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On the Qualitative Analyses of Integro-Differential Equations with Constant Time Lag

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Abstract: In this article, some new investigations are done on stability, asymptotically stability and instability of the zero solution and boundedness, integrability of solutions and integrability of derivatives of solutions of certain nonlinear Volterra delayed integro-differential equations (DIDEs) by using the Lyapunov's functional method. To fulfill the aim of this work, three meaningful Lyapunov functionals are defined as main tools. Then, five new results are given on the mentioned qualitative properties of solutions of the considered DIDEs. Our results have contributions to the relevant literature and they have sufficient conditions. They include and improve some former results that can be found in [1].

Keywords: DIDE; constant time lag; asymptotically stability, boundedness; integrability; instability; Lyapunov functional.

1 Introduction

Linear and non-linear Volterra integral equations and Volterra integro-differential equations with and without retardation(s) have many applications in sciences and engineering, for example, in physics, mechanics, heat transfer, viscoelasticity, electrical circuit, electro-chemistry, population dynamics, control theory and so on (see Burton [2], Hristova and Tunç [3], Lakshmikantham and Rama Mohana Rao [4], Peschel and Mende [5], Staffans [6], Wazwaz [7]). Thanks to such applications, during the last three decades, stability, asymptotic stability, uniform stability, boundedness, exponentially stability and convergence of solutions, existence of periodic solutions and some other properties of solutions of linear and non-linear Volterra integral equations and integro-differential equations with and without retardation(s) have been discussed by many researches in the relevant literature (see [1]-[63]).

Next, solving these kind of equations or finding their analytical solutions explicitly, except numerically, is a very difficult task. In the relevant literature to overcome these problems, some methods have been constructed or improved to get information about the qualitative analyses of solutions of these kind of equations without solving them. Some of these methods are called as fixed point method, second method of Lyapunov function, Lyapunov's functional method, formula of variations of parameters, perturbation approaches and so on. Here, we would not like to give the details of these methods.

However, nearly in all of the papers mentioned above, the Lyapunov's methods are used as effective tools to prove the basic results on the qualitative analyses of solutions of the mathematical models considered in these sources. However, the fixed point method, formula of variations of parameters and perturbation approaches are used in a little number of work on the qualitative analyses of solutions of that kind of equations.

This paper is motivated by the mentioned papers and that can be available in the literature, the books and in particular by the paper of Wang [1].

We should state that , in 1998, Wang [1] considered the following scalar Volterra integro-differential equation (IDE)

$$x'(t) = A(t)x(t) + \int_0^t C(t,s)x(s), \tag{1}$$

where t is non-negative and real variable, that is, $t \in [0,\infty)$, $x \in \Re$, $\Re = (-\infty,\infty)$, $A(t)$ and $C(t,s)$ are continuous scalar functions for $0 \le t < \infty$, and $0 \le s \le t < \infty$, respectively.

In [1], a study has been made on the stability, uniform stability and asymptotically stability of the zero solution of IDE (1) by the second method of Lyapunov. The study made in [1] leads to some new results which are in simple forms and they can be easily verified and applied.

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In this paper, instead of IDE (1), we consider the following DIDE,

$$x'(t) = -A(t)x(t) + \int_{t-\tau}^{t} C(t,s,h(x(s)))f(x(s))ds, \quad (2)$$

where $t \in [0,\infty)$, $x \in \Re$, $A(t)$ and $C(t,s,h(x(s)))$ are continuous functions for their respective arguments such that $0 \le s \le t < \infty$, $h(0) = 0$, $h(x) \ne 0$ when $x \ne 0$, $C(t,s,0) = 0$, $h, f : \Re \to \Re$, $f(0) = 0$, are continuous functions and $\tau > 0$ is constant delay. Hence, DIDE (2) includes the zero solution.

Let $x(t,t_0,\phi)$, $t \ge t_0$ be a solution of DIDE (2) on $[t_0 - \tau, \beta)$, $\beta > 0$, such that $x(t) = \phi(t)$ on $\phi \in [t_0 - \tau, t_0]$ and $|\phi(t)| = \sup_{t\in[t_0-\tau,t_0]} |\phi(t)|$, where $\phi : [t_0 - \tau, t_0] \to \Re$ is a continuous initial function.

Here, we discuss the stability, asymptotically stability and instability of the zero solution and boundedness, integrability of solutions and integrability of derivatives of solutions of DIDEs in the form of DIDE (2). The contributions of this paper can be explained as the following:

1) Wang [1] studied a linear IDE (1) without delay. We studied nonlinear DIDEs of the form of DIDE (2). This is a clear contribution of this paper from the linear case and the case without delay to the case nonlinear and with constant delay. Next, DIDE (2) includes and improves IDE (1) when $\tau = 0$. This is an additional contribution of this paper.

2) In the literature, there are only a few results on the instability of solution of DIDEs. In this paper, we give a result on the instability of the zero solution of DIDE (2) when $A(t) < 0$. Thanks to this information, this paper has a contribution to the results on the instability of solutions (see [1]-[63]).

3) In this paper, Theorem 2 is an additional result for the integrability of solutions of DIDE (2) on $[0,\infty)$. This is a third additional contribution of this paper to the results of Wang [1] and the results in the references of this paper.

First, let us give basic definitions.

Definition 1([2]). The zero solution of DIDE (2) is stable, if for each $\varepsilon > 0$ and $t_0 \ge 0$ there exists $\delta = \delta(t_0,\varepsilon) > 0$ such that if $|\phi(t)| < \delta$ on $[t_0 - \tau, t_0]$, we have $|x(t,\phi)| < \varepsilon$, $\forall t \ge t_0$.

Definition 2([2]). The zero solution of DIDE (2) is uniformly stable if δ is independent of t_0.

Definition 3([2]). The zero solution of DIDE (2) is asymptotically stable if it is stable and if for each $t_0 \ge 0$, there is a $\delta > 0$ such that , $t \ge t_0$, $|\phi(t)| < \delta$ on $[0,t_0]$ implies $|x(t,\phi)| \to 0$ as $t \to \infty$.

Through this paper, without mention, x and x' wikk represent $x(t)$ and $x'(t)$, respectively.

This paper is organized as follows: Section 2 consists of qualitative analyses of solutions such as asymptotic stability, integrability, boundedness, convergence and instability of the solutions of the considered DIDE (2). In Section 3, we give a conclusion that summarizes the contribution of this paper to the relevant literature.

2 Qualitative analyses of solutions

We now present our basic assumptions for qualitative analyses of solutions

A. Assumptions

Suppose the following assumptions hold:

(A1) There exists a positive constant β such that

$$f(0) = 0, |f(x)| \le \beta|x|, \forall x \in \Re$$

(A2) There exists a positive constant α such that

$$h(0) = 0, h(x) \ne 0 \; if \; x \ne 0,$$
$$C(t,s,0) = 0, |C(t,s,h(x(s))| \le \alpha|K(t,s)|,$$

and

$$\int_0^\infty |K(t,s)|ds < N < \infty$$

for some positive constant N, $N \in \Re$.

(A3) There exists a positive constant μ such that

$$A(t) > 0, A(t) - \alpha\beta \int_0^\infty |K(t,s)|ds \ge \mu,$$

$\forall 0 \le s \le t \le \infty, x \in \Re$.

(A4) The following integral is convergent:

$$\int_0^{t_0} \int_{t_0}^\infty |K(t,s)|duds < M < \infty,$$

$t_0 \ge 0$, $M > 0$, and t_0, $M \in \Re$.

(A5) There exists a positive constant K_2 such that

$$A(t) < 0, -A(t) - \alpha\beta \int_0^\infty |K(u,t)|ds \ge K_2 > 0,$$

$\forall t, u \in [0,\infty)$

Firstly, we give a new asymptotically stability result for the zero solution of DIDE (2).

Theorem 1. Let assumptions (A1)-(A4) hold. Then, the zero solution of DIDE (2) is asymptotically stable.

Proof. We define a Lyapunov functional by

$$V = V(t,x(.)) = |x| + k_0 \int_0^t \int_t^\infty |C(u,s,h(x(s))|du|x(s)|ds,$$

where k_0 is a positive constant and we choose it later.

Clearly, we can see that this Lyapunov functional is positive definite. Next, we can write that

$$V(t,0) = 0, V(t,x(.)) \ge |x|.$$

By the time derivative of the Lyapunov functional V along solutions of DIDE (2) and hypotheses (A1)-(A4), we can derive

$$\begin{aligned}\frac{d}{dt}V(t,x(.)) =& \frac{dx}{dt}sgnx(t+0) + k_0|x| \int_t^\infty |C(u,t,h(x(t)))|dt \\ & - k_0 \int_0^t |C(t,s,h(x(s)))||x(s)|ds \\ =& -A(t)x(t)sgnx(t+0) \\ & + sgnx(t+0) \int_{t-\tau}^t C(t,s,h(x(s)))f(x(s))ds\end{aligned}$$

$$
\begin{aligned}
&+k_0|x|\int_t^{\infty}|C(u,t,h(x(t)))|dt\\
&-k_0\int_0^t|C(t,s,h(x(s)))||x(s)|ds\\
\le& -A(t)|x|+\int_{t-\tau}^t|C(t,s,h(x(s)))||f(x(s))|ds\\
&+k_0|x|\int_t^{\infty}|C(u,t,h(x(t)))|du\\
&-k_0\int_0^t|C(t,s,h(x(s)))||x(s)|ds\\
\le& -A(t)|x|+\beta\int_0^t|C(t,s,h(x(s)))||x(s)|ds\\
&+k_0|x|\int_t^{\infty}|C(u,t,h(x(t)))|du\\
&-k_0\int_0^t|C(t,s,h(x(s)))||x(s)|ds.
\end{aligned}
$$

Let us choose the constant k_0 such that $k_0=\beta$. Then, by this choice and assumption (A3), it follows that

$$
\begin{aligned}
\frac{d}{dt}V(t,x(.)) &\le -[A(t)-\beta\int_0^t|C(t,s,h(x(s))|ds]|x|\\
&\le -A(t)|x|+\alpha\beta|x|\int_0^t|K(t,s)|ds\\
&\le -\left[A(t)-\alpha\beta\int_0^{\infty}|K(t,s)|ds\right]|x|\\
&\le -\mu|x|, \mu\in\Re, \mu>0.
\end{aligned}
$$

From our discussion it is now notable that

$$V(t,0)=0, V(t,x(.))\ge|x|$$

and

$$\frac{d}{dt}V(t,x(.))\le-\mu|x|, \mu>0.$$

In view of the Lyapunov's stability theorems (see Burton [2]), we can conclude that the zero solution of DIDE (2) is asymptotically stable. Hence, it is stable, too. The proof is completed.

Secondly, we present a new integrability result for the solutions of DIDE (2).

Theorem 2. Let assumptions (A1)-(A4) hold. Then the solutions of DIDE (2) are integrable on $[0,\infty)$.

Proof. From Theorem 1, we have

$$\frac{d}{dt}V(t,x(.))\le-\mu|x|, \forall t\ge t_0.$$

Integrating this inequality from t_0 to t, we get

$$V(t,x(t))-V(t_0,x(t_0))\le-\mu\int_{t_0}^t|x(s)|ds.$$

From the above discussion, it can be seen that $V(t,x(.))$ is a positive definite and decreasing functional. Therefore, we can assume

$$V(t_0,x(t_0))=A_0, A_0>0, A_0\in\Re.$$

Hence, it is clear that

$$\mu\int_{t_0}^t|x(s)|ds\le V(t_0,x(t_0))-V(t,x(t))\le V(t_0,x(t_0))=A_0.$$

Thus, we can conclude that

$$\int_{t_0}^{\infty}|x(s)|ds\le\mu^{-1}A_0<\infty.$$

This result completes the proof of Theorem 2.

Next, we introduce a new boundedness result for the solutions of DIDE (2).

Theorem 3. Let assumptions (A1)-(A4) hold. Then the solutions of DIDE (2) are bounded.

Proof. Since the Lyapunov functional V is decreasing, then we can write the following:

$$
\begin{aligned}
|x(t,t_0,\phi(t_0))| &\le V(t,x(.))\le V(t_0,\phi(t_0))\\
&=|\phi(t_0)|+\int_0^{t_0}\int_{t_0}^{\infty}|K(t,s)|duds\\
&\le|\phi(t_0)|+M=K, \forall t\ge t_0.
\end{aligned}
$$

Hence, the proof of Theorem 3 is completed.

We now give a new result on the integrability and convergence of the solutions of DIDE (2).

Theorem 4. Let assumptions (A1)-(A4) hold. Then the solutions of DIDE (2) and their first order derivatives are integrable on $[0,\infty)$, that is, $x(t)\in L^1(0,\infty)$ and $x'(t)\in L^1(0,\infty)$. In addition, we have $x(t)\to0$ as $t\to\infty$.

Proof. We now reconsider DIDE (2). Hence, it can be written that

$$
\begin{aligned}
|x'| &\le A(t)|x|+\int_{t-\tau}^t|C(t,s,h(x(s)))||f(x(s))|ds\\
&\le A(t)|x|+\beta\int_{t-\tau}^t|C(t,s,h(x(s)))||x(s)|ds.
\end{aligned}
$$

Let a constant $K_0>0$ to be given. Then, it follows from DIDE (2) and the discussion made above that

$$K_0|x'|\le K_0A(t)|x|+K_0\beta\int_{t-\tau}^t|C(t,s,h(x(s)))||x(s)|ds$$

so that

$$-K_0A(t)|x|\le K_0\beta\int_{t-\tau}^t|C(t,s,h(x(s)))||x(s)|ds-K_0|x'|$$

We now define a new Lyapunov functional by

$$
\begin{aligned}
V_1(t,x(.)) =&(1+K_0)|x|\\
&+K_1\beta\int_0^t\int_t^{\infty}|C(u,s,h(x(s)))|du|x(s)|ds.
\end{aligned}
\tag{3}
$$

If we calculate time derivative of functional (3) along solutions of DIDE (2), then

$$\begin{aligned}
\frac{d}{dt}V_1^+(t,x(.)) =&(1+K_0)\frac{dx}{dt}sgnx(t+0)\\
&+K_1\beta\int_t^\infty |C(u,t,h(x(t)))|du|x(t)|\\
&-K_1\beta\int_0^t |C(t,s,h(x(s)))||x(s)|ds\\
\le&-(1+K_0)A(t)|x|\\
&+(1+K_0)\int_{t-\tau}^t |C(t,s,h(x(s)))||f(x(s))|ds\\
&+K_1\beta\int_t^\infty |C(u,t,h(x(t)))|du|x(t)|\\
&-K_1\beta\int_0^t |C(t,s,h(x(s)))||x(s)|ds\\
\le&-(1+K_0)A(t)|x|\\
&+(1+K_0)\beta\int_{t-\tau}^t |C(t,s,h(x(s)))||x(s)|ds\\
&+K_1\beta\int_t^\infty |C(u,t,h(x(t)))|du|x(t)|\\
&-K_1\beta\int_0^t |C(t,s,h(x(s)))||x(s)|ds\\
\le&-(1+K_0)A(t)|x|\\
&+(1+K_0-K_1)\beta\int_0^t |C(t,s,h(x(s)))||x(s)|ds\\
&+K_1\beta\int_t^\infty |C(u,t,h(x(t)))|du|x(t)|\\
\le&-A(t)|x|+(1+2K_0-K_1)\\
&\times\beta\int_0^t |C(t,s,h(x(s)))||x(s)|ds\\
&+K_1\beta\int_t^\infty |C(u,t,h(x(t)))|du|x(t)|-K_0|x'|
\end{aligned}$$

Let $K_0=\frac{(K_1-1)}{4}$, $K_1>1$. By the assumptions of Theorem 4 and the definition of the constant K_0 , we can derive

$$\begin{aligned}
\frac{d}{dt}V_1^+(t,x(.))\le&-[A(t)-K_1\beta\int_t^\infty |C(u,t,h(x(t)))|du]|x|\\
&-K_0|x'|\\
\le&-[A(t)-K_1\beta\int_t^\infty |K(u,t)|du]|x|-K_0|x'|\\
\le&-\mu|x|-K_0|x'|.
\end{aligned}$$

By integrating this last inequality from t_0 to t, we have

$$V_1(t,x(t))\le V_1(t_0,\phi(t_0))-\mu\int_{t_0}^t |x(s)|ds-K_0\int_{t_0}^t |x'(s)|ds.$$

Since Lyapunov functional $V_1(t,x(.))$ is positive definite, then we can say

$$V_1(t_0,\phi(t_0))=K_1, K_1\in\Re, K_1>0.$$

Hence, in view of the definition of the Lyapunov functional $V_1(t,x(.))$ and the discussion we have just made above, it follows that

$$\begin{aligned}
0\le(1+K_0)|x(t,t_0,\phi(t_0))|&\le V_1(t,x(t))\\
&\le V_1(t_0,\phi(t_0))-\mu\int_{t_0}^t |x(s)|ds-K_0\int_{t_0}^t |x'(s)|ds
\end{aligned}$$

so that

$$\begin{aligned}
\mu\int_{t_0}^t |x(s)|ds+K_0\int_{t_0}^t |x'(s)|ds\le&(1+K_0)|x(t,t_0,\phi(t_0))|\\
&+\mu\int_{t_0}^t |x(s)|ds+K_0\int_{t_0}^t |x'(s)|ds\\
\le&V_1(t_0,\phi(t_0))=K_1.
\end{aligned}$$

Hence, it is clear that

$$\mu\int_{t_0}^t |x(s)|ds+K_0\int_{t_0}^t |x'(s)|ds\le V_1(t_0,\phi(t_0))=K_1.$$

Then, we can derive

$$\int_{t_0}^\infty |x(s)|ds\le\mu^{-1}K_1<\infty$$

and

$$\int_{t_0}^\infty |x'(s)|ds\le K_0^{-1}K_1<\infty.$$

Therefore, we can conclude that $x(t)\in L^1(0,\infty)$, $x'(t)\in L^1(0,\infty)$.

Next, we have the following estimates:

$$V_1(t,0)=0, V_1(t,x(.))\ge(1+K_0)|x|$$

and

$$\frac{d}{dt}V_1^+(t,x(.))\le-\mu|x|-K_0|x'|.$$

Thus, we can easily arrive at $x(t)\to0$ as $t\to\infty$. These results complete the proof of Theorem 4.

Theorem 5. If we assume that assumptions (A1),(A2) and (A4)-(A5) hold, then the zero solution of DIDE (2) is unstable.

Proof. We now define a new Lyapunov functional by

$$V_2(t,x(.))=|x|-\int_0^t\int_t^\infty |C(u,s,h(x(s))|du|f(x(s))|ds. \tag{4}$$

The derivative of the Lyapunov functional (4) along solutions of DIDE (2) gives

$$\begin{aligned}
\frac{d}{dt}V(t,x(.)) =&\frac{dx}{dt}sgnx(t+0)\\
&-\int_t^\infty |C(u,t,h(x(t)))|du|f(x(t)|\\
&+\int_0^t |C(t,s,h(x(s)))||f(x(s))|ds\\
=&[-A(t)x(t)\\
&+\int_{t-r}^t C(t,s,h(x(s)))f(x(s))ds]\\
&\times sgnx(t+0)\\
&-\int_t^\infty |C(u,t,h(x(t)))|du|f(x(t))|\\
&+\int_0^t |C(t,s,h(x(s)))||f(x(s))|ds\\
\geq& -A(t)|x(t)|-\int_0^t |C(t,s,h(x(s)))||f(x(s))|ds\\
&-\int_t^\infty |C(u,t,h(x(t)))|du|f(x(t))|\\
&+\int_0^t |C(t,s,h(x(s)))||f(x(s))|ds\\
=&-A(t)|x(t)|-\int_0^t |C(u,t,h(x(t)))|du|f(x(t))|\\
\geq&-[A(t)+\beta\int_t^\infty |C(u,t,h(x(t)))|du]|x(t)|\\
\geq&[-A(t)-\alpha\beta\int_0^\infty |K(u,t)|ds]|x(t)|\\
\geq&K_2|x|\geq 0.
\end{aligned}$$

The rest of the proof can be done by following a similar procedure as in [1]. We omit the rest of the proof.

3 Conclusion

In this work, certain DIDEs of first order with constant time-lag are taken into consideration. Qualitative analyses such as stability, asymptotic stability, instability boundedness of solutions and integrability of solutions and their derivatives of that DIDEs are proceeded by the functional method of Lyapunov. To reach the aim of this paper, three suitable auxiliary functionals are defined and hence five new and original results are proved by help of that auxiliary functionals. The results obtained have a contribution to the literature, and they improve or generalize the results of Wang [47].

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