or other equivalent conditions can be used to show the efficiency of method as follows

$$\left|\psi - \psi_d^{\alpha_k}\right|, \tag{18}$$

where ψ is an exact solution and $\psi_d^{\alpha_k}$ shows the regularized approximate solution of iteration d. This condition can not be acceptable. It depends on the exact solution. Also, condition (18) should be applied for special iteration of iterative methods or it should depend a parameter like ε for stopping the numerical algorithms such as

$$\left|\psi - \psi_d^{\alpha_k}\right| \le \varepsilon, \tag{19}$$

which can not be acceptable too. If ε is chosen very large, the iterations are stopped before getting access to a suitable approximation. If ε is chosen very small, then unnecessary iterations are done without improving the accuracy of the results.

In order to modify these problems the following criterion based on the SA is applied as

$$\left|\psi_d^{\alpha_k} - \psi_d^{\alpha_{k+1}}\right| = @.0, \tag{20}$$

where $\psi_d^{\alpha_k}$ and $\psi_d^{\alpha_{k+1}}$ are the approximate solutions for fixed iteration d and two successive regularized parameters α_k and α_{k+1}. By condition (20) we do not need to have the exact solution and apply the parameter ε. Theorem 1 permits to apply the condition (20) instead of (18) or (19).

In this section, some applicable examples of the linear and non-linear first kind IEs are presented. Several tables of numerical results are presented to find the optimal value of regularization parameter α based on Algorithm 2. This algorithm is written by CADNA library with C++ on Linux operating system.

Algorithm 2:
Step 1- Enter $\psi^\alpha(v), \Delta\alpha, v$ and initial value α_0.
Step 2- Let $k = 1$.
Step 3- Do the following steps while $|\psi^{\alpha_k} - \psi^{\alpha_{k-1}}| \ne @.0$
　　{
　　Step 3-1- Print $k, \alpha_k, Y(v), \psi^{\alpha_k}(v)$,
　　$|\psi^{\alpha_k}(v) - \psi^{\alpha_{k-1}}(v)|, |\psi(v) - \psi^{\alpha_{k-1}}(v)|$.
　　Step 3-2- $\alpha_{k+1} = \alpha_k + \Delta\alpha$.
　　}

Example 1: Consider the following linear Fredholm IE of the first kind [20,25]

$$\frac{\pi}{2}\cos v = \int_0^\pi \cos(v-s)Y(s)ds. \qquad (21)$$

Now, the RM is applied to transform the first kind IE (21) to the second kind as follows

$$\alpha\psi_\alpha(v) = \frac{\pi}{2}\cos v - \int_0^\pi \cos(v-s)\psi_\alpha(s)ds,$$

and we can write

$$\psi_\alpha(v) = \frac{\pi}{2\alpha}\cos v - \frac{1}{\alpha}\int_0^\pi \cos(v-s)\psi_\alpha(s)ds. \qquad (22)$$

By using Eq. (8) the following relations can be obtained

$$\begin{cases} \psi_0^\alpha(v) = \frac{1}{\alpha}g(v) = \frac{\pi}{2\alpha}\cos v, \\[2mm] \psi_1^\alpha(v) = -\frac{1}{\alpha}\int_0^\pi \cos(v-s)\psi_0^\alpha(s)ds = -\frac{\pi^2}{4\alpha^2}\cos v, \\[2mm] \psi_2^\alpha(v) = -\frac{1}{\alpha}\int_0^\pi \cos(v-s)\psi_1^\alpha(s)ds = \frac{\pi^3}{8\alpha^3}\cos v, \\[2mm] \vdots \\[2mm] \psi_d^\alpha(v) = -\frac{1}{\alpha}\int_0^\pi \cos(v-s)\psi_{d-1}^\alpha(s)ds, \quad d \ge 3, \end{cases} \qquad (23)$$

and the regularized series solution is in the following form

$$\psi^\alpha(v)$$

$$= \sum_{d=0}^\infty \psi_d^\alpha(v)$$

$$= \frac{\pi}{2\alpha}\cos v - \frac{\pi^2}{4\alpha^2}\cos v + \frac{\pi^3}{8\alpha^3}\cos v + \cdots = \frac{\pi\cos v}{2\alpha + \pi}. \qquad (24)$$

The numerical results for $v = 0.2$ and tolerance $\alpha_k - \alpha_{k-1} = -0.0000001$ are presented in Table 1. When the number of significant digits between $|\psi^{\alpha_k}(v) - \psi^{\alpha_{k-1}}(v)|$ and $|\psi(v) - \psi^{\alpha_k}(v)|$ is zero then the CADNA library is stopped and denoted by @.0 which is called the informatical zero. This step is the optimal value of α. In this example, we have $\alpha_{opt} = 0.0009997$.

Therefore the approximate solution of IE (21) is $Y^{\alpha_4}(v) = \frac{\pi\cos v}{2(0.0009997)+\pi}$.

Example 2: The following non-linear IE [19,20]

$$v\ln 2 = \int_0^{\frac{\pi}{3}} vY^4(s)ds, \qquad (25)$$

is considered. By putting $\psi(v) = Y^4(v)$ and applying the RM, the non-linear IE (25) can be transformed to the

following regularized linear IE

$$\psi^\alpha(v) = \frac{1}{\alpha}v\ln 2 - \frac{1}{\alpha}\int_0^{\frac{\pi}{3}} v\psi^\alpha(s)ds. \qquad (26)$$

The series solution of the regularization-homotopy method is obtained as

$$\psi^\alpha(v)$$

$$= \sum_{d=0}^\infty \psi_d^\alpha(v) = \frac{v\ln 2}{\alpha} - \frac{\pi^2 v\ln 2}{18\alpha^2} + \frac{\pi^4 v\ln 2}{324\alpha^3} + \cdots$$

$$= \frac{18v\ln 2}{\pi^2 + 18\alpha}.$$

Now the CESTAC method is applied to find the optimal regularization parameter α. According to Table 2, optimal α of the HRM for $v = 0.8$ and $\Delta\alpha = -0.000001$ is $\alpha_{opt} = 0.000918$. The approximate solution of IE (26) is $\psi^{\alpha_{83}}(v) = \frac{18v\ln 2}{\pi^2+18(0.000918)}$. Therefore $Y(v) \approx \sqrt[4]{\psi^{\alpha_{83}}(v)} = \sqrt[4]{\frac{18v\ln 2}{\pi^2+18(0.000918)}}$.

Example 3: Consider the non-linear Abel IE [21]

$$\frac{4}{3}v^{\frac{3}{2}} = \int_0^v \frac{\ln(Y(s))}{\sqrt{v-s}}ds. \qquad (27)$$

By variations

$$\psi(v) = \ln(Y(s)), \ Y(v) = e^{\psi(v)},$$

and applying the RM, the IE (27) can be converted to linear and second kind IE

$$\psi^\alpha(v) = \frac{4}{3\alpha}v^{\frac{3}{2}} - \frac{1}{\alpha}\int_0^v \frac{\psi^\alpha(s)}{\sqrt{v-s}}ds. \qquad (28)$$

By applying the HAM and the following iteration formulas

$$\begin{cases} \psi_0^\alpha(v) = \frac{1}{\alpha}g(v) = \frac{4}{3\alpha}v^{\frac{3}{2}}, \\[2mm] \psi_1^\alpha(v) = -\frac{1}{\alpha}\int_0^v \frac{\psi_{\alpha,0}(s)}{\sqrt{v-t}}ds = -\frac{\pi v^2}{2\alpha^2}, \\[2mm] \psi_2^\alpha(v) = -\frac{1}{\alpha}\int_0^v \frac{\psi_{\alpha,1}(s)}{v^{\frac{3}{2}}}ds = \frac{8\pi v^{\frac{5}{2}}}{15\alpha^3}, \\[2mm] \vdots \\[2mm] \psi_d^\alpha(v) = -\frac{1}{\alpha}\int_0^v \frac{\psi_{\alpha,d-1}(s)}{v^{\frac{3}{2}}}ds, \quad d \ge 3, \end{cases}$$

Table 1: The numerical results of example 1 for $v = 0.2$ and tolerance $\Delta\alpha = -0.0000001$.

k	α_k	$Y(v)$	$\psi^{\alpha_k}(v)$	$\|\psi^{\alpha_k}(v) - \psi^{\alpha_{k-1}}(v)\|$	$\|\psi(v) - \psi^{\alpha_{k-1}}(v)\|$
1	0.0010000	$0.9800665E+000$	$0.9794430E+000$	$0.9794430E+000$	$0.6235E-003$
2	0.0009999	$0.9800665E+000$	$0.9794430E+000$	$0.5960464E-007$	$0.6234E-003$
3	0.0009998	$0.9800665E+000$	$0.9794431E+000$	$0.5960464E-007$	$0.6234E-003$
4	0.0009997	$0.9800665E+000$	$0.9794432E+000$	$@.0$	$0.6233E-003$

Table 2: The numerical results of example 2 for $v = 0.8$ and tolerance $\Delta\alpha = -0.000001$.

k	α_k	$\psi(v)$	$\psi^{\alpha_k}(v)$	$\|\psi^{\alpha_k}(v) - \psi^{\alpha_{k-1}}(v)\|$	$\|\psi(v) - \psi^{\alpha_{k-1}}(v)\|$
1	0.001000	$0.101131E+001$	$0.1009477E+001$	$0.1009477E+001$	$0.1841E-002$
2	0.000999	$0.101131E+001$	$0.1009479E+001$	$0.1E-005$	$0.1839E-002$
3	0.000998	$0.101131E+001$	$0.1009481E+001$	$0.1E-005$	$0.1837E-002$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
40	0.000961	$0.101131E+001$	$0.1009549E+001$	$0.1E-005$	$0.1769E-002$
41	0.000960	$0.101131E+001$	$0.1009551E+001$	$0.1907348E-005$	$0.1767E-002$
42	0.000959	$0.101131E+001$	$0.1009553E+001$	$0.1788139E-005$	$0.1765E-002$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
80	0.000921	$0.101131E+001$	$0.1009623E+001$	$0.1E-005$	$0.1695E-002$
81	0.000920	$0.101131E+001$	$0.1009624E+001$	$0.1E-005$	$0.1693E-002$
82	0.000919	$0.101131E+001$	$0.1009626E+001$	$0.1E-005$	$0.1692E-002$
83	0.000918	$0.101131E+001$	$0.100962E+001$	$@.0$	$0.169E-002$

Table 3: The numerical results of example 3 for $v = 0.5$ and tolerance $\Delta\alpha = -0.000001$.

k	α_k	$\psi(v)$	$\psi^{\alpha_k}(v)$	$\|\psi^{\alpha_k}(v) - \psi^{\alpha_{k-1}}(v)\|$	$\|\psi(v) - \psi^{\alpha_{k-1}}(v)\|$
1	0.0001000	$0.5000000E+000$	$0.4999624E+000$	$0.4999624E+000$	$0.375E-004$
2	0.0000990	$0.5000000E+000$	$0.4999628E+000$	$0.3E-006$	$0.372E-004$
3	0.0000980	$0.5000000E+000$	$0.4999631E+000$	$@.0$	$0.368E-004$

the regularized series solution of IE (28) can be produced as

$$\psi^\alpha(v)$$

$$= \sum_{d=0}^\infty \psi_d^\alpha(v)$$

$$= \frac{4}{3\alpha}v^{\frac{3}{2}} - \frac{\pi v^2}{2\alpha^2} + \frac{8\pi v^{\frac{5}{2}}}{15\alpha^3} + \cdots = 1.33v^{\frac{3}{2}}\left(\frac{1}{\alpha + 1.33v^{\frac{1}{2}}}\right).$$

The numerical results of regularized solution $\psi^\alpha(v)$ for different values of α and $\Delta\alpha = -0.000001$ are presented in Table 3. These results are obtained based on the SA and the CADNA library. The optimal value of α is $\alpha_{opt} = 0.0000980$, therefore the approximate solution is in the following form

$$Y(v) = \exp(\psi^{\alpha_3}(v)) = \exp\left(1.33v^{\frac{3}{2}}\left(\frac{1}{0.0000980 + 1.33v^{\frac{1}{2}}}\right)\right).$$

5 Conclusion

An approach based on a stochastical process is presented to find the optimal regularization parameter of the HRM for solving the IEs of the first kind. To this end, the CESTAC method is applied based on the SA. Also, instead of applying the common mathematical packages, the CADNA library is applied to validate the results and find the optimal solution. The presented theorem illustrates the accuracy of the approach and to apply the optimal termination criterion in the proposed algorithm. Some sample singular IEs of the first kind are presented to show the efficiency and importance of using the CESTAC method and the proposed algorithm.

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