Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/060101

Asymptotic Behavior of Solutions of Higher Order Fractional Differential Equations with a Caputo-Type Hadamard Derivative

John R. Graef[1,*], *Said R. Grace*[2] *and Ercan Tunç*[3]

[1] Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403, USA
[2] Department of Engineering Mathematics, Faculty of Engineering, Cairo University, Orman, Giza 12613, Egypt
[3] Department of Mathematics, Faculty of Arts and Sciences, Gaziosmanpasa University, 60240, Tokat, Turkey

Received: 2 Mar. 2019, Revised: 12 Jun. 2019, Accepted: 17 Jun. 2019
Published online: 1 Jan. 2020

Abstract: The present paper investigates the behavior of nonoscillatory solutions of the higher order fractional differential equation

$$^{C,H}\mathscr{D}_a^r y(t) = e(t) + f(t,x(t)), \quad a > 1,$$

where $^{C,H}\mathscr{D}_a^r$ is a Caputo-type Hadamard derivative. The authors address the two cases $y(t) = x^{(k)}(t)$ with k a positive integer, and $y(t) = \big(c(t)\,(x'(t))^{\mu}\big)'$ with $\mu \geq 1$ being the ratio of odd positive integers. Here, $r = n + \alpha - 1$, $\alpha \in (0,1)$, and $n \in \mathbb{Z}^+$.

Keywords: Asymptotic behavior, nonoscillatory solutions, Caputo type Hadamard derivative, fractional differential equations.

1 Introduction

We handle the fractional initial value problem (IVP)

$$\begin{cases} ^{C,H}\mathscr{D}_a^r y(t) = e(t) + f(t,x(t)), & a > 1, \\ \delta^N y(a) = b_N, \end{cases} \tag{1}$$

where b_N with $N = 0,1,\ldots,n-1$ are constants, $y(t) = x^{(k)}(t)$ with k a positive integer, or $y(t) = \big(c(t)\,(x'(t))^{\mu}\big)'$ with $\mu \geq 1$ the ratio of odd positive integers, $r = n+\alpha-1$, $\alpha \in (0,1)$, and $n \in \mathbb{Z}^+$. Here $\delta = t\frac{d}{dt}$, and $^{C,H}\mathscr{D}_a^r y(t)$ is a Caputo-type Hadamard fractional derivative of y on the interval $[a,\infty)$.

In what follows, we assume:

(i) $c : [a,\infty) \to (0,\infty)$ is continuous;
(ii) $e : [a,\infty) \to \mathbb{R}$ is continuous;
(iii) $f : [a,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and there is a continuous function $h : [a,\infty) \to (0,\infty)$ and constants γ, $\lambda \in \mathbb{R}$ with $0 < \lambda < 1$ such that

$$0 \leq x f(t,x) \leq \left(\ln\frac{t}{a}\right)^{\gamma-1} h(t)\,|x|^{\lambda+1} \quad \text{for } x \neq 0 \text{ and } t > a.$$

Only the solutions of equation (1) that are continuable and nontrivial in any neighborhood of ∞ are addressed. Moreover, this solution is called *oscillatory* if it has arbitrarily large zeros and *nonoscillatory* otherwise.

Fractional differential equations have grabbed the attention of numerous researchers because they provide excellent models for the problems involving memory or hereditary properties. They play important roles in several research areas, such as physics, chemical technology, population dynamics, biotechnology, and economics. Examples of applications and new developments in the area are illustrated, for example, in [1,2,3,4,5,6,7,8,9,10,11,12,13].

* Corresponding author e-mail: John-Graef@utc.edu

Much work on problems involving fractional differential equations centered on either the Riemann-Liouville or the Caputo fractional derivative. Hadamard introduced the Hadamard fractional derivative in 1892 (see [14]). It differs from other ones in that the kernel of the integral in the definition contains a logarithmic function with an arbitrary exponent. Background material on Hadamard fractional derivatives and integrals are involved in [15,16,17,18,19,20].

Results on the asymptotic behavior of solutions of fractional differential equations with Caputo-type Hadamard derivatives are relatively scarce in the literature. Moreover, few examples involve equations of the form (1). In [21], we obtained some new results on the oscillatory and asymptotic behavior of the solutions of equation (1) in the case where $y(t)=x(t)$. The present paper aims to establish some new criteria for the asymptotic behavior of all nonoscillatory solutions of equation (1) with $y(t)=x^{(k)}(t)$, where k is a positive integer, and for $y(t)=\left(c(t)\left(x'(t)\right)^{\mu}\right)'$ where $\mu\geq 1$ is the ratio of odd positive integers.

We begin with some basic notations and definitions from the fractional calculus; see, for example, [8,11].

Definition 1. *The Hadamard fractional integral of order $\alpha\in\mathbb{R}^{+}$ of a function $f(t)$ is defined as follows:*

$$J_a^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1}f(s)\frac{ds}{s},\quad t>a>0,$$

where Γ is the Euler-Gamma function.

Definition 2. *The Hadamard derivative of order $\alpha\in[n-1,n)$, $n\in\mathbb{Z}^{+}$, of a function $f(t)$ is given by*

$$\begin{aligned}{}^{H}\mathscr{D}_a^{\alpha}f(t)&=\delta^{n}(J_a^{n-\alpha}f)(t)\\&=\frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^{n}\int_a^t\left(\ln\frac{t}{s}\right)^{n-\alpha-1}f(s)\frac{ds}{s},\ t>a>0,\end{aligned}$$

where $\delta:=t\frac{d}{dt}$, $\delta^{0}f(t)=f(t)$, and $n=[\alpha]+1$ where $[\alpha]$ denotes the integer part of the real number α.

Definition 3. *[23] For an n-times differentiable function $f:[a,\infty)\to\mathbb{R}$, the Caputo-type Hadamard derivative of fractional order α is defined as follows:*

$$\begin{aligned}{}^{C,H}\mathscr{D}_a^{\alpha}f(t)&=\frac{1}{\Gamma(n-\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{n-\alpha-1}\delta^{n}f(s)\frac{ds}{s}\\&=J_a^{n-\alpha}\delta^{n}f(t),\quad n-1<\alpha<n,\end{aligned}$$

where $t>a>0$, $\delta=t\frac{d}{dt}$, and $n=[\alpha]+1$.

Based on what can be obtained from Lemma 2.5 in [23], a Volterra type equation corresponding to the IVP (1) has the form

$$y(t)=\sum_{N=0}^{n-1}\frac{b_N}{N!}\left(\ln\frac{t}{a}\right)^{N}+\frac{1}{\Gamma(r)}\int_a^t\left(\ln\frac{t}{s}\right)^{r-1}[e(s)+f(s,x(s))]\frac{ds}{s}. \tag{2}$$

2 Asymptotic behavior of equation (1.1) with $y(t)=x^{(k)}(t)$

We begin this section with two lemmas required to prove our results in this case.

Lemma 1. *[21] Let β, γ, and p be positive constants such that*

$$p(\beta-1)+1>0\ \ and\ \ p(\gamma-1)+1>0.$$

Then

$$\int_a^t\left(\ln\frac{t}{s}\right)^{p(\beta-1)}\left(\ln\frac{s}{a}\right)^{p(\gamma-1)}\frac{ds}{s}=B\left(\ln\frac{t}{a}\right)^{\theta},\quad t>a>0, \tag{3}$$

where

$$B:=B\left[p(\gamma-1)+1,p(\beta-1)+1\right],\quad B[\xi,\eta]=\int_0^1 s^{\xi-1}(1-s)^{\eta-1}ds,$$

$$\xi>0,\quad \eta>0,\quad and\quad \theta=p(\beta+\gamma-2)+1.$$

Lemma 2.[22] *If X and Y are nonnegative and $0<\lambda<1$, then*

$$X^{\lambda}-(1-\lambda)Y^{\lambda}-\lambda XY^{\lambda-1}\le 0, \tag{4}$$

with equality holding if and only if $X=Y$.

We now give sufficient conditions whereby which any nonoscillatory solution x of (1) satisfies

$$\frac{|x(t)|}{t^k}=O\left(\left(\ln\frac{t}{a}\right)^{n-1}\right)\quad \text{as } t\to\infty,$$

where k is a positive integer.

Theorem 1.*Let conditions (ii) and (iii) hold and assume that there exist $p>1$, $r>0$, $q=\frac{p}{p-1}$ and $\gamma=(n-r)+\frac{1}{q}$ such that*

$$p(r-1)+1>0 \;\; and \;\; p(\gamma-1)+1>0. \tag{5}$$

In addition, assume that there is a continuous function $m:[a,\infty)\to(0,\infty)$ such that

$$\lim_{t\to\infty}\left(\ln\frac{t}{a}\right)^{1-n}\int_a^t\left(\ln\frac{t}{s}\right)^{r-1}\left(\ln\frac{s}{a}\right)^{\gamma-1}m^{\lambda/(\lambda-1)}(s)h^{1/(1-\lambda)}(s)\frac{ds}{s}<\infty, \tag{6}$$

$$\int_a^{\infty}\left(\ln\frac{s}{a}\right)^{(n-1)q}s^{kq}m^q(s)ds<\infty, \tag{7}$$

and

$$\lim_{t\to\infty}\left(\ln\frac{t}{a}\right)^{1-n}\int_a^t\left(\ln\frac{t}{s}\right)^{r-1}|e(s)|\frac{ds}{s}<\infty. \tag{8}$$

If $x(t)$ is a nonoscillatory solution of (1), then

$$\limsup_{t\to\infty}\left(\ln\frac{t}{a}\right)^{1-n}\frac{|x(t)|}{t^k}<\infty,$$

where k is a positive integer.

*Proof.*Let $x(t)$ be a nonoscillatory solution of equation (1); say $x(t)>0$ for $t\ge t_1$ for some $t_1\ge a$. Since equation (1) is equivalent to equation (2), we see from (2) that there exists a constant $M_1>0$ such that

$$x^{(k)}(t)\le M_1\left(\ln\frac{t}{a}\right)^{n-1}+\frac{1}{\Gamma(r)}\int_a^t\left(\ln\frac{t}{s}\right)^{r-1}[e(s)+f(s,x(s))]\frac{ds}{s}. \tag{9}$$

Letting $F(t)=e(t)+f(t,x(t))$, in view of (ii) and (iii), equation (9) can be written as

$$\begin{aligned}x^{(k)}(t)\le\ & M_1\left(\ln\frac{t}{a}\right)^{n-1}+\frac{1}{\Gamma(r)}\int_a^{t_1}\left(\ln\frac{t}{s}\right)^{r-1}|F(s)|\frac{ds}{s}\\ &+\frac{1}{\Gamma(r)}\int_{t_1}^t\left(\ln\frac{t}{s}\right)^{r-1}|e(s)|\frac{ds}{s}\\ &+\frac{1}{\Gamma(r)}\int_{t_1}^t\left(\ln\frac{t}{s}\right)^{r-1}\left(\ln\frac{s}{a}\right)^{\gamma-1}\left[h(s)x^{\lambda}(s)-m(s)x(s)\right]\frac{ds}{s}\\ &+\frac{1}{\Gamma(r)}\int_{t_1}^t\left(\ln\frac{t}{s}\right)^{r-1}\left(\ln\frac{s}{a}\right)^{\gamma-1}m(s)x(s)\frac{ds}{s}.\end{aligned} \tag{10}$$

Applying Lemma 2 with

$$X=h^{1/\lambda}(s)x(s)\;\;\text{and}\;\;Y=\left(\frac{1}{\lambda}m(s)h^{-1/\lambda}(s)\right)^{1/(\lambda-1)},$$

we obtain

$$h(s)x^{\lambda}(s)-m(s)x(s)\le(1-\lambda)\lambda^{\lambda/(1-\lambda)}m^{\lambda/(\lambda-1)}(s)h^{1/(1-\lambda)}(s). \tag{11}$$

Using (11) in (10) yields

$$x^{(k)}(t) \le M_1\left(\ln\frac{t}{a}\right)^{n-1} + \frac{1}{\Gamma(r)}\int_a^{t_1}\left(\ln\frac{t}{s}\right)^{r-1}|F(s)|\frac{ds}{s}$$
$$+\frac{1}{\Gamma(r)}\int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{r-1}|e(s)|\frac{ds}{s}$$
$$+\frac{(1-\lambda)\lambda^{\lambda/(1-\lambda)}}{\Gamma(r)}\int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{r-1}\left(\ln\frac{s}{a}\right)^{\gamma-1}m^{\lambda/(\lambda-1)}(s)h^{1/(1-\lambda)}(s)\frac{ds}{s}$$
$$+\frac{1}{\Gamma(r)}\int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{r-1}\left(\ln\frac{s}{a}\right)^{\gamma-1}m(s)x(s)\frac{ds}{s}. \tag{12}$$

In view of (6) and (8), inequality (12) takes the form

$$x^{(k)}(t) \le M_2\left(\ln\frac{t}{a}\right)^{n-1} + \frac{1}{\Gamma(r)}\int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{r-1}\left(\ln\frac{s}{a}\right)^{\gamma-1}m(s)x(s)\frac{ds}{s} := w(t), \tag{13}$$

for some constant $M_2 > 0$ and all $t \ge t_2$ for $t_2 \ge t_1$ sufficiently large. Integrating (13) from t_1 to t consecutively k-times and noting that $w(t)$ is an increasing function, we see that

$$x(t) \le M_3 t^{k-1} + t^k w(t), \tag{14}$$

for some constant $M_3 > 0$ and $t \ge t_3$ for some $t_3 \ge t_2$. From the definition of $w(t)$, inequality (14) yields

$$\frac{x(t)}{t^k} \le \frac{M_3}{t} + w(t)$$
$$= \frac{M_3}{t} + M_2\left(\ln\frac{t}{a}\right)^{n-1} + \frac{1}{\Gamma(r)}\int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{r-1}\left(\ln\frac{s}{a}\right)^{\gamma-1}m(s)x(s)\frac{ds}{s}$$

or

$$\left(\ln\frac{t}{a}\right)^{1-n}\frac{x(t)}{t^k} := z(t)$$
$$\le 1 + M + \frac{(\ln\frac{t}{a})^{1-n}}{\Gamma(r)}\int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{r-1}\left(\ln\frac{s}{a}\right)^{\gamma-1}m(s)x(s)\frac{ds}{s}, \tag{15}$$

for some constant $M > 0$ and $t \ge t_4$ for some $t_4 \ge t_3$.

If we apply Lemma 3 and Hölder's inequality, we obtain

$$\int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{r-1}\left(\ln\frac{s}{a}\right)^{\gamma-1}m(s)x(s)\frac{ds}{s}$$
$$\le \left(\int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{p(r-1)}\left(\ln\frac{s}{a}\right)^{p(\gamma-1)}\frac{ds}{s}\right)^{1/p}\left(\int_{t_1}^{t}m^q(s)x^q(s)ds\right)^{1/q}$$
$$\le \left(\int_{a}^{t}\left(\ln\frac{t}{s}\right)^{p(r-1)}\left(\ln\frac{s}{a}\right)^{p(\gamma-1)}\frac{ds}{s}\right)^{1/p}\left(\int_{t_1}^{t}m^q(s)x^q(s)ds\right)^{1/q}$$
$$= \left(B\left(\ln\frac{t}{a}\right)^{\theta}\right)^{1/p}\left(\int_{t_1}^{t}m^q(s)x^q(s)ds\right)^{1/q}, \tag{16}$$

where $B := B[p(\gamma-1)+1, p(r-1)+1]$ and $\theta = p(r+\gamma-2)+1$. Since $\gamma = (n-r)+\frac{1}{q}$, inequality (16) becomes

$$\int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{r-1}\left(\ln\frac{s}{a}\right)^{\gamma-1}m(s)x(s)\frac{ds}{s} \le B^{1/p}\left(\ln\frac{t}{a}\right)^{n-1}\left(\int_{t_1}^{t}m^q(s)x^q(s)ds\right)^{1/q}. \tag{17}$$

From (15) and (17) we observe that

$$z(t) \le 1 + M + \frac{B^{1/p}}{\Gamma(r)}\left(\int_{t_1}^{t}\left(\ln\frac{s}{a}\right)^{(n-1)q}s^{kq}m^q(s)z^q(s)ds\right)^{1/q}. \tag{18}$$

Applying the inequality

$$(x+y)^q \le 2^{q-1}(x^q+y^q), \ \text{ for } x, \ y \ge 0 \text{ and } q \ge 1, \tag{19}$$

to (18) gives

$$z^q(t) \le 2^{q-1}(1+M)^q + 2^{q-1}\left(\frac{B^{1/p}}{\Gamma(r)}\right)^q \int_{t_1}^t \left(\ln\frac{s}{a}\right)^{(n-1)q} s^{kq} m^q(s) z^q(s) ds.$$

If we set $P = 2^{q-1}(1+M)^q$, $Q = 2^{q-1}\left(B^{1/p}/\Gamma(r)\right)^q$, and $u(t) = z^q(t)$, i.e, $z(t) = u^{1/q}(t)$, the above-mentioned inequality becomes

$$u(t) \le P + Q\int_{t_1}^t \left(\ln\frac{s}{a}\right)^{(n-1)q} s^{kq} m^q(s) u(s) ds$$

for $t \ge t_4$. Now condition (7) and Gronwall's inequality imply that $u(t)$ is bounded. Thus, $z(t)$ is bounded and

$$\limsup_{t\to\infty}\left(\ln\frac{t}{a}\right)^{1-n}\frac{x(t)}{t^k} < \infty.$$

The proof in the case where $x(t)$ is eventually negative is similar.

Similar to the sublinear case, we present the following interesting result in which we do not need condition (6) in Theorem 1.

Theorem 2.*Let $\lambda = 1$ and the hypotheses of Theorem 1 except for (6) hold with $m(t) = h(t)$. Then the conclusion of Theorem 1 holds.*

*Proof.*Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t) > 0$ for $t \ge t_1$ for some $t_1 \ge a$. Since equation (1) is equivalent to equation (2), we likewise attain (9) as in the proof of Theorem 1. Letting $F(t) = e(t) + f(t,x(t))$, in view of (iii) with $h(t) = m(t)$, equation (9) can be written as

$$\begin{aligned} x^{(k)}(t) \le\, & M_1\left(\ln\frac{t}{a}\right)^{n-1} + \frac{1}{\Gamma(r)}\int_a^{t_1}\left(\ln\frac{t}{s}\right)^{r-1}|F(s)|\frac{ds}{s} \\ & + \frac{1}{\Gamma(r)}\int_{t_1}^t \left(\ln\frac{t}{s}\right)^{r-1}|e(s)|\frac{ds}{s} \\ & + \frac{1}{\Gamma(r)}\int_{t_1}^t \left(\ln\frac{t}{s}\right)^{r-1}\left(\ln\frac{s}{a}\right)^{\gamma-1} m(s)x(s)\frac{ds}{s}. \end{aligned} \tag{20}$$

In view of (8), inequality (20) takes the form

$$x^{(k)}(t) \le D\left(\ln\frac{t}{a}\right)^{n-1} + \frac{1}{\Gamma(r)}\int_{t_1}^t \left(\ln\frac{t}{s}\right)^{r-1}\left(\ln\frac{s}{a}\right)^{\gamma-1} m(s)x(s)\frac{ds}{s} := w_1(t), \tag{21}$$

for some constant $D > 0$ and $t \ge t_2$ for some $t_2 \ge t_1$. The rest of the proof is similar to that of Theorem 1 and is omitted.

3 Asymptotic behavior of equation (1.1) with $y(t) = \left(c(t)\left(x'(t)\right)^\mu\right)'$

In this case, we give sufficient conditions whereby any nonoscillatory solution x of equation (1) satisfies

$$\left(\frac{|x(t)|}{R(t)}\right)^\mu = O\left(\left(\ln\frac{t}{a}\right)^{n-1}\right) \quad \text{as } t \to \infty,$$

where

$$R(t) = \int_a^t \left(\frac{s}{c(s)}\right)^{1/\mu} ds,$$

and we assume that

$$R(t) \to \infty \quad \text{as} \quad t \to \infty. \tag{22}$$

To obtain our results in this section, we adopt the following lemma.

Lemma 3.*[Young's inequality] Let X and Y be nonnegative, $u>1$, and $\frac{1}{u}+\frac{1}{v}=1$. Then*

$$XY \le \frac{1}{u}X^u + \frac{1}{v}Y^v, \tag{23}$$

where equality holds if and only if $Y = X^{u-1}$.

Theorem 3.*Let conditions (i)–(iii) and (22) hold and assume that there exist $p>1$, $r>0$, $q=\frac{p}{p-1}$, and $\gamma=(n-r)+\frac{1}{q}$ such that (5) holds. In addition, assume there is a continuous function $m:[a,\infty)\to(0,\infty)$ with*

$$\lim_{t\to\infty}\left(\ln\frac{t}{a}\right)^{1-n}\int_a^t\left(\ln\frac{t}{s}\right)^{r-1}\left(\ln\frac{s}{a}\right)^{\gamma-1}m^{\lambda/(\lambda-\mu)}(s)h^{\mu/(\mu-\lambda)}(s)\frac{ds}{s}<\infty, \tag{24}$$

$$\int_a^\infty\left(\ln\frac{s}{a}\right)^{(n-1)q}R^{\mu q}(s)m^q(s)ds<\infty, \tag{25}$$

and condition (8) holds. If $x(t)$ is a nonoscillatory solution of equation (1),

$$\limsup_{t\to\infty}\left(\ln\frac{t}{a}\right)^{1-n}\left(\frac{|x(t)|}{R(t)}\right)^{\mu}<\infty.$$

*Proof.*Let $x(t)$ be a nonoscillatory solution of (1), say $x(t)>0$ for $t\ge t_1$ for some $t_1\ge a$. Since (1) is equivalent to (2), from (2) we see that there exists $N_1>0$ such that

$$\left(c(t)\left(x'(t)\right)^{\mu}\right)'\le N_1\left(\ln\frac{t}{a}\right)^{n-1}+\frac{1}{\Gamma(r)}\int_a^t\left(\ln\frac{t}{s}\right)^{r-1}[e(s)+f(s,x(s))]\frac{ds}{s}. \tag{26}$$

Once more let $F(t)=e(t)+f(t,x(t))$. In view of (i)–(iii), similar to (10), inequality (26) can be written as

$$\begin{aligned}\left(c(t)\left(x'(t)\right)^{\mu}\right)'&\le N_1\left(\ln\frac{t}{a}\right)^{n-1}+\frac{1}{\Gamma(r)}\int_a^{t_1}\left(\ln\frac{t}{s}\right)^{r-1}|F(s)|\frac{ds}{s}\\&+\frac{1}{\Gamma(r)}\int_{t_1}^t\left(\ln\frac{t}{s}\right)^{r-1}|e(s)|\frac{ds}{s}\\&+\frac{1}{\Gamma(r)}\int_{t_1}^t\left(\ln\frac{t}{s}\right)^{r-1}\left(\ln\frac{s}{a}\right)^{\gamma-1}\left[h(s)x^{\lambda}(s)-m(s)x^{\mu}(s)\right]\frac{ds}{s}\\&+\frac{1}{\Gamma(r)}\int_{t_1}^t\left(\ln\frac{t}{s}\right)^{r-1}\left(\ln\frac{s}{a}\right)^{\gamma-1}m(s)x^{\mu}(s)\frac{ds}{s}.\end{aligned} \tag{27}$$

Applying Lemma 3 with

$$u=\frac{\mu}{\lambda}>1,\quad X=x^{\lambda}(t),\quad Y=\frac{\lambda}{\mu}\frac{h(t)}{m(t)},\quad v=\frac{\mu}{\mu-\lambda},$$

we see that

$$h(t)x^{\lambda}(t)-m(t)x^{\mu}(t)\le\frac{\mu-\lambda}{\lambda}\left(\frac{\lambda}{\mu}\right)^{\mu/(\mu-\lambda)}h^{\mu/(\mu-\lambda)}(t)m^{\lambda/(\lambda-\mu)}(t). \tag{28}$$

Using (28) in (27) gives

$$\begin{aligned}\left(c(t)\left(x'(t)\right)^{\mu}\right)'&\le N_1\left(\ln\frac{t}{a}\right)^{n-1}+\frac{1}{\Gamma(r)}\int_a^{t_1}\left(\ln\frac{t}{s}\right)^{r-1}|F(s)|\frac{ds}{s}\\&+\frac{1}{\Gamma(r)}\int_{t_1}^t\left(\ln\frac{t}{s}\right)^{r-1}|e(s)|\frac{ds}{s}\\&+\frac{\mu-\lambda}{\lambda}\left(\frac{\lambda}{\mu}\right)^{\mu/(\mu-\lambda)}\frac{1}{\Gamma(r)}\int_{t_1}^t\left(\ln\frac{t}{s}\right)^{r-1}\left(\ln\frac{s}{a}\right)^{\gamma-1}h^{\mu/(\mu-\lambda)}(s)m^{\lambda/(\lambda-\mu)}(s)\frac{ds}{s}\\&+\frac{1}{\Gamma(r)}\int_{t_1}^t\left(\ln\frac{t}{s}\right)^{r-1}\left(\ln\frac{s}{a}\right)^{\gamma-1}m(s)x^{\mu}(s)\frac{ds}{s}.\end{aligned} \tag{29}$$

In view of (8) and (24), inequality (29) has the form

$$\left(c(t)\left(x'(t)\right)^{\mu}\right)' \leq N_2\left(\ln\frac{t}{a}\right)^{n-1} + \frac{1}{\Gamma(r)}\int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{r-1}\left(\ln\frac{s}{a}\right)^{\gamma-1}m(s)x^{\mu}(s)\frac{ds}{s} := \psi(t), \tag{30}$$

for some constant $N_2 > 0$ and $t \geq t_2$ for some $t_2 \geq t_1$. Integrating (30) from t_1 to $t \geq t_2$ and adopting the fact that $\psi(t)$ is an increasing function, we obtain

$$c(t)\left(x'(t)\right)^{\mu} \leq c(t_1)\left(x'(t_1)\right)^{\mu} + \int_{t_1}^{t}\psi(s)ds \leq N_3 + t\psi(t), \tag{31}$$

where $N_3 = c(t_1)\left|x'(t_1)\right|^{\mu}$.

Thus, there exist a constant $N_4 > 1$ and $t_3 \geq t_2$ such that

$$c(t)\left(x'(t)\right)^{\mu} \leq N_4 t\psi(t). \tag{32}$$

for $t \geq t_3$. From (32), we have

$$x'(t) \leq N_4^{1/\mu}\left(\frac{t}{c(t)}\right)^{1/\mu}\psi^{1/\mu}(t).$$

Integrating the last inequality from t_1 to $t \geq t_3$ and adopting the fact that $\psi(t)$ is an increasing function, we see that

$$\begin{aligned} x(t) &\leq x(t_1) + N_4^{1/\mu}R(t)\psi^{1/\mu}(t) \\ &\leq \left[\frac{x(t_1)}{R(t)} + N_4^{1/\mu}\psi^{1/\mu}(t)\right]R(t) \\ &\leq \left[\frac{x(t_1)}{R(t_1)} + N_4^{1/\mu}\psi^{1/\mu}(t)\right]R(t), \end{aligned}$$

or

$$\frac{x(t)}{R(t)} \leq N_5 + N_6\psi^{1/\mu}(t), \tag{33}$$

for some positive constants N_5 and N_6.

Applying inequality (19) to (33), we obtain

$$\left(\frac{x(t)}{R(t)}\right)^{\mu} \leq 2^{\mu-1}N_5^{\mu} + 2^{\mu-1}N_6^{\mu}\psi(t) = d_1 + d_2\psi(t), \tag{34}$$

where $d_1 = 2^{\mu-1}N_5^{\mu}$ and $d_2 = 2^{\mu-1}N_6^{\mu}$. From (30) and (34), we obtain

$$\begin{aligned} \left(\frac{x(t)}{R(t)}\right)^{\mu} &\leq d_1 + d_2\psi(t) \\ &= d_1 + d_2N_2\left(\ln\frac{t}{a}\right)^{n-1} + \frac{d_2}{\Gamma(r)}\int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{r-1}\left(\ln\frac{s}{a}\right)^{\gamma-1}m(s)x^{\mu}(s)\frac{ds}{s}, \end{aligned}$$

or

$$\left(\ln\frac{t}{a}\right)^{1-n}\left(\frac{x(t)}{R(t)}\right)^{\mu} := z(t) \leq 1 + N + \frac{d_2\left(\ln\frac{t}{a}\right)^{1-n}}{\Gamma(r)}\int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{r-1}\left(\ln\frac{s}{a}\right)^{\gamma-1}m(s)x^{\mu}(s)\frac{ds}{s} \tag{35}$$

for some constant $N > 0$ and $t \geq t_3$.

An application of Hölder's inequality and Lemma 3 shows that

$$\begin{aligned}
&\int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{r-1}\left(\ln\frac{s}{a}\right)^{\gamma-1} m(s)x^{\mu}(s)\frac{ds}{s} \\
&\le \left(\int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{p(r-1)}\left(\ln\frac{s}{a}\right)^{p(\gamma-1)}\frac{ds}{s}\right)^{1/p}\left(\int_{t_1}^{t} m^q(s)x^{\mu q}(s)ds\right)^{1/q} \\
&\le \left(\int_{a}^{t}\left(\ln\frac{t}{s}\right)^{p(r-1)}\left(\ln\frac{s}{a}\right)^{p(\gamma-1)}\frac{ds}{s}\right)^{1/p}\left(\int_{t_1}^{t} m^q(s)x^{\mu q}(s)ds\right)^{1/q} \\
&= \left(B\left(\ln\frac{t}{a}\right)^{\theta}\right)^{1/p}\left(\int_{t_1}^{t} m^q(s)x^{\mu q}(s)ds\right)^{1/q}, \qquad (36)
\end{aligned}$$

where $B := B[p(\gamma-1)+1, p(r-1)+1]$ and $\theta = p(r+\gamma-2)+1$. Presently, $\gamma = (n-r)+\frac{1}{q}$, so inequality (36) becomes

$$\int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{r-1}\left(\ln\frac{s}{a}\right)^{\gamma-1} m(s)x^{\mu}(s)\frac{ds}{s} \le B^{1/p}\left(\ln\frac{t}{a}\right)^{n-1}\left(\int_{t_1}^{t} m^q(s)x^{\mu q}(s)ds\right)^{1/q}. \qquad (37)$$

It follows from (35) and (37) that

$$z(t) \le 1+N+\frac{d_2B^{1/p}}{\Gamma(r)}\left(\int_{t_1}^{t}\left(\ln\frac{s}{a}\right)^{(n-1)q} R^{\mu q}m^q(s)z^q(s)ds\right)^{1/q}. \qquad (38)$$

Applying inequality (19) to (38), we see that

$$z^q(t) \le 2^{q-1}(1+N)^q + 2^{q-1}\left(\frac{d_2B^{1/p}}{\Gamma(r)}\right)^q \int_{t_1}^{t}\left(\ln\frac{s}{a}\right)^{(n-1)q} R^{\mu q}(s)m^q(s)z^q(s)ds.$$

With $P = 2^{q-1}(1+N)^q$, $Q = 2^{q-1}\left(\frac{d_2B^{1/p}}{\Gamma(r)}\right)^q$, and $u(t) = z^q(t)$, i.e., $z(t) = u^{1/q}(t)$, the above-mentioned inequality becomes

$$u(t) \le P + Q\left(\int_{t_1}^{t}\left(\ln\frac{s}{a}\right)^{(n-1)q} R^{\mu q}(s)m^q(s)u(s)ds\right).$$

According to Gronwall's inequality and (25), we see that $u(t)$ is bounded. Thus,

$$\limsup_{t\to\infty}\left(\ln\frac{t}{a}\right)^{1-n}\left(\frac{x(t)}{R(t)}\right)^{\mu} < \infty.$$

The proof in case $x(t)$ is eventually negative is similar.

Similar to the sublinear case, we can obtain the following result in which we do not need condition (24).

Theorem 4. *Let $\lambda = \mu = 1$ and the conditions of Theorem 3 except for (24) hold with $m(t) = h(t)$. Then the conclusion of Theorem 3 holds.*

Proof. The proof is similar to that of Theorem 2. Thus, we omit the details.

Finally, we give an example illustrating Theorem 1.

Example 1. Consider the equation

$${}^{C,H}\mathscr{D}_e^{1/2}y(t) = \frac{1}{t}\left(\ln\frac{t}{e}\right)^{\gamma-1} + \left(\ln\frac{t}{e}\right)^{\gamma-1} h(t)\,|x(t)|^{\lambda-1}x(t), \qquad (39)$$

with $0 < \lambda < 1$. Here we have $a = e$, $f(t,x(t)) = \left(\ln\frac{t}{e}\right)^{\gamma-1} h(t)\,|x(t)|^{\lambda-1}x(t)$, $e(t) = \frac{1}{t}\left(\ln\frac{t}{e}\right)^{\gamma-1}$, and $r = 1/2$. Since $r = n+\alpha-1$, $\alpha \in (0,1)$ and $n \in \mathbb{Z}^+$, we have $\alpha = 1/2$ and $n = 1$. Since $n = 1$, equation (39) with the initial value

$$\delta^N y(a) = \delta^0 y(e) = y(e) = b_0$$

is equivalent to the equation (see (2))

$$y(t) = b_0 + \frac{1}{\Gamma(r)} \int_e^t \left(\ln \frac{t}{s}\right)^{r-1} \left[\frac{1}{s}\left(\ln \frac{s}{e}\right)^{\gamma-1} + \left(\ln \frac{s}{e}\right)^{\gamma-1} h(s)\,|x(s)|^{\lambda-1} x(s)\right] \frac{ds}{s}. \tag{40}$$

Letting $p = 3/2 > 1$, we see that $q = p/(p-1) = 3$, $\gamma = n - r + 1/q = 5/6$, $p(r-1)+1 = 1/4 > 0$, $p(\gamma-1)+1 = 3/4 > 0$, and $\theta = p(r+\gamma-2)+1 = 0$.

With $h(t) = m(t) = 1/t^3$, we have

$$\begin{aligned}
\int_a^t \left(\ln \frac{t}{s}\right)^{r-1} \left(\ln \frac{s}{a}\right)^{\gamma-1} m^{\lambda/(\lambda-1)}(s) h^{1/(1-\lambda)}(s) \frac{ds}{s} &= \int_e^t \left(\ln \frac{t}{s}\right)^{r-1} \left(\ln \frac{s}{e}\right)^{\gamma-1} h(s) \frac{ds}{s} \\
&\le \left(\int_e^t \left(\ln \frac{t}{s}\right)^{p(r-1)} \left(\ln \frac{s}{e}\right)^{p(\gamma-1)} \frac{ds}{s}\right)^{1/p} \left(\int_e^t h^q(s) ds\right)^{1/q} \\
= \left(B\left(\ln \frac{t}{e}\right)^{\theta}\right)^{2/3} \left(\int_e^t \frac{1}{s^9} ds\right)^{1/3} &= B^{2/3} \left(\int_e^t \frac{1}{s^9} ds\right)^{1/3} < \infty \text{ as } t \to \infty.
\end{aligned}$$

So condition (6) holds.

Since

$$\begin{aligned}
\left(\ln \frac{t}{a}\right)^{1-n} \int_a^t \left(\ln \frac{t}{s}\right)^{r-1} |e(s)| \frac{ds}{s} &= \int_e^t \left(\ln \frac{t}{s}\right)^{r-1} \left(\ln \frac{s}{e}\right)^{\gamma-1} \frac{1}{s}\frac{ds}{s} \\
\le \left(\int_e^t \left(\ln \frac{t}{s}\right)^{p(r-1)} \left(\ln \frac{s}{e}\right)^{p(\gamma-1)} \frac{ds}{s}\right)^{1/p} &\left(\int_e^t \frac{1}{s^q} ds\right)^{1/q} \\
= \left(B\left(\ln \frac{t}{e}\right)^{\theta}\right)^{2/3} &\left(\int_e^t \frac{1}{s^3} ds\right)^{1/3} \\
= B^{2/3} \left(\int_e^t \frac{1}{s^3} ds\right)^{1/3} &< \infty \text{ as } t \to \infty,
\end{aligned}$$

(8) is satisfied.

Finally, letting $y(t) = x^{(k)}(t) = x''(t)$, we have $k = 2$. Thus, and so condition (7) becomes

$$\int_a^\infty \left(\ln \frac{s}{a}\right)^{(n-1)q} s^{kq} m^q(s) ds = \int_e^\infty \frac{s^6}{s^9} ds = \int_e^\infty \frac{1}{s^3} ds < \infty.$$

That is, condition (7) holds. All conditions of Theorem 1 are satisfied, so every nonoscillatory solution x of equation (39) satisfies

$$\frac{|x(t)|}{t^2} = O(1) \quad \text{as} \quad t \to \infty.$$

To conclude, the results are presented in a high degree of generality. Accordingly, this approach can be used to obtain analogous results for different choices of the form of the function $y(t)$. It would also be of interest to study equation (1) for f satisfying condition (iii) with $\lambda > 1$.

Acknowledgments

J. R. Graef's research was partially supported by the University of Tennessee at Chattanooga SimCenter – Center of Excellence in Applied Computational Science and Engineering (CEACSE) grant.

References

[1] R. P. Agarwal, Y. Zhou and Y. He, Existence of fractional neutral functional differential equations, *Comput. Math. Appl.* **59**, 1095–1100 (2010).

[2] B. Ahmad and J. J. Nieto, Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions, *Bound. Val. Probl.* **2011**, Article ID 36, 9 pages (2011).

[3] B. Ahmad and J. J. Nieto, Boundary value problems for a class of sequential integro differential equations of fractional order, *J. Func. Space Appl.* **2013**, Article ID 149659, 8 pages (2013).

[4] B. Ahmad, S. K. Ntouyas and A. Alsaedi, New existence results for nonlinear fractional differential equations with three-point integral boundary conditions, *Adv. Differ. Equ.* **2011**, Article ID 107384, 11 pages (2011).

[5] B. Ahmad, S. K. Ntouyas and A. Alsaedi, A study of nonlinear fractional differential equations of arbitrary order with Riemann-Liouville type multistrip boundary conditions, *Math. Probl. Eng.* **2013**, Article ID 320415, 9 pages (2013).

[6] D. Băleanu, K. Diethelm, E. Scalas and J. J. Trujillo, *Fractional calculus models and numerical methods*, World Scientific, Singapore, 2012.

[7] D. Băleanu, O. G. Mustafa, and R. P. Agarwal, On L^p -solutions for a class of sequential fractional differential equations, *Appl. Math. Comput.* **218**, 2074–2081 (2011).

[8] A. A. Kilbas, H. M. Srivastava and J. T. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Vol. 204, Elsevier, Amsterdam, 2006.

[9] X. Liu, M. Jiaa and W. Ge, Multiple solutions of a p-Laplacian model involving a fractional derivative, *Adv. Differ. Equ.* **2013**, Article ID 126, 12 pages (2013).

[10] D. O'Regan and S. Staněk, Fractional boundary value problems with singularities in space variables, *Nonlinear Dyn.*, **71**, 641–652 (2013).

[11] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, 1999.

[12] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives; theory and applications*, Gordon and Breach Science Publishers, Yverdon, Switzerland, 1993.

[13] L. Zhang, B. Ahmad, G. Wang and R. P. Agarwal, Nonlinear fractional integro-differential equations on unbounded domains in a Banach space, *J. Comput. Appl. Math.* **249**, 51–56 (2013).

[14] J. Hadamard, Essai sur l'étude des fonctions données par leur dévelopment de Taylor, *J. Math. Pure. Appl.* **8**, 101–186 (1892).

[15] P. L. Butzer, A. A. Kilbas and J. J. Trujillo, Compositions of Hadamard-type fractional integration operators and the semigroup property, *J. Math. Anal. Appl.* **269**, 387–400 (2002).

[16] P. L. Butzer, A. A. Kilbas and J. J. Trujillo, Fractional calculus in the Mellin setting and Hadamard-type fractional integrals, *J. Math. Anal. Appl.* **269**, 1–27 (2002).

[17] P. L. Butzer, A. A. Kilbas and J. J. Trujillo, Mellin transform analysis and integration by parts for Hadamard-type fractional integrals, *J. Math. Anal. Appl.* **270**, 1–15 (2002).

[18] W. T. Coffey, Y. P. Kalmykov and J. T. Waldron, *The Langevin equation*, 2nd Edition, World Scientific, Singapore, 2004.

[19] A. A. Kilbas, Hadamard-type fractional calculus, *J. Korean Math. Soc.* **38**, 1191–1204 (2001).

[20] A. A. Kilbas and J. J. Trujillo, Hadamard-type integrals as G-transforms, *Int. Transf. Spec. Funct.* **14**, 413–427 (2003).

[21] J. R. Graef, S. R. Grace and E. Tunç, Asymptotic behavior of solutions of nonlinear fractional differential equations with Caputo-type Hadamard derivatives, *Fract. Calc. Appl. Anal.* **20**, 71–87 (2017).

[22] G. H. Hardy, I. E. Littlewood and G. Polya, *Inequalities*, Reprint of the 1952 edition, Cambridge University Press, Cambridge, 1988.

[23] F. Jarad, T. Abdeljawad and D. Băleanu, Caputo-type modification of the Hadamard fractional derivatives, *Adv. Differ. Equ.* **2012**, Article ID 142, 8 pages (2012).