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A Convenience Approximate Method for Solving an Inverse Heat Conduction Problem

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Abstract: The present research addresses, fractional type one-dimensional inverse heat conduction problem (FIHCP). This problem is devoted to calculating the temperature distribution in its range and the thermal flux on the bound, while the temperature is clear at some domain points. The new approach of homotopic perturbation method (*NHPM*) is employed to recover unknown functions and to obtain a solution for the problem. In the end, some appropriate examples are given for introducing and implementing the proposed approach in solving FIHCP.

Keywords: New homotopy perturbation method, Heat conduction problem, Fractional inverse problem

1 Introduction

Paying attention to the problems of both mathematical and physical aspects is necessary in numerous situations. In such cases, some of the information required for a complete mathematical or physical description of a model of the missing one is uncharted.

From the perspective of theory, the inverse problem (IP) can be considered as a category of problems so called ill-posed problems. Because of the small variations in the measured data, large deviations occur. Thus, considering the effect of small variations in input data, we will see the truthfulness of the solutions in the general assumptions of stability, existence and uniqueness.

Among the practical applications of the IP, one can find them in fields, including geophysics [1], optics [2], astronomy [3], quantum mechanics [4], medical imaging as well as nondestructive testing [5], photo-elasticity [6], X- ray tomography [7] and elastoplasticity model [8].

Recently, the IP in fractional calculus has also been of great interest. In [9], Sakamoto and Yamamoto investigated the uniqueness and existence of weak solutions to Diffusion-wave equation of non integer order and its asymptomatic behavior when $t \to \infty$. With the help of some methods, such as weighted difference scheme and generalized factorization method [10], Kansa-type method [11] and simplified Tikhonov regularization method [12], the researchers presented numerical solutions to the diffusion problem. Zhang et al. used Levenberg -Marquardt method to give numerical solutions for fractional Fokker-Planck problems [13]. Shakeri and Dehghan [14] employed He's homotopic perturbation to obtain the solution of diffusion equations of non integer order. FIHCP was explored by Murio [15], and [16,17,18,19,20,21,22].

The present paper investigates FIHCP. This problem is devoted to calculating the temperature distribution in its range and the thermal flux on the bound, while the temperature is clear at some of the domain points.

Let us consider the famous equation "*Heat conduction*"

$$D_t^{\mu} v(\zeta,s) = M(\zeta)\frac{\partial^2 v(\zeta,s)}{\partial \zeta^2}, \quad 0<\zeta<a,\ 0<s<b,\ 0<\mu\leq 1, \tag{1}$$

with initial condition

$$v(\zeta,0)=\psi(\zeta), \quad \zeta\in[0,a], \tag{2}$$

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and Dirichlet boundary conditions

$$v(0,s) = \varphi(\zeta), \quad s \in [0,b], \tag{3}$$

$$-k\frac{\partial v(b,s)}{\partial \zeta} = q(\zeta), \quad s \in [0,b], \tag{4}$$

in which k is a constant and it is assumed that the functions $\psi(\zeta), \varphi(\zeta)$ and $q(\zeta)$ are piecewise continuous.

In Eq. (1) the operator D^{μ} indicates to fractional Caputo derivative. Fractional calculus are generalizations of differentiation and integration to non-integer orders. Interested readers are encouraged to use [24,25,26,27] for further reading of these calculus.

Looking at the two unknown functions, temperature distribution $v(\zeta,s)$ and $q(\zeta)$ in Neumann boundary condition is goal in the IP.

For this purpose, we add the overspecified condition at the fixed point $\zeta = \zeta_p$ as

$$v(\zeta_p,s) = \varphi_p(\zeta), \quad s \in (0,b). \tag{5}$$

The framework of this research is defined as follows: The model of FIHCP is described in Section Two. The numerical method (*NHPM*) is described in Section Three. In Section Four, the proposed method is utilized to obtain the approximate solution of the problem.

We use *NHPM* [28] as one of the numerical methods. Interested readers can read some other papers referenced in [29, 30,31,32,33,34].

Before introducing the main topic, we present some definitions that are required later.

2 Methodology

A new approach supported by one of the most widely used methods, i.e. HPM, to define an answer for the desired problem has been proposed in this part. We call this new assertion (*NHPM*).

First, we describe the method in general terms. Then, we execute it for solving the problem (1)-(2).

2.1 Description of the method

Let's propound the following initial value problem and elaborate (*NHPM*) on it:

$$D_t^{\mu} v(\zeta,s) + N[v(\zeta,s)] = M(\zeta,s), \qquad 0<\zeta<a,\ 0<s<b,\ n-1<\mu\leq n, \tag{6}$$

$$v_j(\zeta,s_0) = g_i(\zeta), \qquad j = 0,1,...n-1. \tag{7}$$

Here, "$\zeta,\ s$" are independent variables, the function $u(\zeta,s)$ is unknown function, $M(\zeta,s)$ denotes inhomogeneous term and D^{μ} signifies fractional derivative of Caputo sense with the order μ.

Now, we handle the method in 3 steps.

Step 1. Create the following homotopy

$$(1-q)\left(D_s^{\mu} V(\zeta,s) - v_0(\zeta,s)\right) + q\left(D_s^{\mu} V(\zeta,s) + N(V(\zeta,s)) - M(\zeta,s)\right) = 0, \tag{8}$$

or

$$D_s^{\mu} V(\zeta,s) = v_0(\zeta,s) - q\left(v_0(\zeta,s) + N(V(\zeta,s)) - M(\zeta,s)\right). \tag{9}$$

By Incorporating of the operator $L^{-1} = I_s^{\mu}(.)$ on both sides of the (9), we have the next sequel

$$V(\zeta,s) = V(\zeta,s_0) + I_s^{\mu} v_0(\zeta,s) - q I_s^{\mu}\left(v_0(\zeta,s) + N(V(\zeta,s)) - M(\zeta,s)\right), \tag{10}$$

and

$$V(\zeta,s_0) = \sum_{i=0}^{n-1} g_i(\zeta)\,\frac{(s-s_0)^i}{i!}. \tag{11}$$

Step 2. Consider the answer of Eq. (10):

$$V(\zeta,s) = V_0(\zeta,s) + qV_1(\zeta,s) + q^2 V_2(\zeta,s) + \cdots. \tag{12}$$

where $V_j(\zeta, s)$, $j = 0, 1, 2, 3, \ldots$, are functions that will specified later.
Step 3. Suppose that the primary approximation of the solution can be presented using the following form

$$v_0(\zeta, s) = \sum_{j=0}^{\infty} c_j(\zeta) q_j(s), \tag{13}$$

and $c_j(\zeta)$, $j = 0, 1, 2, 3, \ldots$ are indeterminate coefficients, $q_j(s)$, $j = 0, 1, 2, 3, \ldots$ are specific functions.

It should be noted that in the case of analyticity (around $s = 0$) $v_0(\zeta, s)$ and $M(\zeta, s)$, we will consider their expansion of the Taylor series, i.e.

$$v_0(\zeta, s) = \sum_{j=0}^{\infty} c_j(\zeta) t^{j\mu}, \tag{14}$$

$$M(\zeta, s) = \sum_{j=0}^{\infty} c_j^*(\zeta) t^{j\mu}. \tag{15}$$

From (10) featuring (12) and (13) the classification of sentences in terms of powers q and ultimately zeroing of the coefficients of q, we shall obtain

$$\begin{aligned}
&q^0: \ V_0(\zeta, s) = V(\zeta, s_0) + \sum_{n=0}^{\infty} c_n(\zeta) I_s^{\mu}\left(q_n(s)\right), \\
&q^1: \ V_1(\zeta, s) = -\sum_{n=0}^{\infty} c_n(\zeta) I_s^{\mu}\left(q_n(s)\right) - I_s^{\mu}\left(N\left(V_0(\zeta, s) - M(\zeta, s)\right)\right), \\
&q^2: \ V_2(\zeta, s) = -I_s^{\mu}\left(N\left(V_0(\zeta, s), V_1(\zeta, s)\right)\right), \\
&\qquad \vdots \\
&q^k: \ V_j(\zeta, s) = -I_s^{\mu}\left(N\left(V_0(\zeta, s), V_1(\zeta, s) \ldots, V_{j-1}(\zeta, s)\right)\right),
\end{aligned} \tag{16}$$

Determine answer for these equations, so $V_1(\zeta, s) = 0$, then Eqs. (16) refers to $V_j(\zeta, s) = 0$, $k = 2, 3, \cdots$. Accordingly, we will obtain the accurate solution as what follows:

$$v(\zeta, s) = V_0(\zeta, s) = V(\zeta, s_0) + \sum_{k=0}^{\infty} c_k(\zeta) \left(I_s^{\mu} q_k(s)\right). \tag{17}$$

2.2 Run the procedure for fractional heat conduction problem:

Consider the FHCP

$$D_t^{\mu} v(\zeta, s) + M(\zeta) \frac{\partial^2 v(\zeta, s)}{\partial \zeta^2} = 0, \quad 0 < \zeta < a, \ 0 < s < b, \ 0 < \mu \leq 1, \tag{18}$$

$$v(\zeta, s_0) = g(\zeta). \tag{19}$$

We assume

$$v_0(\zeta, s) = \sum_{j=0}^{\infty} c_j(\zeta) s^{j\alpha}, \tag{20}$$

By executing the procedure steps for the equation (18) along with (20), the coefficients $c_j(\zeta)$ are as follows:

$$\begin{aligned}
&c_0(\zeta) = -M(\zeta) v(\zeta, s_0), \\
&\qquad c_1(\zeta) = -M(\zeta) \left[c_0(\zeta)\right]_{\zeta\zeta}, \\
&\qquad c_2(\zeta) = -M(\zeta) \frac{\Gamma(\mu+1)}{\Gamma(2\mu+1)} \left[c_1(\zeta)\right]_{\zeta\zeta}, \\
&\qquad \vdots
\end{aligned} \tag{21}$$

Putting these coefficients in (20), the solution of (18)-(19) is obtained.
Note: In (18), whenever $M(\zeta) = c$, where "c" is constant, we obtain the following result:

$$\begin{aligned} c_0(\zeta) =& c[v(\zeta, s_0)]_{\zeta\zeta}, \\ & c_1(\zeta) = \frac{c^2}{\Gamma(\mu+1)}[v(\zeta, s_0)]_{\zeta\zeta\zeta\zeta}, \\ & \vdots \\ & c_n(\zeta) = \frac{c^{n+1}}{\Gamma(n\mu+1)}[v(\zeta, s_0)]\underbrace{\zeta\zeta...\zeta\zeta}_{2(n+1)-order}. \end{aligned} \tag{22}$$

Therefore, the solution is obtained as follows:

$$v(\zeta, s) = V_0(\zeta, s) = \sum_{n=0}^{\infty}[v(\zeta, s_0)]\underbrace{\zeta\zeta...\zeta\zeta}_{2n-order}\frac{(-ct^\mu)^n}{\Gamma(n\mu+1)}. \tag{23}$$

3 Examples

Example 1. Regarding the q-FHIVP below:

$$\begin{cases} D_t^\mu v(\zeta, s) + \frac{1}{2}\zeta^2 v_{\zeta\zeta}(\zeta, s) = 0, \quad s \in [0,2], \quad \zeta \in [0,2], \\ v(\zeta, 0) = \zeta^2, \\ v(0, s) = 0, \end{cases} \tag{24}$$

along with the over specified condition

$$v(\zeta_p, s) = v(1, s) = E_\mu(-s^\mu),$$

and the boundary condition

$$q(s) = -\frac{1}{4}v_\zeta(2, s).$$

Regarding (21), we achieve

$$c_0(\zeta) = -\zeta^2, \quad c_1(\zeta) = \frac{\zeta^2}{\Gamma(\mu+1)}, \cdots, c_n(\zeta) = \frac{(-1)^{n+1}\zeta^2}{\Gamma(n\mu+1)}.$$

Now, from (20) we conclude that

$$v(\zeta, s) = \zeta^2 \sum_{n=0}^{\infty} \frac{(-s)^\mu}{\Gamma(n\mu+1)} = \zeta^2 E_\mu(-s^\mu). \tag{25}$$

From (25), the Neumann boundary condition $q(s)$ is as follows:

$$q(s) = -\frac{1}{4}v_\zeta(2, s) = -E_\mu(-s^\mu). \tag{26}$$

Example 2. Regarding the q-FHIVP below:

$$\begin{cases} D_t^\mu v(\zeta, s) - 0.1\, v_{\zeta\zeta}(\zeta, s) = 0, \quad s \in [0,2], \quad \zeta \in [0,1.2], \\ v(\zeta, 0) = e^{1-\zeta}, \\ v(0, s) = eE_\mu(0.1\, s^\mu), \end{cases}$$

along with the over specified condition

$$v(\zeta_p, s) = v(1, s) = E_\mu(0.1s^\mu),$$

and the boundary condition

$$q(s) = -2v_\zeta(1.2, s).$$

Considering (22), we achieve

$$c_0(\zeta) = 0.1e^{1-\zeta}, \quad c_1(\zeta) = \frac{(0.1)^2}{\Gamma(\mu+1)}e^{1-\zeta}, \cdots, c_n(\zeta) = \frac{(0.1)^{n+1}}{\Gamma(n\mu+1)}e^{1-\zeta}.$$

Thus, we conclude from (23) that

$$v(\zeta, s) = \sum_{n=0}^{\infty} \frac{(0.1s^{\mu})^n}{\Gamma(n\mu+1)} e^{1-\zeta} = e^{1-\zeta} E_{\mu}(0.1s^{\mu}). \tag{27}$$

Now, the Neumann boundary condition $q(s)$ can be reached from (27) as

$$q(s) = -2v_{\zeta}(1.2, s) = 2e^{-0.2} E_{\mu}(0.1s^{\mu}). \tag{28}$$

Example 3. Regarding the q-FHIVP below:

$$\begin{cases} D_t^{\mu} v(\zeta, s) - v_{\zeta\zeta}(\zeta, s) = 0, \quad s \in [0,2], \quad \zeta \in [0,1.2], \\ v(\zeta, 0) = \frac{\zeta^4}{4!}, \\ v(0, s) = \frac{t^{2\mu}}{\Gamma(2\mu+1)}, \end{cases}$$

along with the over specified condition

$$v(\zeta_p, s) = v(1, s) = \frac{\zeta^4}{4!} + \frac{t^{\mu}}{2\Gamma(\mu+1)} + \frac{t^{2\mu}}{\Gamma(2\mu+1)},$$

and Neumann boundary condition

$$q(s) = -v_{\zeta}(2, s).$$

It is easy to notice that

$$v(\zeta, s) = \frac{\zeta^4}{4!} + \frac{\zeta^2 s^{\mu}}{2\Gamma(\mu+1)} + \frac{t^{2\mu}}{\Gamma(2\mu+1)}. \tag{29}$$

Now, the Neumann boundary condition $q(s)$ obtained from (29) is as follows:

$$q(s) = -v_{\zeta}(2, s) = -\frac{4}{3} - 2\frac{s^{\mu}}{\Gamma(\mu+1)}. \tag{30}$$

4 Conclusion

The present paper has investigated FIHCP. This problem is devoted to calculating the temperature distribution in its range and the thermal flux on the bound, while the temperature is clear at some domain points.

The new viewpoint of homotopic perturbation method (*NHPM*) is applied to recover the unknown functions and obtain a solution for the problem. For other similar nonlinear inverse problems of non integer order, this view may be employed to get solution.

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