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Another Generalized Transmuted Power Function Distribution for Modelling Lifetime Data

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Abstract: A more flexible four-parameter distribution (i.e.) another generalized transmuted power function distribution is introduced. The sub-models and some mathematical properties such as moment, order statistics, quantile function, entropy, etc. of the distribution are considered. The maximum likelihood estimation method is used to estimate the parameters of the distribution. The potential of the distribution is demonstrated through the application of real data set.

Keywords: Transmuted distribution, Power function distribution, Trimmed L-moments, Lifetime data

1 Introduction

The Transmuted class (T-G) of distributions was introduced by [1]. A random variable X has T-G class of distributions if the distribution function (cdf) is of the form

$$G(x) = (1+\lambda)F(x) - \lambda F^2(x); \;\; |\lambda| \leq 1. \tag{1}$$

where $F(x)$ is the baseline distribution. Many distributions have been proposed using equation (1) for different baselines. Aryal et al. [2] introduced and examined transmuted Weibull, [3] considered transmuted log-logistics, transmuted Quasi Lindley distribution was introduced by [4], the transmuted generalized inverse Weibull was proposed by [5], transmuted generalized Rayleigh was also introduced and studied by [6], transmuted Kumaraswamy was introduced by [7] and Haq et al.[8] introduced the transmuted power function. Shahzad and Asghar [9] also proposed transmuted power function but with scale parameter in [8] equal to one. Abdul-Moniem and Diab [10] proposed a generalized transmuted power function in which the power function is a three-parameter distribution.

In this paper, we proposed a new distribution for modelling the failure times of independent component of a system connected in series. Suppose a random variable X_i; $i = 1,2,...,\alpha$ such that $X \in \Re^+$ denotes the failure times of i independent components of a system connected in series. The failure time of the system is the minimum of the individual component potential failure times [11]. However, the probability that individual component of the system survives up to time x is defined by the survivor function $P(X_i) = 1-F(x)$, where $F(x)$ is the distribution function of the individual components. The distribution function of the failure time is given by

$$\begin{aligned} F(y) &= 1 - P(X_i > x) \\ &= 1 - P(X_1 > x).P(X_2 > x)...P(X_\alpha > x) \\ &= 1 - [P(X \leq x)]^\alpha \\ &= 1 - [\bar{F}(x)]^\alpha \end{aligned} \tag{2}$$

Assuming that X_i follows Power function distribution with distribution function (cdf) is given by

$$F(x) = \left(\frac{x}{\beta}\right)^\theta; \;\; \theta > 0, 0 < x < \beta. \tag{3}$$

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Equation (3) was proposed by[12] as a simple distribution for modelling the reliability of electronic components. The remaining part of the paper is structured as follows: The derivation of the new distribution and density functions are presented in Section 2. Section 3 addresses the derivation of the mathematical properties of the distribution. The entropy and the pdf of the i^{th} ordered statistics are discussed in Sections 4 and 5. The estimation of parameters of the distribution is discussed in Section 6. Application of real-life data to illustrate the potentials of the new distribution is presented in Section 7. Section 8, is devoted to conclusion.

2 The new distribution

Suppose a random variable X denotes the failure time of a system that has α independent components connected in series; Substituting equation (3) in (2), then in equation (1), the cdf of the new distribution i.e. Another Generalized Transmuted Power Function (AGTPF) is given by

$$G(x) = (1+\lambda)\left\{1-\left[1-\left(\frac{x}{\beta}\right)^{\theta}\right]^{\alpha}\right\} - \lambda\left\{1-\left[1-\left(\frac{x}{\beta}\right)^{\theta}\right]^{\alpha}\right\}^{2}; \quad (4)$$

where $|\lambda| \leq 1, \alpha, \theta, \beta > 0$ and $0 < x < \beta$.
The density function (pdf) corresponding to equation (4) is given by

$$g(x) = \frac{\alpha\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta-1}\left[1-\left(\frac{x}{\beta}\right)^{\theta}\right]^{\alpha-1}\left(1+\lambda-2\lambda\left\{1-\left[1-\left(\frac{x}{\beta}\right)^{\theta}\right]^{\alpha}\right\}\right) \quad (5)$$

However, equation (4) follows the cdf of class of distribution introduced by [13]. The associated hazard rate function (hrf) is given below

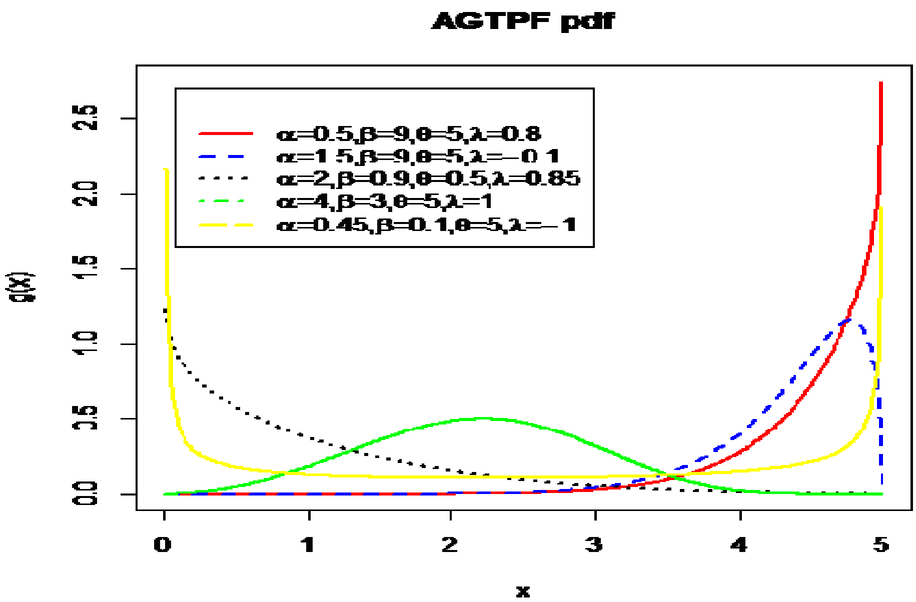

Fig. 1: Pdf plot of AGTPF for some values of parameter

$$hrf = \frac{\alpha\theta x^{\theta-1}\left[1-\left(\frac{x}{\beta}\right)^{\theta}\right]^{\alpha-1}\left(1+\lambda-2\lambda\left\{1-\left[1-\left(\frac{x}{\beta}\right)^{\theta}\right]^{\alpha}\right\}\right)}{\beta^{\theta}\left(1-\left[(1+\lambda)\left\{1-\left[1-\left(\frac{x}{\beta}\right)^{\theta}\right]^{\alpha}\right\}-\lambda\left\{1-\left[1-\left(\frac{x}{\beta}\right)^{\theta}\right]^{\alpha}\right\}^{2}\right]\right)}$$

Figures 1 and 2 show some possible shapes including monotone decreasing and increasing, left-skewed, symmetric and bathtub shapes for the pdf and monotone increasing, left-skewed, symmetric and bathtub shapes for the hrf.

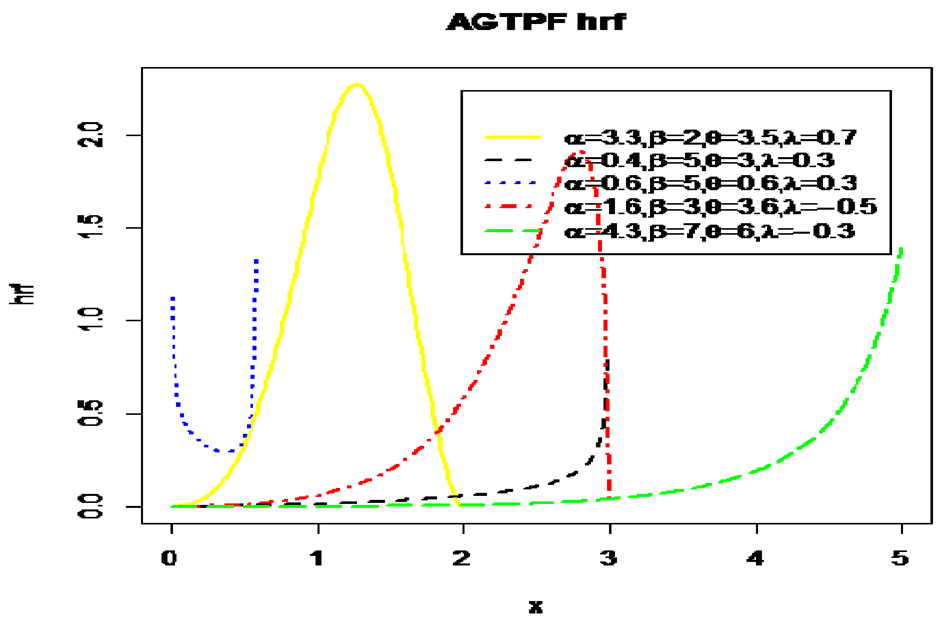

Fig. 2: Hazard rate function plot of AGTPF for some values of parameter

2.1 Relationship with other distributions

Another generalized transmuted power function relates to other distributions such that:
i) If $\theta = 1$ AGTPF reduces to Another Generalized Transmuted Uniform distribution due to [13]
ii) If $\lambda = 0$ AGTPF reduces to exponentiated power function (EPF) distribution
iii) If $\alpha = 1$ AGTPF reduces to transmuted power function (TPF) distribution due to [5]
If $\alpha = 1$ and $\lambda = 0$ AGTPF reduces to power function distribution due to [12]
iv) If $\alpha = 1$, $\lambda = 0$, $\beta = 1$ and $\theta = 1$ AGTPF reduces to Uniform distribution

3 Mathematical properties

3.1 Quantile function

The quantile function plays an important role in statistics. Random number generation and fractiles can be obtained through quantile function.

Theorem 1*If $X \sim AGTPF$, then the U^{th} quantile function is given below as* $H(U) = \beta \left\{ 1 - \left[\frac{\lambda - 1 + \sqrt{(1+\lambda)^2 - 4\lambda U}}{2\lambda} \right]^{\frac{1}{\alpha}} \right\}^{\frac{1}{\theta}}$;

where $U \in (0,1)$.

Corollary 11*If $X \sim AGTPF$, then the median $\boldsymbol{M}$ of X is given by*

$\boldsymbol{M} = \beta \left\{ 1 - \left[\frac{\lambda - 1 + \sqrt{(1+\lambda)^2 - 2\lambda U}}{2\lambda} \right]^{\frac{1}{\alpha}} \right\}^{\frac{1}{\theta}}$.

3.2 Moments

The k^{th} moments of a random variable X that has AGTPF distribution is defined as

$$
\begin{aligned}
E(X^k) &= \int_0^\beta x^k g(x)dx \\
&= \int_0^\beta x^k \frac{\alpha\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta-1}\left[1-\left(\frac{x}{\beta}\right)^\theta\right]^{\alpha-1} \\
&\quad \times \left(1+\lambda-2\lambda\left\{1-\left[1-\left(\frac{x}{\beta}\right)^\theta\right]^\alpha\right\}\right)dx \\
&= \frac{\alpha\theta}{\beta}(1+\lambda)\int_0^\beta x^k\left(\frac{x}{\beta}\right)^{\theta-1}\left[1-\left(\frac{x}{\beta}\right)^\theta\right]^{\alpha-1}dx - 2\lambda\frac{\alpha\theta}{\beta} \\
&\quad \times \int_0^\beta x^k\left(\frac{x}{\beta}\right)^{\theta-1}\left[1-\left(\frac{x}{\beta}\right)^\theta\right]^{\alpha-1}dx \\
&\quad -2\lambda\frac{\alpha\theta}{\beta}\int_0^\beta x^k\left(\frac{x}{\beta}\right)^{\theta-1}\left[1-\left(\frac{x}{\beta}\right)^\theta\right]^{2\alpha-1}dx
\end{aligned}
$$

Let $y = \left(\frac{x}{\beta}\right)^\theta$, $dx = \frac{\beta}{\theta}y^{\frac{1}{\theta}-1}dy$; then simplifyingg further

$$
\begin{aligned}
E(X^k) &= \alpha\beta^k(1+\lambda)\int_0^1 y^{\frac{k}{\theta}}(1-y)^{\alpha-1}dy - 2\lambda\alpha\beta^k\int_0^1 y^{\frac{k}{\theta}-1}(1-y)^{\alpha-1}dy \\
&\quad + 2\lambda\alpha\beta^k\int_0^1 y^{\frac{k}{\theta}}(1-y)^{\alpha-1}dy \\
&= \alpha\beta^k(1+\lambda)\mathbf{B}\left(\frac{k}{\theta}+1,\alpha\right) - 2\lambda\alpha\beta^k\mathbf{B}\left(\frac{k}{\theta}+1,\alpha\right) + 2\lambda\alpha\beta^k \\
&\quad \times\mathbf{B}\left(\frac{k}{\theta}+1,2\alpha\right) \\
&= \alpha\beta^k\left[(1-\lambda)\mathbf{B}\left(\frac{k}{\theta}+1,\alpha\right)+\mathbf{B}\left(\frac{k}{\theta}+1,2\alpha\right)\right] \qquad (6)
\end{aligned}
$$

Based on equation (6), the first four moments of the AGTPF, TPF, and EPF are evaluated for some parameter values. Tables 1, 2, and 3 show the first four moments for AGTPF, TPF, and EPF, respectively. The values in the tables indicate that AGTPF has higher moments, followed by EPF, then TPF.

Table 1: First, Second, third and fourth moment of AGTPF for some parameter values.

α	λ	θ	β	μ_1'	μ_2'	μ_3'	μ_4'
0.2	-0.9	2.6	5	231.43	1131.85	5555.47	27335.80
	-0.2			220.66	1046.28	5021.97	24289.70
	0.7			206.82	936.26	4336.04	20373.40
0.9	-0.9	2.6	5	64.99	282.02	1249.02	5614.37
	-0.2			58.77	239.39	1014.94	4419.07
	0.7			50.78	184.59	713.97	2880.97
0.9	-0.9	1.2	5.80	22.84	94.69	405.81	
	-0.2			4.88	17.81	70.73	294.69
	0.7			3.70	11.35	39.91	151.84
1.5	-0.9	1.2	1.80	6.07	22.17	85.41	
	-0.2			1.48	4.58	15.97	59.77
	0.7			1.07	2.67	7.99	26.79

Table 2: First, Second, third and fourth moment of TPF for selected parameter values.

α	λ	θ	β	μ_1'	μ_2'	μ_3'	μ_4'
1	-0.9	2.6	5	54.31	231.96	1013.12	4498.66
	-0.2			48.95	195.88	817.93	3514.58
	0.7			42.07	149.49	566.97	2249.33
1	-0.9	1.2	5	4.76	18.23	73.91	310.94
	-0.2			3.98	14.11	54.75	223.87
	0.7			2.99	8.82	30.11	111.94

Table 3: First, Second, third and fourth moment of EPF for selected parameter values.

α	λ	θ	β	μ_1'	μ_2'	μ_3'	μ_4'
0.2	0	2.6	5	217.59	1021.83	4869.54	23419.4
0.5				120.82	524.49	2347.88	10720.9
0.7				82.68	343.17	1480.74	6554.27
1.5	0	0.3	5	0.098	0.217	0.639	2.163
1.7				0.059	0.118	0.307	1.054
1.9				0.036	0.065	0.169	0.521

By the computation of the first four moments in Tables 1, 2, and 3; the coefficient of variation (CV), skewness ψ_1, and kurtosis ψ_2 for each of the distributions can be evaluated. The CV, ψ_1, and ψ_2 are respectively given by

$$CV = \sqrt{\frac{\mu_2'}{\mu_1'^2} - 1}$$

$$\psi_1 = \frac{\mu_{'}^{3} - 3\mu_2'\mu_1' + 2\mu_1'^3}{\left(\mu_2' - \mu_1'^2\right)^{\frac{3}{2}}}$$

and

$$\psi_2 = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4}{\left(\mu_2' - \mu_1'^2\right)^2}$$

3.3 TL-moments

The Trimmed L (TL) -moments which are a robust generalization of the L-moments were introduced by [14]. For the symmetric case, the TL-moments are defined by

$$\gamma_n^t = \frac{1}{n}\sum_{k=0} n-1(-1)^k \binom{n-1}{k} E\left(X_{n+t-k:n+2t}\right);\ n=1,2,\ldots$$

where

$$E\left(X_{n+t-k:n+2t}\right) = \frac{(n+2t)!}{(n+t-k-1)!(t+k)!} \times \int_{\forall x} xg(x)G(x)^{n+t-k-1}\left[1-G(x)\right]^{t+k} dx \quad (7)$$

Equation (7) can be re-written as

$$E\left(X_{n+t-k:n+2t}\right) = \frac{(n+2t)!}{(n+t-k-1)!(t+k)!} \times \sum_{i=0}^{t+k}(-1)^i \int_{\forall x} xg(x)G(x)^{n+t-k+i-1} dx \quad (8)$$

Substituting equations (4) and (5) in equation (8) and simplifying, the TL-moments for AGTPF are obtained as

$$\gamma_n^t = \frac{1}{n}\sum_{k=0} n-1(-1)^k \binom{n-1}{k} \frac{(n+2t)!}{(n+t-k-1)!(t+k)!} \times\beta\alpha(1+\lambda)^{n+t-k+i-1}\sum_{i=0}^{\infty}\psi_i \mathbf{B}\left(\frac{1}{\theta}+1,\alpha(a+1)\right) \times\left[\binom{\Phi^*}{a} - \frac{2}{(1+\lambda)}\binom{\Phi^{**}}{a}\right] \quad (9)$$

where $\psi_i = (-1)^i \sum_{s,a=0}^{\infty}(-1)^{s+a}\left(\frac{\lambda}{1+\lambda}\right)^s \binom{\Phi}{s}$, $\Phi = n+t-k+i-1$, $\Phi^* = n+t-k+i+s-1$, and $\Phi^{**} = n+t-k+i+s$. For$t=1$ and $n=1,2,3,4$ the population measures of location, scale, skewness, and kurtosis can be obtained.

3.4 Mode

The mode of AGTPF is obtained as the value of x that will satisfy the non-linear equation (10). This implies that there exists more than one root to the equation. The numerical values can be obtained for any given values of the parameters using the uniroot package in R. If $x = x_0$is one of the roots, then $\vartheta(x_0) < 0$, $\vartheta(x_0) > 0$ and $\vartheta(x_0) = 0$ correspond to the local maxima, the local minima, and point of inflexion, respectively.

$$\frac{\partial log\left[g(x)\right]}{\partial x} = \frac{\theta-1}{x} - \frac{2\lambda\theta\alpha x^{\theta-1}\left[1-\left(\frac{x}{\beta}\right)^{\theta}\right]^{\alpha-1}}{\beta^{\theta}\left(1-\lambda\left\{1-2\left[1-\left(\frac{x}{\beta}\right)^{\theta}\right]^{\alpha}\right\}\right)} + \frac{(\alpha-1)\theta x^{\theta-1}}{\left[1-\left(\frac{x}{\beta}\right)^{\theta}\right]} = 0 \quad (10)$$

4 Order Statistics

Suppose $X_1 < X_2 < ... < X_n$ is an ordered sample from AGTPF population, the pdf of the i^{th} ordered statistics is given as

$$
\begin{aligned}
g(x_{i:n}) &= \frac{n!}{(i-1)!(n-i)!} g(x) G(x)^{i-1} [1-G(x)]^{n-1} \\
&= \frac{n!}{(i-1)!(n-i)!} g(x) \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} G(x)^{i+j-1}
\end{aligned}
\tag{11}
$$

Substituting (4) and (5) in (11), and applying general binomial series expansion yields

$$
g(x_{i:n}) = \Psi \left[\sum_{m=0}^{\infty} w_{m_1} \left(\frac{x}{\beta}\right)^{\theta(n+1)-1} - 2\lambda \sum_{m=0}^{\infty} w_{m_2} \left(\frac{x}{\beta}\right)^{\theta(n+1)-1} \right];
\tag{12}
$$

where

$$
\Phi = \frac{\alpha\theta}{\beta} \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \sum_{k=0} \infty (-1)^k \binom{i+j-1}{k},
$$

$$
w_{m_1} = (-1)^m \sum_{n=0}^{\infty} \binom{i+j+k-1}{m} \binom{\alpha+\alpha m-1}{n},
$$

and

$$
w_{m_2} = (-1)^m \sum_{n=0}^{\infty} \binom{i+j+k}{m} \binom{\alpha+\alpha m-1}{n}.
$$

5 Entropy

The entropy of a random variable X with density function $f(x)$ measures the amount of uncertainty [15]. For any real parameter $\gamma > 0$ and $\gamma \neq 1$, the Renyi's entropy is given by

$$
I_R(\gamma) = \frac{1}{1-\gamma} log \int_{\forall x} g^{\gamma}(x) dx.
\tag{13}
$$

Substituting (5) in (13) and applying general binomial series expansion, then the Renyi's entropy for a random variable $X \sim$ AGTPF is given by

$$
\begin{aligned}
I_R(\gamma) &= \frac{1}{1-\gamma} log \left[\left(\frac{x}{\beta}\right)^{\gamma} (1+\lambda)^{\gamma} \sum_{k=0}^{\infty} \int_0^{\beta} \left(\frac{x}{\beta}\right)^{\gamma(\theta-1)+k\theta} \right] \\
&= \frac{1}{1-\gamma} log \left[\left(\frac{x}{\beta}\right)^{\gamma} (1+\lambda)^{\gamma} \sum_{k=0}^{\infty} \frac{\beta}{1+k\theta+\gamma(\theta-1)} \right] \\
&= \frac{\gamma}{1-\gamma} [log(\alpha\theta) - log(\beta)] + \frac{\gamma}{1-\gamma} log(1+\lambda) \\
&\quad + \frac{1}{1-\gamma} log \left[\sum_{k=0}^{\infty} w_k \frac{\beta}{1+k\theta+\gamma(\theta-1)} \right].
\end{aligned}
$$

6 Parameter estimation

Suppose $X_1, X_2, \ldots, X_n$ is a random sample of size n from a population having AGTPF density function, the log-likelihood function is given by

$$\ell(\Theta) = nlog(\alpha) + nlog(\theta) + n\theta log(\beta) + (\theta - 1)\sum_{i=1}^{n} log(x_i) + (\alpha - 1)\sum_{i=1}^{n} log\left[1 - \left(\frac{x_i}{\beta}\right)^{\theta}\right] + \sum_{i=1}^{n} log\left(1 + \lambda - 2\lambda\left\{1 - \left[1 - \left(\frac{x_i}{\beta}\right)^{\theta}\right]^{\alpha}\right\}\right) \quad (14)$$

The score functions which correspond to equating the first-order partial derivative of equation (14) to zero are given by

$$\frac{\partial \ell(\Theta)}{\partial \lambda} = \frac{2\left[1 - \left(\frac{x_i}{\beta}\right)^{\theta}\right]^{\alpha} - 1}{1 - \lambda\left\{1 - 2\left[1 - \left(\frac{x_i}{\beta}\right)^{\theta}\right]^{\alpha}\right\}} = 0$$

$$\frac{\partial \ell(\Theta)}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} log\left[1 - \left(\frac{x_i}{\beta}\right)^{\theta}\right] + \sum_{i=1}^{n} \frac{2\lambda\left[1 - \left(\frac{x_i}{\beta}\right)^{\theta}\right]^{\alpha} log\left[1 - \left(\frac{x_i}{\beta}\right)^{\theta}\right]}{1 - \lambda\left\{1 - 2\left[1 - \left(\frac{x_i}{\beta}\right)^{\theta}\right]^{\alpha}\right\}} = 0$$

$$\frac{\partial \ell(\Theta)}{\partial \theta} = \frac{n}{\theta} - nlog(\beta) + \sum_{i=1}^{n} log(x_i) - \sum_{i=1}^{n} \frac{\left(\frac{x_i}{\beta}\right)^{\theta} log\left(\frac{x_i}{\beta}\right)}{1 - \left(\frac{x_i}{\beta}\right)^{\theta}} - \sum_{i=1}^{n} \frac{2\lambda\alpha\left[1 - \left(\frac{x_i}{\beta}\right)^{\theta}\right]^{\alpha - 1}\left(\frac{x_i}{\beta}\right)^{\theta} log\left(\frac{x_i}{\beta}\right)}{1 - \lambda\left\{1 - 2\left[1 - \left(\frac{x_i}{\beta}\right)^{\theta}\right]^{\alpha}\right\}} = 0$$

$$\frac{\partial \ell(\Theta)}{\partial \beta} = \frac{n\theta}{\beta} + \sum_{i=1}^{n} \frac{(\alpha - 1)\theta x_i^{\theta}}{\beta^{\theta + 1}\left[1 - ()^{\theta}\right]} + \frac{2\lambda\theta\alpha x_i^{\theta}\left[1 - \left(\frac{x_i}{\beta}\right)^{\theta}\right]^{\alpha - 1}}{\beta^{\theta + 1}\left(1 - \lambda\left\{1 - 2\left[1 - \left(\frac{x_i}{\beta}\right)^{\theta}\right]^{\alpha}\right\}\right)} = 0$$

The solutions, say $\hat{\Theta}$ of the score functions correspond to the maximum likelihood estimators of the AGTPF distribution. However, the score functions are non-linear functions and the numerical values of the maximum likelihood estimates can be obtained using BFGS iterative optimisation method. Furthermore, the negative expected value of the second-order partial derivative of the log-likelihood equation which corresponds to the Fisher's Information Matrix is given by

$$\upsilon(\Theta) = -E\left[\frac{\partial^2 \ell(\Theta)}{\partial \Theta_{i,j}}\right];\ i,j = 1,2,3,4.$$

where i,j index the parameters of the AGTPF.
However, under some regularity conditions, $\sqrt{n}\left(\hat{\Theta} - \Theta\right) \sim N_4(0,\Psi)$ [16]. The variance-covariance matrix Ψ is obtained by taking the inverse of the Fisher's Information Matrix $\upsilon\left(\hat{\Theta}\right)^{-1}$ [17]. Hence, the confidence interval for the estimators can be obtained at any $(1-\alpha)100\%$ as $\hat{\Theta} \pm Z_{\frac{\alpha}{2}}\sqrt{\Psi_{ii}}$.

7 Application

The potential of the new distribution is illustrated by modelling a real-life data set. Two distributions, the transmuted Power function, and Power function distributions are also used to model the same data set. The Cramer-von Mises and Anderson-Darling goodness-of-fit statistics are used to determine how well the distributions fit the data set. The Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and the negative log-likelihood values of the distributions are also computed. Chen and Balakrishnan [18] noted that distribution with the least values of the estimated Cramer-von Mises (W*) and Anderson-Darling (A*) statistics has a better fit. Data set: This is life of fatigue fracture of Kevlar 373/epoxy (pressure 90%) reported by [19]. The data is as follows: 0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.

The pdfs of compared transmuted power function due to [8] and power function due to [12] are respectively given by
$g(x) = \frac{\theta}{\beta}\left(\frac{x}{\beta}\right)^{\theta-1}\left[1+\lambda-2\lambda\left(\frac{x}{\beta}\right)^{\theta}\right];\ |\lambda| \leq 1, \theta > 0, \beta > 0, 0 < x < \beta.$
and
$g(x) = \theta\beta^{-1}x^{\theta-1};\ \theta > 0, 0 < x < \beta.$

Tables 4 and 5 show the goodness-of-fit statistics and the parameter estimates including the standard errors in parentheses of the competing distributions, respectively. The values in Table 4 indicate that the AGTPF distribution has the least values of the goodness-of-fit statistics which imply that it has a better fit among the competing distributions.

The graphical fit of the competing distributions on the data set is shown in Figure 3. The estimated pdf plot of AGTPF depicts the shape of the histogram of the data set and the cdf plot is closest to the empirical cdf plot. Figure 3 also confirms the goodness-of-fit statistics values in Table 4 that the AGTPF has a better fit.

Table 4: Summary of the Parameter estimates (Standard error in parentheses) for data set.

Distribution	Parameter estimates			
AGTPF$(\lambda,\alpha,\theta,\beta)$	0.8077	3.4247	1.235	max(x)=9.096
	(0.1779)	(0.9060)	(0.1284)	
TPF (λ,θ,β)	0.9481	0.7729	max(x) = 9.096	
	(0.0516)	(0.0672)		
PF(θ,β)	0.5348	max(x) = 9.096		
	(0.0614)			

We considered the likelihood ratio test to select the preferred distribution from the nested distributions of the new distribution. The likelihood ratio hypotheses are as follows: $H_0 : \alpha = 1$ versus $H_a : \alpha \neq 1$, and $H_0 : \alpha = 1, \lambda = 0$ versus $H_a : \alpha \neq 1, \lambda \neq 1$ with the test statistic given by $w = 2\left[\ell(\hat{\Theta}_a) - \ell(\hat{\Theta}_0)\right]$ which follows $\chi^2_{\alpha(n)}$; where $n = k_1 - k_0$, k_1, and k_0 are the number of parameters associated with the alternative and the null hypothesis respectively. Based on the P-values

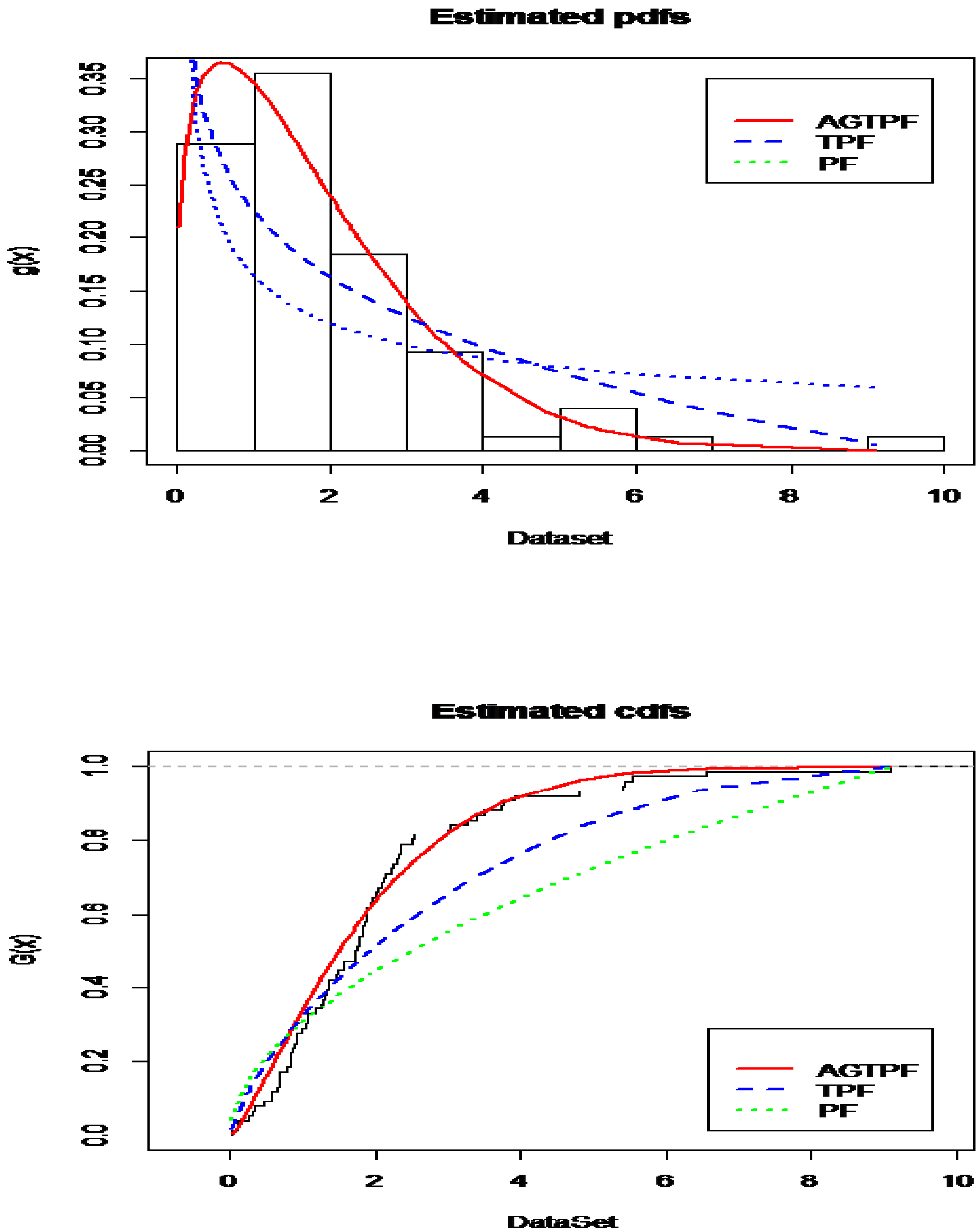

Fig. 3: The plots of estimated pdfs and cdfs with empirical cdf

Table 5: Summary of the Likelihood ratio test.

Null hypothesis	Alternative hypothesis	w	P-value
$H_0 : \alpha = 1$	$H_a : \alpha \neq 1$	50.4816	6.515×10^{-6}
$H_0 : \alpha = 1, \lambda = 0$	$H_a : \alpha \neq 1, \lambda \neq 0$	20.3306	1.091×10^{-11}

in Table 6, the AGTPF is preferred to the transmuted Power function and power function distribution respectively; with respect to the data set.

8 Conclusion

In this paper, another Generalized Transmuted Power function distribution for modelling lifetime data was introduced. Some of the mathematical properties such as quantile function, ordinary moments, and mode of the new distribution were established. Its entropy and pdf of the i^{th} ordered statistics were obtained. We considered the estimation of the parameters of the new distribution using the maximum likelihood method. The usefulness and potentials of the new distribution were demonstrated by comparing its fit to a real-life data set with other distributions. The goodness-of-fit statistics indicated that the new distribution has a better fit than other competing distributions. The likelihood ratio test also indicated the preference of the new distribution to its sub-distributions.

Conflict of Interest

The authors declare that they have no conflict of interest.

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