Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/100211

Estimation for the Generalized Linear Exponential Distribution Based on Type-II Hybrid Censored Data

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Received: 14 Dec. 2019, Revised: 7 Apr. 2020, Accepted: 10 Apr. 2020
Published online: 1 Jul. 2021

Abstract: A hybrid censoring scheme is a mixture of Type-I and Type-II censoring schemes, is quite common in life-testing. In this paper,we study the problem of point and interval estimations for the generalized linear exponential distribution (GLED) using type-II hybrid censored sample.The maximum likelihood (ML) and Bayes methods are utilized for estimating the unknown parameters as well as some lifetime parameters (reliability, hazard function and reversed hazard function). Also, we apply Markov chain Monte Carlo (MCMC) technique and Lindely's approximation technique to carry out a Bayesian estimation. Bayes estimates and the credible intervals are obtained under the assumptions of informative and non informative priors. Different methods have been compared using Monte Carlo simulations. Real data set has been studied for illustrative purpose.

Keywords: Hybrid censoring scheme,Generalized Linear Exponential Distribution (GLE) Bayes Estimators, Maximum Likelihood (ML),Lindely approximation, Markov Chain Monte Carlo (MCMC)

1 Introduction

Experiments often terminate before all units on test have failed due to cost and time considerations. In such experiments failure information is available only on a part of the sample, the data are said to be censored data. The two most regular censoring schemes are Type-I and Type-II censoring schemes. A mixture of Type-I and Type-II censoring schemes, known as hybrid censoring scheme. It has been discussed to overcome the disadvantages of those two types separately. This scheme was first introduced by [1] and [2], and it has been discussed extensively in the reliability literature. In the Type-I hybrid censoring scheme, the experiment is terminated as soon as a pr-specified number r out of n items has failed or a pr-fixed time x_0 on test has been reached. In contrast, in the Type-II hybrid censoring scheme, the life-testing experiment gets terminated whenever the later of the two stopping rules is reached. Hybrid censored lifetime data have been discussed by several authors, including [3], [4],[5], [6], [7],[8], [9], [10], [11], [12], [13] and [14].[15] discussed E-Bayesian method, maximum likelihood and the Bayesian estimation methods of the shape parameter, and the reliability function from one parameter Burr-X distribution based on generalized type-II hybrid censored data.[16] studied maximum likelihood, Bayes and percentile bootstrap method for unknown parameter, failure rate function, the survival function and the coefficient of variation of the exponential Rayleigh distribution with generalized type-II hybrid censored scheme. One of the disadvantages of Type-I hybrid censoring scheme is that there may be very few failures occurring up to the pr-fixed time T . Because of this, [17] proposed a new hybrid censoring scheme which can be described as follows. Put n identical units on test, and then stop the experiment at the random time $T^* = MaxT, m$ where T and m are a prefixed numbers and $x_{m:n}$ point to the time of m-th failure in a sample of size n. Under the Type-II hybrid censoring scheme, we have one of the following three kinds of observations:

Case I: $\{x_{1:n} < ... < x_{m:n}\}$ if $x_{m:n} > T$.

Case II: $\{x_{1:n} < ... < x_{d:n} < T < x_{d+1:n}\}$ if $m \leq d < n$ and $x_{d:n} < T < x_{d+1:n}$.

Case III: $\{x_{1:n} < ... < x_{n:n} < T\}$ where $x_{1:n} < ... < x_{m:n}$ indicate the observed ordered failure times of the experimental units.

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The analysis of Type-II hybrid censored scheme from generalized linear exponential distribution (GLE) is considered in this paper.First, we consider the ML of the unknown parameters. Second,Bayes estimator for the unknown parameters are considered. Lindley's approximation and MCMC method are considered to approximate the Bayes estimator. A simulation study to compare the performance of ML and Bayes estimators are executed. Real data are used for illustrative purpose. The paper is organized as follows. In the first section the probability density function of GLE and its important. The ML estimators of the unknown parameters, reliability and hazard functions are presented in section 3. The corresponding approximate confidence interval for the parameters are given. Section 4 deal with description of the priors, posteriors, Gibbs sampling and Metropolis-Hasting (MH) algorithm. In the same section the proposed hybrid algorithm with resulting Bayes estimators are discussed. Numerical example and real data sets have been analyzed in Section 5 and 6. Finally, conclusion in section 7.

2 Generalized Linear Exponential Distribution

This distribution was originally proposed by [18]. The linear exponential distribution, having exponential and Rayleigh distributions as special cases, is a well-known distribution of reliability and medical studies lifetime data modeling. It is also models phenomena with increasing failure rate. The linear exponential distribution does not provide a reasonable fit for modeling phenomena with non-monotonic distributions such as bathtub shaped ones, which are common in reliability and biological studies; see for example [19]. Models that present bathtub shaped failure rates are very useful in survival analysis. Many authors have proposed new distributions that generalize exponential, Rayleigh and linear exponential distributions. [20] proposed a generalization of the exponential distribution. The Burr type X distribution, also known as the generalized Rayleigh distribution (GRD), was proposed by [21]. [22] introduced the generalized linear failure rate distribution, while [23] the estimation of the parameters of the generalized linear failure rate distribution. The three parameter generalized linear exponential distribution has a probability density function (pdf)

$$f(x;\alpha,\theta,\lambda)=\alpha\left(\lambda x+\frac{\theta}{2}x^{2}\right)^{\alpha-1}(\lambda+\theta x)e^{-\left(\lambda x+\frac{\theta}{2}x^{2}\right)^{\alpha}},\alpha,\theta>0 \text{ and } \lambda\geq 0, \tag{1}$$

cumulative distribution function (cdf)

$$F(x,\alpha,\theta,\lambda)=1-e^{-\left(\lambda x+\frac{\theta}{2}x^{2}\right)^{\alpha}},\alpha,\theta>0 \text{ and } \lambda\geq 0, \tag{2}$$

the survival function R(t)

$$R(t)=e^{-\left(\lambda t+\frac{\theta}{2}t^{2}\right)^{\alpha}},t>0, \tag{3}$$

the hazard function H(t)

$$H(t)=\alpha(\lambda+\theta t)\left(\lambda t+\frac{\theta}{2}t^{2}\right)^{\alpha-1},t>0, \tag{4}$$

and the revised hazard function Q(t)

$$Q(t)=\frac{\alpha(\lambda+\theta t)\left(\lambda t+\frac{\theta}{2}t^{2}\right)^{\alpha-1}e^{-\left(\lambda t+\frac{\theta}{2}t^{2}\right)^{\alpha}}}{1-e^{-\left(\lambda t+\frac{\theta}{2}t^{2}\right)^{\alpha}}},t>0, \tag{5}$$

3 Maximum Likelihood Estimation

maximum likelihood estimation (ML) is often the most feasible method to use when doing statistical inference. In this section, we study MLEs of the GLE distribution based on Type-II hybrid censored scheme with pdf given in (1). Also, we construct Approximate confidence intervals (ACIs) of the parameters of GLE distribution based on Type-II hybrid censored scheme. The likelihood function in case I is given

$$L(\alpha,\theta,\lambda)=\frac{n!}{(n-R)!}\prod_{i=1}^{R}f(x_i)\left[1-F(x_R)\right]^{n-R}, \tag{6}$$

for case II,

$$L(\alpha,\theta,\lambda)=\frac{n!}{(n-d)!}\prod_{i=1}^{d}f(x_i)\left[1-F(T)\right]^{n-d}, \tag{7}$$

and for case III,

$$L(\alpha,\theta,\lambda)=\prod_{\imath=1}^{n} f(x_i), \tag{8}$$

here f(x) is presented in (1). We present likelihood function (6) , (7) and (8) by:

$$L(\alpha,\theta,\lambda)=\frac{n!}{(n-r)!}\prod_{\imath=1}^{r} f(x_i)\left[1-F(c)\right]^{n-r}, \tag{9}$$

where

$$r=\begin{cases} R & \text{for case I} \\ d & \text{for case II} \\ n & \text{for case III} \end{cases}, \tag{10}$$

and

$$c=\begin{cases} x_{R:n} & \text{for case I} \\ T & \text{for case II and III} \end{cases}. \tag{11}$$

By substituting (1) and (2) in Equation (9) and taking the logarithm , the log likelihood function can be written as

$$\begin{aligned} log\, L(\alpha,\theta,\lambda) = const. + r\log\alpha + (\alpha-1)\sum_{i=1}^{r}\log\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)+\sum_{i=1}^{r}\log\left(\lambda+\theta x_i\right) \\ -\sum_{i=1}^{r}\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}-r(n-r)\left(\lambda x_r+\frac{\theta}{2}x_r^2\right)^{\alpha}. \end{aligned} \tag{12}$$

Applying the first derivative with respect to α, θ and λ and equating by zero, then we get the normal equations as follows:

$$\begin{aligned} \frac{r}{\hat{\alpha}}+\sum_{i=1}^{r}\ln\left(\hat{\lambda}x_i+\frac{\hat{\theta}}{2}x_i^2\right)-\sum_{i=1}^{r}\left(\hat{\lambda}x_i+\frac{\hat{\theta}}{2}x_i^2\right)^{\hat{\alpha}}\log\left(\hat{\lambda}x_i+\frac{\hat{\theta}}{2}x_i^2\right) \\ -r(n-r)\left(\hat{\lambda}c+\frac{\hat{\theta}}{2}c^2\right)^{\hat{\alpha}}\log\left(\hat{\lambda}c+\frac{\hat{\theta}}{2}c^2\right)=0, \end{aligned} \tag{13}$$

$$\begin{aligned} (\hat{\alpha}-1)\sum_{i=1}^{r}\frac{x_i^2}{2\left(\hat{\lambda}x_i+\frac{\hat{\theta}}{2}x_i^2\right)}+\sum_{i=1}^{r}\frac{x_i}{\left(\hat{\lambda}+\hat{\theta}x_i\right)}-\hat{\alpha}\sum_{i=1}^{r}\left(\hat{\lambda}x_i+\frac{\hat{\theta}}{2}x_i^2\right)^{\hat{\alpha}-1}\frac{x_i^2}{2} \\ -\hat{\alpha}r(n-r)\left(\hat{\lambda}c+\frac{\hat{\theta}}{2}c^2\right)^{\hat{\alpha}-1}\frac{c^2}{2}=0, \end{aligned} \tag{14}$$

and

$$\begin{aligned} (\hat{\alpha}-1)\sum_{i=1}^{r}\frac{x_i}{\left(\hat{\lambda}x_i+\frac{\hat{\theta}}{2}x_i^2\right)}+\sum_{i=1}^{r}\frac{1}{\left(\hat{\lambda}+\hat{\theta}x_i\right)}-\hat{\alpha}\sum_{i=1}^{r}\left(\hat{\lambda}x_i+\frac{\hat{\theta}}{2}x_i^2\right)^{\hat{\alpha}-1}x_i \\ -\hat{\alpha}r(n-r)\left(\hat{\lambda}c+\frac{\hat{\theta}}{2}c^2\right)^{\hat{\alpha}-1}c=0. \end{aligned} \tag{15}$$

Since, the Equations (13), (14) and (15) are nonlinear equations in three parameter $\hat{\alpha}$, $\hat{\theta}$ and $\hat{\lambda}$.The exact solution is not easy to compute.Therefore a some numerical methods must be employed.

By invariant property of ML estimators, the ML estimators of reliability function R(t), hazard rate function H(t) and reversed hazard rate function Q(t) can be obtained by substituting the MLE's of the parameters α, θ and λ in (3), (4) and (5), respectively. Hence,

$$\hat{R}(t,\hat{\alpha},\hat{\theta},\hat{\lambda})=e^{-\left(\hat{\lambda}t+\frac{\hat{\theta}}{2}t^2\right)^{\hat{\alpha}}}, t>0, \tag{16}$$

$$\hat{H}(t,\hat{\alpha},\hat{\theta},\hat{\lambda}) = \hat{\alpha}(\hat{\lambda}+\hat{\theta}t)\left(\hat{\lambda}t+\frac{\hat{\theta}}{2}t^2\right)^{\hat{\alpha}-1}, t>0, \tag{17}$$

and

$$\hat{Q}(t,\hat{\alpha},\hat{\theta},\hat{\lambda}) = \frac{\hat{\alpha}(\hat{\lambda}+\hat{\theta}t)\left(\hat{\lambda}t+\frac{\hat{\theta}}{2}t^2\right)^{\hat{\alpha}-1} e^{-\left(\hat{\lambda}t+\frac{\hat{\theta}}{2}t^2\right)^{\hat{\alpha}}}}{1-e^{-\left(\hat{\lambda}t+\frac{\hat{\theta}}{2}t^2\right)^{\hat{\alpha}}}}, t>0, \tag{18}$$

3.1 Approximate confidence interval

The asymptotic variance-covariance matrix of the estimators of the parameters $\varphi = (\varphi_1,\dots\varphi_n)$ is obtained by inverting the Fisher information matrix (given by taking the expectation of the second derivative of the log- likelihood functions) in which elements are negatives. In the present situation, ie seems appropriate to approximate the expected values by their maximum likelihood (ML) estimates.Accordingly, the approximate variance-covariance matrix is given by [24]

$$\begin{pmatrix} -\frac{\partial^2 l}{\partial^2\varphi_1^2} & \cdots & -\frac{\partial^2 l}{\partial^2\varphi_1\varphi_n} \\ \vdots & \vdots & \vdots \\ -\frac{\partial^2 l}{\partial^2\varphi_n} & \cdots & -\frac{\partial^2 l}{\partial^2\varphi_n^2} \end{pmatrix}^{-1}_{(\hat{\varphi_1},\dots,\hat{\varphi_n})}$$

From the log-likelihood equation (12), we get

$$\frac{\partial^2 l}{\partial\alpha^2} = -\frac{r}{\alpha^2} - \sum_{i=1}^{r}\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}\left(\log\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)\right)^2 - r(n-r)\left(\lambda c+\frac{\theta}{2}c^2\right)^{\alpha}\left(\log\left(\lambda c+\frac{\theta}{2}c^2\right)\right)^2, \tag{19}$$

$$\frac{\partial^2 l}{\partial\theta^2} = -(\alpha-1)\sum_{i=1}^{r}\frac{1}{\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^2}\left(\frac{x_i^4}{4}\right) - \sum_{i=1}^{r}\frac{x_i^2}{\left(\lambda+\theta x_i^2\right)^2} - \alpha(\alpha-1)\sum_{i=1}^{r}\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha-2}\left(\frac{x_i^4}{4}\right)$$
$$-\alpha r(\alpha-1)(n-r)\left(\lambda c+\frac{\theta}{2}c^2\right)^{\alpha-2}\left(\frac{c^4}{4}\right), \tag{20}$$

$$\frac{\partial^2 l}{\partial\lambda^2} = -(\alpha-1)\sum_{i=1}^{r}\frac{x_i^2}{\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^2} - \sum_{i=1}^{r}\frac{1}{(\lambda+\theta x_i)} - \alpha(\alpha-1)\sum_{i=1}^{r}\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha-2}(x_i^2) - \alpha r(\alpha-1)(n-r)\left(\lambda c+\frac{\theta}{2}c^2\right)^{\alpha-2}c^2, \tag{21}$$

$$\frac{\partial^2 l}{\partial\alpha\theta} = \sum_{i=1}^{r}\frac{1}{\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)}\left(\frac{x_i^2}{2}\right) - \alpha\sum_{i=1}^{r}\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha-1}\left(\frac{x_i^2}{2}\right)\left(\log\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)\right) - \sum_{i=1}^{r}\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha-1}\left(\frac{x_i^2}{2}\right)$$
$$-\alpha r(n-r)\left(\lambda c+\frac{\theta}{2}c^2\right)^{\alpha-1}\left(\log\left(\lambda c+\frac{\theta}{2}c^2\right)\right)\left(\frac{c^2}{2}\right) - r(n-r)\left(\lambda c+\frac{\theta}{2}c^2\right)^{\alpha-1}\left(\frac{c^2}{2}\right), \tag{22}$$

$$\frac{\partial^2 l}{\partial\alpha\lambda} = \sum_{i=1}^{r}\frac{x_i}{\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)} - \alpha\sum_{i=1}^{r}\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha-1}x_i\left(\log\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)\right) - \sum_{i=1}^{r}\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha-1}x_i$$
$$-\alpha r(n-r)\left(\lambda c+\frac{\theta}{2}c^2\right)^{\alpha-1}\left(\log\left(\lambda c+\frac{\theta}{2}c^2\right)\right)c - r(n-r)\left(\lambda c+\frac{\theta}{2}c^2\right)^{\alpha-1}c, \tag{23}$$

$$\frac{\partial^2 l}{\partial\theta\lambda} = (\alpha-1)\sum_{i=1}^{r}\frac{1}{\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)}\left(\frac{x_i^3}{2}\right) - \sum_{i=1}^{r}\frac{x_i}{(\lambda+\theta x_i)^2} - \alpha(\alpha-1)\sum_{i=1}^{r}\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha-2}\left(\frac{x_i^3}{2}\right)$$
$$-\alpha r(n-r)(\alpha-1)\left(\lambda c+\frac{\theta}{2}c^2\right)^{\alpha-2}\left(\frac{c^3}{2}\right). \tag{24}$$

Then, the asymptotic variance-covariance matrix of the estimators of the parameters α, θ and λ is obtained by inverting the Fisher information matrix given by taking the expectation of Equations (19), (20), (21), (22), (23) and (24) in which elements are negatives. In the present situation, it seems appropriate to approximate the expected values by their ML estimates. Accordingly, the approximate variance -covariance matrix is given as

$$\begin{pmatrix} \hat{\sigma_{\alpha\alpha}} & \hat{\sigma_{\alpha\theta}} & \hat{\sigma_{\alpha\lambda}} \\ \hat{\sigma_{\alpha\theta}} & \hat{\sigma_{\theta\theta}} & \hat{\sigma_{\theta\lambda}} \\ \hat{\sigma_{\alpha\lambda}} & \hat{\sigma_{\theta\lambda}} & \hat{\sigma_{\lambda\lambda}} \end{pmatrix} = \begin{pmatrix} -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2\alpha^2} & -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2\alpha\theta} & -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2\alpha\lambda} \\ -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2\alpha\theta} & -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2\theta^2} & -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2\theta\lambda} \\ -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2\alpha\lambda} & -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2\theta\lambda} & -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2\lambda^2} \end{pmatrix}^{-1}_{(\hat{\alpha},\hat{\theta},\hat{\lambda})}. \tag{25}$$

The ACIs for the parameters α, θ and λ are respectively given as:

$$\hat{\alpha} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{\sigma}_{\alpha\alpha}}, \quad \hat{\theta} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{\sigma}_{\theta\theta}} \quad \text{and} \quad \hat{\lambda} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{\sigma}_{\lambda\lambda}},$$

where $Z_{\frac{\alpha}{2}}$ is the percentile of the standard normal distribution with right tail probability $\frac{\alpha}{2}$.

4 Bayesian Estimation of the Parameters

In Bayesian estimation, we study two types of loss functions. The first is the squared error loss function (quadratic loss) which is classified as a symmetric function and associates equal importance to the losses for over estimation and underestimation of equal magnitude. However, in most practical circumstances such a limitation may be impractical. For instance, in the estimation of reliability and failure rate function , an overestimation is usually much more serious than an underestimation. In this situation, the use of symmetrical loss function might be inappropriate as also highlighted by [25]. The second is LINEX (linear-exponential) loss function which is asymmetric, was introduced by [26]. These loss functions was widely used by several authors; among of them [27], [28], [29], [30], [31]. [32] and [33]. This function rises roughly exponentially on one side of zero and roughly linearly on the other side. It can also be noted here that for a specific choice of the loss function parameter, the squared error loss function can be obtained as a specific member of the LINEX loss function. The squared error loss function and the LINEX loss function for a parameter δ are as follows, respectively:

$$L_{BS}\left(\delta,\hat{\delta}\right) = \left(\delta-\hat{\delta}\right)^2, \tag{26}$$

and

$$L_{BL}(\Delta) = a\left(e^{h\Delta} - h\Delta - 1\right) \qquad , \ a>0 \ , \ h\neq 0, \tag{27}$$

where a and h are shape and scale parameters of the loss function, respectively and $\Delta = \left(\delta-\hat{\delta}\right)$ denotes the scalar estimation error in using $\hat{\delta}$ to estimate δ. Generally, the sign and magnitude of h in LINEX loss function affect on the direction and degree of asymmetry. Further properties of this loss function have been investigated by [34]. For small values of h (near to zero), the LINEX loss function is almost the same as the squared error loss function and for the choice of negative or positive values of h , the LINEX loss function gives more weight to overestimation or underestimation.

Bayesian estimates of δ against the squared error loss function and the LINEX loss function are as follows, respectively:

$$\hat{\delta}_{BS} = E\left[\delta \mid x\right], \tag{28}$$

and

$$\hat{\delta}_{BL} = -\frac{1}{h}\log\{E\left[e^{-h\delta} \mid x\right]\}. \tag{29}$$

Now we will propose the Bayesian estimator of parameters $(\alpha, \theta, \lambda)$ as well as reliability R(t), hazard H(t) and reversed hazard Q(t) functions of the GLE. It is assumed that α, θ and λ have the following independent prior:

$$\pi_1(\alpha) \propto \alpha^{w_2-1}e^{-\alpha w_1} \quad , w_1>0, w_2>0 \quad \alpha>0,$$

$$\pi_2(\theta) \propto \theta^{w_4-1} e^{-\theta w_3} \quad , w_3 > 0, w_4 > 0, \ \theta > 0, \tag{30}$$

and

$$\pi_3(\lambda) \propto \lambda^{w_6-1} e^{-\lambda w_5} \quad , w_6 > 0, w_5 > 0, \ \lambda \geq 0,$$

where $w_1, w_2, w_3, w_4, w_5, w_6$ are chosen to reflect prior knowledge about α, θ and λ. When $w_1 = w_2 = w_3 = w_4 = w_5 = w_6 = 0$, there are non-informative priors of α, θ and λ.

From Equations (12) and (30) , the joint prior density function of α, θ and λ is of the form

$$\pi(\alpha,\theta,\lambda) \propto \alpha^{w_2-1} e^{-\alpha w_1} \ \theta^{w_4-1} e^{-\theta w_3} \ \lambda^{w_6-1} e^{-\lambda w_5} \ , \ \alpha, \theta > 0, \ \lambda \geq 0,$$

$$w_1, w_2, w_3, w_4, w_5, w_6 > 0.$$

Then the posterior distribution α, θ and λ can be written as

$$\pi^*(\alpha,\theta,\lambda \mid x) = \frac{1}{k} \alpha^{w_2+r-1} e^{-\alpha w_1} \ \theta^{w_4-1} e^{-\theta w_3} \ \lambda^{w_6-1} e^{-\lambda w_5} \ \times \tag{31}$$

$$\prod_{\imath=1}^{r} \left(\lambda x_i + \frac{\theta}{2} x_i^2\right)^{\alpha-1} (\lambda + \theta x_i) \, e^{-\left(\lambda x_i + \frac{\theta}{2} x_i^2\right)^{\alpha}} e^{-(n-r)\left(\lambda c + \frac{\theta}{2} c^2\right)^{\alpha}},$$

where

$$k = \int_0^\infty \int_0^\infty \int_0^\infty \Big[\alpha^{w_2+r-1} e^{-\alpha w_1} \ \theta^{w_4-1} e^{-\theta w_3} \ \lambda^{w_6-1} e^{-\lambda w_5} \times$$

$$\prod_{\imath=1}^{r} \left(\lambda x_i + \frac{\theta}{2} x_i^2\right)^{\alpha-1} (\lambda + \theta x_i) \, e^{-\left(\lambda x_i + \frac{\theta}{2} x_i^2\right)^{\alpha}} \ e^{-(n-r)\left(\lambda c + \frac{\theta}{2} c^2\right)^{\alpha}}\Big] \, d\alpha \, d\theta \, d\lambda.$$

In Bayesian statistics the posterior distribution $\pi^*(\alpha,\theta,\lambda \mid x)$ contains all information on the unknown parameters given the observed data. All statistical inference can be deduced from the posterior distribution. We observe that equation (31) can not solved explicitly, So two different procedure are introduced Lindley's approximation and MCMC technique can be used to obtain the Bayes estimator for α, θ and λ and the corresponding credible intervals.

For any function $u(\alpha,\theta,\lambda)$ of α, θ and λ, the Bayes estimates is given by

$$\hat{u}(\alpha,\theta,\lambda) = \frac{1}{k} \int_0^\infty \int_0^\infty \int_0^\infty u(\alpha,\theta,\lambda) L(\alpha,\theta,\lambda) \pi(\alpha,\theta,\lambda) \, d\alpha \, d\theta \, d\lambda. \tag{32}$$

4.1 Lindely approximation

Lindley's approximation, which introduced by [35] can approximate the Bayes estimators into a form containing no integral. . For our estimation problem we describe this method below. As noticed the Bayesian estimates involve the ratio of two integrals, we consider $I(x)$ defined as

$$I(x) = E[u(\gamma_1,\gamma_2,\gamma_3)] = \frac{\int\limits_{\gamma_3}\int\limits_{\gamma_2}\int\limits_{\gamma_1} u(\gamma_1,\gamma_2,\gamma_3) \, e^{L(\gamma_1,\gamma_2,\gamma_3)+\rho(\gamma_1,\gamma_2,\gamma_3)} d\gamma_1 d\gamma_2 d\gamma_3}{\int\limits_{\gamma_3}\int\limits_{\gamma_2}\int\limits_{\gamma_1} e^{L(\gamma_1,\gamma_2,\gamma_3)+\rho(\gamma_1,\gamma_2,\gamma_3)} d\gamma_1 d\gamma_2 d\gamma_3},$$

where $u(\gamma_1,\gamma_2,\gamma_3)$ is a function of γ_1, γ_2 or γ_3 only.

$L(\gamma_1,\gamma_2,\gamma_3)$ is log of likelihood function.

$\rho(\gamma_1,\gamma_2,\gamma_3)$ is log joint prior of γ_1, γ_2 and γ_3.

Utilizing the Lindley's method $I(x)$ can be approximated as

$$I(x) = u(\hat{\gamma}_1,\hat{\gamma}_2,\hat{\gamma}_3) + (u_1 a_1 + u_2 a_2 + u_3 a_3 + a_4 + a_5) +$$

$$\frac{1}{2}\left[A\left(u_1\sigma_{11}+u_2\sigma_{12}+u_3\sigma_{13}\right)+B\left(u_1\sigma_{21}+u_2\sigma_{22}+u_3\sigma_{23}\right)\right.$$
$$\left.+C\left(u_1\sigma_{31}+u_2\sigma_{32}+u_3\sigma_{33}\right)\right], \tag{33}$$

where
$\hat{\gamma}_1, \hat{\gamma}_2$ and $\hat{\gamma}_3$ are the MLE of of γ_1, γ_2 and γ_3 respectively.

$$a_i = \rho_1\sigma_{i1}+\rho_2\sigma_{i2}+\rho_3\sigma_{i3},\ i=1,2,3,$$

$$a_4 = u_{12}\sigma_{12}+u_{13}\sigma_{13}+u_{23}\sigma_{23},$$

$$a_5 = \frac{1}{2}\left(u_{11}\sigma_{11}+u_{22}\sigma_{22}+u_{33}\sigma_{33}\right),$$

$$A = \sigma_{11}L_{111}+2\sigma_{12}L_{121}+2\sigma_{13}L_{131}+2\sigma_{23}L_{231}+\sigma_{22}L_{221}+\sigma_{33}L_{331},$$

$$B = \sigma_{11}L_{112}+2\sigma_{12}L_{122}+2\sigma_{13}L_{132}+2\sigma_{23}L_{232}+\sigma_{22}L_{222}+\sigma_{33}L_{332},$$

$$C = \sigma_{11}L_{113}+2\sigma_{12}L_{123}+2\sigma_{13}L_{133}+2\sigma_{23}L_{233}+\sigma_{22}L_{223}+\sigma_{33}L_{333},$$

and subscripts $1,2,3$ on the right-hand sides refer to $\gamma_1, \gamma_2, \gamma_3$ respectively and

$$\rho_i = \frac{\partial\rho}{\partial\gamma i},\quad u_i = \frac{\partial u(\gamma_1,\gamma_2,\gamma_3)}{\partial\gamma i},\ i=1,2,3,\ u_{ij} = \frac{\partial^2 u(\gamma_1,\gamma_2,\gamma_3)}{\partial\gamma_i\partial\gamma_j},\ i,j=1,2,3,$$

$$L_{ij} = \frac{\partial^2 L(\gamma_1,\gamma_2,\gamma_3)}{\partial\gamma i\partial\gamma_j},\ i,j=1,2,3\ ,\quad L_{ijk} = \frac{\partial^3 L(\gamma_1,\gamma_2,\gamma_3)}{\partial\gamma_i\partial\gamma_j\partial\gamma_k},\ i,j,k=1,2,3,$$

and σ_{ij} is the $(i,j)-th$ element of the inverse of the matrix $\{L_{ij}\}$, all evaluated at the MLE of parameters.With the above defined expressions, we can obtain the values of the Bayes estimates of various parameters .

If $u\left(\hat{\alpha},\hat{\theta},\hat{\lambda}\right)=\hat{\alpha}$ then the Bayes estimate of the parameter α under the squared error loss (SEL) function from (33) is

$$\hat{\alpha}_{BS} = \hat{\alpha}+a_1+\frac{1}{2}\left[A\sigma_{11}+B\sigma_{21}+C\sigma_{31}\right] \tag{34}$$

.

If $u\left(\hat{\alpha},\hat{\theta},\hat{\lambda}\right)=\hat{\theta}$ then the Bayes estimate of the parameter θ under the squared error loss (SEL) function is

$$\hat{\theta}_{BS} = \hat{\theta}+a_2+\frac{1}{2}\left[A\sigma_{12}+B\sigma_{22}+C\sigma_{32}\right]. \tag{35}$$

If $u\left(\hat{\alpha},\hat{\theta},\hat{\lambda}\right)=\hat{\lambda}$ then the Bayes estimate of the parameter λ under the squared error loss (SEL) function is

$$\hat{\lambda}_{BS} = \hat{\lambda}+a_3+\frac{1}{2}\left[A\sigma_{13}+B\sigma_{23}+C\sigma_{33}\right]. \tag{36}$$

If $u\left(\hat{\alpha},\hat{\theta},\hat{\lambda}\right)=e^{-h\hat{\alpha}}$ then the Bayes estimate of the parameter α under the LINEX loss function from (33) is

$$\hat{\alpha}_{BL} = -\frac{1}{h}\log\left\{e^{-h\hat{\alpha}}\left[1-ha_1+\frac{1}{2}h^2\sigma_{11}-\frac{1}{2}h\left(A\sigma_{11}+B\sigma_{21}+C\sigma_{31}\right)\right]\right\}. \tag{37}$$

If $u\left(\hat{\alpha},\hat{\theta},\hat{\lambda}\right)=e^{-h\hat{\theta}}$ then the Bayes estimate of the parameter θ under the LINEX loss function is

$$\hat{\theta}_{BL} = -\frac{1}{h}\log\left\{e^{-h\hat{\theta}}\left[1 - ha_2 + \frac{1}{2}h^2\sigma_{22} - \frac{1}{2}h\left(A\sigma_{12} + B\sigma_{22} + C\sigma_{32}\right)\right]\right\}. \tag{38}$$

If $u\left(\hat{\alpha},\hat{\theta},\hat{\lambda}\right) = e^{-h\hat{\lambda}}$ then the Bayes estimate of the parameter λ under the LINEX loss function is

$$\hat{\lambda}_{BL} = -\frac{1}{h}\log\left\{e^{-h\hat{\lambda}}\left[1 - ha_3 + \frac{1}{2}h^2\sigma_{33} - \frac{1}{2}h\left(A\sigma_{13} + B\sigma_{23} + C\sigma_{33}\right)\right]\right\}. \tag{39}$$

The estimators of reliability function R(t), hazard rate function H(t) and reversed hazard rate function Q(t) can be obtained by substituting the Lindely's approximate under the squared error loss (SEL) function and LINEX loss function of the parameters α, θ and λ in (3), (4) and (5).

The approximate Bayes estimator of α, θ and λ can be obtained using Lindley approximation , but it is not possible to construct highest posterior density(HPD) confidence intervals using this method.There for, we using the following Markov Chain Mont Carlo (MCMC) method to generate samples from the posterior density function, and obtain the Bayes estimators and HPD confidence intervals.

4.2 MCMC Method

We study the Markov chain Monte Carlo (MCMC) methods to draw samples from the posterior density function and then compute the Bayes estimators and also construct HPD credible intervals of α, θ and λThere are several conventional methods to define such Markov Chains exist, including Gibbs sampling, Metropolis-Hastings (MH) and reversible jump.Using these algorithms it is possible to implement posterior simulation in essentially any issue which allow point wise evaluation of the prior distribution and likelihood function. From Equation (31), the marginal posterior density of α, is proportional to

$$\pi_1^*\left(\alpha \mid \theta,\lambda,\, x\right) \propto \alpha^{w_2+r-1}e^{-\alpha w_1} \quad \prod_{\iota=1}^{r}\left(\lambda x_i + \frac{\theta}{2}x_i^2\right)^{\alpha-1}\left(\lambda+\theta x_i\right)e^{-\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}}e^{-\left(\lambda c+\frac{\theta}{2}c^2\right)^{\alpha(n-r)}} \tag{40}$$

Similarly, the posterior conditional distribution for θ and λ are respectively

$$\pi_2^*\left(\theta \mid \alpha,\lambda,\, x\right) \propto \theta^{w_4+r-1}e^{-\theta w_3} \quad \prod_{\iota=1}^{r}\left(\lambda x_i + \frac{\theta}{2}x_i^2\right)^{\alpha-1}\left(\lambda+\theta x_i\right)e^{-\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}}e^{-\left(\lambda c+\frac{\theta}{2}c^2\right)^{\alpha(n-r)}} \tag{41}$$

and

$$\pi_3^*\left(\lambda \mid \alpha,\theta,\, x\right) \propto \theta^{w_6+r-1}e^{-\theta w_5} \quad \prod_{\iota=1}^{r}\left(\lambda x_i + \frac{\theta}{2}x_i^2\right)^{\alpha-1}\left(\lambda+\theta x_i\right)e^{-\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}}e^{-\left(\lambda c+\frac{\theta}{2}c^2\right)^{\alpha(n-r)}} \tag{42}$$

We can see the posteriors conditional distributions for α, θ and λ are log-concave and can not be reduced analytically to well known distributions. So, as suggested by [36], a common way to solve this problem is to use the hybird algorithm by combined a Metropolis- Hasting (MH) sampling with Gibbs sampling scheme using normal distribution. There for the algorithm works as follow:

1) Set the initial values of α, θ and λ say (α_0, θ_0,λ_0).
2) Set j=1.
3) Using MH, generate α_1^j from $\pi_1^*(\alpha^{j-1} \mid \theta^{j-1},\lambda^{j-1}$,x) with normal distribution, N(α^{j-1},KαVα).
4) Using MH, generate θ_1^j from $\pi_2^*(\theta^{j-1} \mid \alpha^{j-1},\lambda^{j-1}$,x) with normal distribution, N(θ^{j-1},$K_\theta V_\theta$).
5) Using MH, generate λ_1^j from $\pi_3^*(\lambda^{j-1} \mid \alpha^{j-1},\theta^{j-1}$,x) with normal distribution, N(λ^{j-1},$K_\lambda V_\theta$), where Kα,K_θ and K_λ are scaling factor and Vα , V_θ and V_θ are variances-covariances matrix.
6) Set j=j+1.
7) Repeat steps from 1 to 5 N times.
8) The Bayes estimators of u(α, θ, λ) can be approximated as :

$$\hat{u}_{Mc} \approx \frac{\frac{1}{N}\sum_{i=1}^{N} u\left(\alpha_j,\theta_j,\lambda_j\right) f_1(\alpha_j,\theta_j,\lambda_j)}{\frac{1}{N}\sum_{i=1}^{N} f_1(\alpha_j,\theta_j,\lambda_j)} \tag{43}$$

where

$$f_1(\alpha_j,\theta_j,\lambda_j) = \prod_{\iota=1}^{r}\left(\lambda x_i + \frac{\theta}{2}x_i^2\right)^{\alpha-1}\left(\lambda+\theta x_i\right)e^{-\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}}e^{-\left(\lambda c+\frac{\theta}{2}c^2\right)^{\alpha(n-r)}},$$

9) Ordered α_j , θ_j and λ_j ,$j = 1,\ldots N$ and suppose that we would like to construct the HPD credible intervals of α, θ and λ Now, we construct all 100(1-α)% credible intervals of α say $(\alpha_{[1]}, \alpha_{[N(1-\alpha)]}), \ldots (\alpha_{[N\alpha]}, \alpha_{[N]})$.

Here [X] denotes the largest integer less than or equal to X. Then the HPD credible interval of α is that interval which has the shortest length. Similarity, the HPD credible interval of θ and λ can also be constructed. Now, two loss functions to determine the Bayes estimates based on MCMC method from (35).

4.2.1 Bayes estimate based on MCMC under LINEX loss function

1- For estimating α, consider $u\ (\alpha_j, \theta_j, \lambda_j) = \exp[-h\alpha_j]$, therefore

$$\hat{\alpha}_{BL} = \frac{-1}{h} \log \left[\frac{\frac{1}{N} \sum_{i=1}^{N} u\ (\alpha_j, \theta_j, \lambda_j)\ f_1(\alpha_j, \theta_j, \lambda_j)}{\frac{1}{N} \sum_{i=1}^{N}\ f_1(\alpha_j, \theta_j, \lambda_j)} \right]. \tag{44}$$

2- For estimating θ, consider $u\ (\alpha_j, \theta_j, \lambda_j) = \exp[-h\theta_j]$, therefore

$$\hat{\theta}_{BL} = \frac{-1}{h} \log \left[\frac{\frac{1}{N} \sum_{i=1}^{N} u\ (\alpha_j, \theta_j, \lambda_j)\ f_1(\alpha_j, \theta_j, \lambda_j)}{\frac{1}{N} \sum_{i=1}^{N}\ f_1(\alpha_j, \theta_j, \lambda_j)} \right]. \tag{45}$$

3-For estimating λ, consider $u\ (\alpha_j, \theta_j, \lambda_j) = \exp[-h\lambda_j]$, then

$$\hat{\lambda}_{BL} = \frac{-1}{h} \log \left[\frac{\frac{1}{N} \sum_{i=1}^{N} u\ (\alpha_j, \theta_j, \lambda_j)\ f_1(\alpha_j, \theta_j, \lambda_j)}{\frac{1}{N} \sum_{i=1}^{N}\ f_1(\alpha_j, \theta_j, \lambda_j)} \right]. \tag{46}$$

The estimators of reliability function R(t), hazard rate function H(t) and reversed hazard rate function Q(t) can be obtained by substituting the MCMC under the LINEX loss function of the parameters α, θ and λ in (3), (4) and (5).

4.2.2 Bayes estimate based on MCMC under Squared Error loss function

1- For estimating α, consider $u\ (\alpha_j, \theta_j, \lambda_j) = \alpha_j$, therefore

$$\hat{\alpha}_{SL} = \frac{\frac{1}{N} \sum_{i=1}^{N} u\ (\alpha_j, \theta_j, \lambda_j)\ f_1(\alpha_j, \theta_j, \lambda_j)}{\frac{1}{N} \sum_{i=1}^{N}\ f_1(\alpha_j, \theta_j, \lambda_j)}. \tag{47}$$

2- For estimating θ, consider $u\ (\alpha_j, \theta_j, \lambda_j) = \theta_j$, therefore

$$\hat{\theta}_{SL} = \frac{\frac{1}{N} \sum_{i=1}^{N} u\ (\alpha_j, \theta_j, \lambda_j)\ f_1(\alpha_j, \theta_j, \lambda_j)}{\frac{1}{N} \sum_{i=1}^{N}\ f_1(\alpha_j, \theta_j, \lambda_j)}. \tag{48}$$

3-For estimating λ, consider $u\ (\alpha_j, \theta_j, \lambda_j) = \lambda_j$, then

$$\hat{\lambda}_{SL} = \frac{\frac{1}{N} \sum_{i=1}^{N} u\ (\alpha_j, \theta_j, \lambda_j)\ f_1(\alpha_j, \theta_j, \lambda_j)}{\frac{1}{N} \sum_{i=1}^{N}\ f_1(\alpha_j, \theta_j, \lambda_j)}. \tag{49}$$

The estimators of reliability function R(t), hazard rate function H(t) and reversed hazard rate function Q(t) can be obtained by substituting the MCMC under the Squared Error loss(SEL) function of the parameters α, θ and λ in (3), (4) and (5).

5 Simulation Study

In this section, we carry out a simulation study to compare the performance of ML estimators and Bays estimators. We estimate the unknown parameters using the ML estimate and Bayes estimators obtained by Lindley approximations and MCMC method. The performances of different estimators with MSE are compared. Also, the average length of the asymptotic confidence intervals and the HPD confidence intervals are obtained.

The comparison between the estimates is taking place according to the following steps:

1- For given the parameter w_1, w_2, w_3, w_4, w_5 and w_6 generate random values of α, θ and λ from Gamma distribution.

2- For given values of n and r with initial values of α, θ and λ given in step 1, we generate random samples from inverse cumulative distribution function of GLE distribution and then ordered them.

3- The ML estimates of α, θ and λ are then obtained by solving numerically the nonlinear equation (11), (12) and (13) .

4- The ML estimates of the hazard function, survival function and reversed hazard function are obtained from the equations (16),(17) and (18) with t=0.25.

5- The Bayes estimates of α, θ and λ by using Lindley's approximation forms under Secured Error (SE) loss function given by (34) - (36), and under LINEX loss function given by (37) - (39).

6- The Bayes estimates of α, θ , λ, hazard function, survival function and reversed hazard function are computed by applying MCMC method with 1100 observations under Secured Error (SE) loss function and under LINEX loss function .

7- The quantities $(\hat{\kappa}-\kappa)^2$ are computed where $\hat{\kappa}$ stands for an estimate of κ (ML or Bayes).

Steps 1-7 are repeated at least 1000 times for informative prior and non-informative prior and for different sample sizes n and r at T=2 .

A simulation data for hybird type-II censored sample from GLED is generated with $\alpha = 1.2, \theta = 1$, $\lambda = 1.5$ and $T = 2$ for different choices of n and r with informative priors $w_1 = 1, w_2 = 1, w_3 = 0.05, w_4 = 0.05, w_5 = 4$, $w_6 = 4$, and non-informative priors $w_1 = w_2 = w_3 = w_4 = w_5 = w_6 = 0$,the MSE's, the average asymptotic confidence intervals and the HPD confidence intervals length from the MCMC technique are computed with informative priors and non-informative priors .

The MSE of the estimates were estimated by

$$MSE(\hat{\kappa}) = \sum_{1}^{1000} \frac{(\hat{\kappa}-\kappa)^2}{1000}.$$

First, the informative priors are considered.

Table 1: Estimates of the parameters α and the corresponding MSE for informative priors

n	r	MLEs	BAYES					MCMC				
			SEL	LINEX				SEL	LINEX			
				c1=-5	c2=-2	c3=1	c4=5		c1=-5	c2=-2	c3=1	c4=5
10	7	0.9644	0.8396	0.9837	0.8927	0.8214	0.7937	0.9644	0.9644	0.9644	0.9644	0.9644
		0.0555	0.1299	0.0468	0.0944	0.1433	0.1651	0.0555	0.0555	0.0555	0.0555	0.0555
40	25	1.2622	1.022	1.3686	1.1175	0.9592	0.9683	1.2621	1.2621	1.2621	1.2621	1.2621
		0.0039	0.0317	0.0284	0.0068	0.058	0.0485	0.0038	0.0038	0.0038	0.0038	0.0038
90	75	1.1699	1.1516	1.1728	1.16	1.1476	1.1339	1.17	1.17	1.17	1.17	1.17
		0.0009	0.0023	0.0007	0.0016	0.0027	0.0044	0.0009	0.0009	0.0009	0.0009	0.0009

Table 2: Estimates of the parameters θ and the corresponding MSE for informative priors

n	r	MLEs	BAYES					MCMC				
			SEL	LINEX				SEL	LINEX			
				c1=-5	c2=-2	c3=1	c4=5		c1=-5	c2=-2	c3=1	c4=5
10	7	1.2634	0.9642	1.747	1.6752	0.6932	0.7323	1.2661	1.2662	1.266	1.266	1.266
		0.0694	0.0013	0.5583	0.4559	0.0941	0.0717	0.06873	0.06876	0.0688	0.0688	0.0688
40	25	1.1216	0.9683	1.3611	1.2069	0.8775	0.8059	1.1211	1.1211	1.1211	1.1211	1.1211
		0.0148	0.001	0.1304	0.0428	0.015	0.0377	0.0143	0.0143	0.0143	0.0143	0.0143
90	75	0.9918	0.9954	1.1535	1.0318	0.9861	0.86	0.9846	0.9846	0.9846	0.9846	0.9846
		0.0001	0.0000	0.024	0.001	0.0002	0.0058	0.0002	0.0002	0.0002	0.0002	0.0002

Table 3: Estimates of the parameters λ and the corresponding MSE for informative priors

n	r	MLEs	BAYES					MCMC				
			SEL	LINEX				SEL	LINEX			
				c1=-5	c2=-2	c3=1	c4=5		c1=-5	c2=-2	c3=1	c4=5
10	7	1.0218	1.001	1.3042	1.2148	0.8822	0.7292	1.0231	1.0231	1.0231	1.0231	1.0231
		0.2287	0.2495	0.0383	0.0813	0.3817	0.5941	0.22776	0.22776	0.22776	0.22776	0.22776
40	25	1.3533	1.2964	1.4119	1.3463	1.2746	1.224	1.3537	1.3537	1.3537	1.3537	1.3537
		0.0215	0.0415	0.0078	0.0236	0.0508	0.0762	0.02144	0.02144	0.02144	0.02144	0.02144
90	75	1.5261	1.4855	1.5568	1.5149	1.47	1.44	1.5264	1.5264	1.5264	1.5264	1.5264
		0.0007	0.0002	0.0005	0.0002	0.0008	0.0041	0.0007	0.0007	0.0007	0.0007	0.0007

Table 4: Estimates of survival function R(t) and the corresponding MSE for informative priors

n	r	MLEs	BAYES					MCMC				
			SEL	LINEX				SEL	LINEX			
				c1=-5	c2=-2	c3=1	c4=5		c1=-5	c2=-2	c3=1	c4=5
10	7	0.6861	0.6652	0.6131	0.6186		0.7156	0.6858	0.6858	0.6858	0.6858	0.6858
		0.0013	0.0003	0.0014	0.0041		0.0042	0.0013	0.0013	0.0013	0.0013	0.0013
40	25	0.7495	0.7055	0.6192	0.586	0.7023	0.7508	0.7494	0.7494	0.7494	0.7494	0.7494
		0.001	0.0000	0.0009	0.0011	0.0027	0.01	0.001	0.001	0.001	0.0013	0.001
90	75	0.6395	0.6469	0.6292	0.6386	0.6504	0.6573	0.6397	0.6397	0.6397	0.6397	0.6397
		0.0001	0.0000	0.0005	0.0001	0.0000	0.0000	0.0001	0.0001	0.0001	0.0001	0.0001

Table 5: Estimates of hazard function H(t) and the corresponding MSE for informative priors

n	r	MLEs	BAYES					MCMC				
			SEL	LINEX				SEL	LINEX			
				c1=-5	c2=-2	c3=1	c4=5		c1=-5	c2=-2	c3=1	c4=5
10	7	1.4005	1.2853	2.2231	1.6744	1.1131	1.0011	1.4025	1.4025	1.4025	1.4025	1.4025
		0.227	0.3497	0.1201	0.0409	0.583	0.7665	0.22628	0.22628	0.22628	0.22628	0.22628
40	25	1.5926	1.6101	1.9012	1.7564	1.5526	1.469	1.5931	1.5931	1.5931	1.5931	1.5931
		0.0259	0.071	0.001	0.0145	0.105	0.1662	0.0260	0.02603	0.02603	0.02603	0.02603
90	75	1.8981	1.8107	1.9924	1.8947	1.77	1.7	1.896	1.896	1.896	1.896	1.896
		0.0005	0.0043	0.000	0.0003	0.0104	0.0305	0.0005	0.0005	0.0005	0.0005	0.0005

Table 6: Estimates of reversed hazard function Q(t) and the corresponding MSE for informative priors

n	r	MLEs	BAYES					MCMC				
			SEL	LINEX				SEL	LINEX			
				c1=-5	c2=-2	c3=1	c4=5		c1=-5	c2=-2	c3=1	c4=5
10	7	3.0614	2.5532	2.989	2.7154	2.5041	2.5192	3.0607	3.0607	3.0607	3.0607	3.0607
		0.1858	0.882	0.2534	0.6038	0.9766	0.9471	0.18566	0.18566	0.18566	0.18566	0.18566
40	25	4.7648	3.6994	3.881	3.7863	3.6619	2.9762	4.7636	4.7636	4.7636	4.7636	4.7636
		0.1788	0.0429	0.151	0.02	0.03	0.2664	0.1786	0.1786	0.1786	0.1786	0.1786
90	75	3.3674	3.3171	3.3807	3.3591	3.4025	3.2654	3.3663	3.3663	3.3663	3.3663	3.3663
		0.0156	0.0307	0.0125	0.01	0.05	0.0515	0.0157	0.0157	0.0157	0.0157	0.0157

Table 7: The 95% ACI's and HPD credible intervals and the corresponding length of α, θ and λ for informative priors.

n	r	parameter	ACI	HPD
10	7	α	1.782	0.0004
		θ	8.35021	0.0114
		λ	5.1938	0.0059
40	25	α	1.7918	0.0005
		θ	7.9846	0.0093
		λ	2.4734	0.0007
90	75	α	0.8756	0.0002
		θ	5.5741	0.0033
		λ	1.8729	0.0009

Second, non-informative priors

Table 8: Estimates of the parameters α and the corresponding MSE for non-informative priors

n	r	MLEs	BAYES					MCMC				
			SEL	LINEX				SEL	LINEX			
				c1=-5	c2=-2	c3=1	c4=5		c1=-5	c2=-2	c3=1	c4=5
10	7	0.9644	0.8375	0.9817	0.8902	0.8195	0.7927	0.9644	0.9644	0.9644	0.9644	0.9644
		0.0555	0.1314	0.0477	0.0959	0.1448	0.1659	0.0555	0.0555	0.0555	0.0555	0.0555
40	25	1.2622	1.2191	1.2745	1.241	1.2094	1.1819	1.2621	1.2621	1.2621	1.2621	1.2621
		0.0039	0.0004	0.0056	0.0017	0.0001	0.0003	0.0038	0.0038	0.0038	0.0038	0.0038
90	75	1.1699	1.1836	1.174	1.1612	1.1488	1.1349	1.17	1.17	1.17	1.17	1.17
		0.0009	0.0002	0.0007	0.0015	0.0000	0.0000	0.0009	0.0009	0.0009	0.0009	0.0009

Table9 : Estimates of the parameters θ and the corresponding MSE for non-informative priors

n	r	MLEs	BAYES					MCMC				
			SEL	LINEX				SEL	LINEX			
				c1=-5	c2=-2	c3=1	c4=5		c1=-5	c2=-2	c3=1	c4=5
10	7	1.2634	0.974	1.7481	1.6795	0.6988	0.733	1.2661	1.2662	1.266	1.266	1.266
		0.0694	0.001	0.5596	0.4617	0.0907	0.0713	0.06873	0.06876	0.0688	0.0688	0.0688
40	25	1.1216	0.9696	1.3615	1.208	0.8785	0.8062	1.1211	1.1211	1.1211	1.1211	1.1211
		0.0148	0.0009	0.1307	0.0433	0.0148	0.0376	0.0143	0.0143	0.0143	0.0143	0.0143
90	75	0.9918	0.8953	1.1534	1.0318	0.84	0.76	0.9846	0.9846	0.9846	0.9846	0.9846
		0.0001	0.000	0.064	0.017	0.0038	0.02	0.0002	0.0002	0.0002	0.0002	0.0002

Table10 : Estimates of the parameters λ and the corresponding MSE for non-informative priors

n	r	MLEs	BAYES					MCMC				
			SEL	LINEX				SEL	LINEX			
				c1=-5	c2=-2	c3=1	c4=5		c1=-5	c2=-2	c3=1	c4=5
10	7	1.0218	1.0224	1.3095	1.2295	0.9015	0.7344	1.0231	1.0231	1.0231	1.0231	1.0231
		0.2287	0.2281	0.0363	0.0732	0.3583	0.5862	0.22776	0.22776	0.22776	0.22776	0.22776
40	25	1.3533	1.3486	1.4474	1.3966	1.324	1.2534	1.3537	1.3537	1.3537	1.3537	1.3537
		0.0215	0.0229	0.0028	0.0107	0.031	0.0608	0.02144	0.02144	0.02144	0.02144	0.02144
90	75	1.5261	1.5225	1.5889	1.5548	1.51	1.46	1.5264	1.5264	1.5264	1.5264	1.5264
		0.0007	0.0005	0.0000	0.003	0.0001	0.0013	0.0007	0.0007	0.0007	0.0007	0.0007

Table 11: Estimates of survival function R(t) and the corresponding MSE for non- informative priors

n	r	MLEs	BAYES					MCMC				
			SEL	LINEX				SEL	LINEX			
				c1=-5	c2=-2	c3=1	c4=5		c1=-5	c2=-2	c3=1	c4=5
10	7	0.6861	0.6599	0.62	0.6152	0.6875	0.7141	0.6858	0.6858	0.6858	0.6858	0.6858
		0.0013	0.0001	0.001	0.0012	0.0014	0.0041	0.0013	0.0013	0.0013	0.0013	0.0013
40	25	0.7495	0.6866	0.6646	0.6735	0.6927	0.704	0.7494	0.7494	0.7494	0.7494	0.7494
		0.001	0.0000	0.0002	0.001	0.001	0.0029	0.001	0.001	0.001	0.0013	0.001
90	75	0.6395	0.6402	0.6231	0.6309	0.6428	0.652	0.6397	0.6397	0.6397	0.6397	0.6397
		0.0001	0.0000	0.0000	0.0005	0.0001	0.000	0.0001	0.0001	0.0001	0.0001	0.0001

Table 12: Estimates of hazard rate function H(t) and the corresponding MSE for non- informative priors

n	r	MLEs	BAYES					MCMC				
			SEL	LINEX				SEL	LINEX			
				c1=-5	c2=-2	c3=1	c4=5		c1=-5	c2=-2	c3=1	c4=5
10	7	1.4005	1.3054	2.1985	1.6867	1.1303	1.0056	1.4025	1.4025	1.4025	1.4025	1.4025
		0.227	0.3263	0.1036	0.0361	0.557	0.7587	0.22628	0.22628	0.22628	0.22628	0.22628
40	25	1.5926	1.6764	1.9507	1.8229	1.6142	1.5041	1.5931	1.5931	1.5931	1.5931	1.5931
		0.0259	0.0401	0.01	0.0029	0.0688	0.1387	0.0260	0.02603	0.02603	0.02603	0.02603
90	75	1.8981	1.929	2.0331	1.944	1.82	1.73	1.896	1.896	1.896	1.896	1.896
		0.0005	0.0027	0.0036	0.0000	0.0004	0.0115	0.0005	0.0005	0.0005	0.0005	0.0005

Table 13: Estimates of reversed hazard rate function Q(t) and the corresponding MES for non-informative priors

n	r	MLEs	BAYES					MCMC				
			SEL	LINEX				SEL	LINEX			
				c1=-5	c2=-2	c3=1	c4=5		c1=-5	c2=-2	c3=1	c4=5
10	7	3.0614	2.5331	2.9781	2.6968	2.4864	2.512	3.0607	3.0607	3.0607	3.0607	3.0607
		0.1858	0.9202	0.2645	0.633	1.012	0.9612	0.18566	0.18566	0.18566	0.18566	0.18566
40	25	4.7648	3.6735	3.8647	3.7601	3.6383	3.1828	4.7636	4.7636	4.7636	4.7636	4.7636
		0.1788	0.0328	0.1386	0.0717	0.0213	0.0958	0.1786	0.1786	0.1786	0.1786	0.1786
90	75	3.3674	3.4327	3.3608	3.3235	3.2794	3.2501	3.3663	3.3663	3.3663	3.3663	3.3663
		0.0156	0.0036	0.0129	0.0228	0.0038	0.05	0.0157	0.0157	0.0157	0.0157	0.0157

Table 14 The 95% ACI's and HPD credible intervals and the corresponding length of α, θ and λ for non -informative priors

n	r	parameter	ACI	HPD
10	7	α	1.782	0.0006
		θ	8.35021	0.0176
		λ	5.1938	0.0056
40	27	α	1.7918	0.001
		θ	7.9846	0.0113
		λ	2.4734	0.0012
90	72	α	0.8756	0.0001
		θ	5.5741	0.0056
		λ	1.8729	0.0006

We observe from the simulation study ,from Tables The performance the MLE's and Bayes estimators using MCMC are very similar in all aspects.

6 Real Data Analysis

In this section, we use the lifetime data set given by Table 15 to compare between proposed methods. The data set given in Table 3 represents the relief times of twenty patients receiving an analgesic.

This data set was taken from [37].

Table 15. Relief times of twenty patients.

1.1	1.4	1.3	1.7	1.9	1.8	1.6	2.2	1.7	2.7
4.1	1.8	1.5	1.2	1.4	3.0	1.7	2.3	1.6	2.0

Before progressing, first we would like to check whether the GLED fit this data or not. The calculated value of the K-S test is 0.18497 for the GLE distribution and this value is smaller than their corresponding values expected at 5% significance level, which is 0.29407 at $n = 20$. We have just plotted the empirical survival function and the fitted survival functions in Figure 1. Observe that the GLED can be a good model fitting this data. Figure 2 shows that a Q-Q plot for the data .

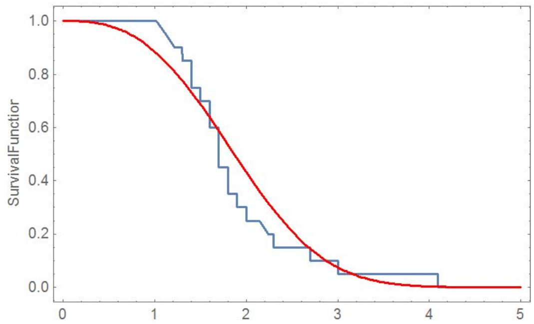

Fig. 1: Empirical and fitted distribution function for completed data set.

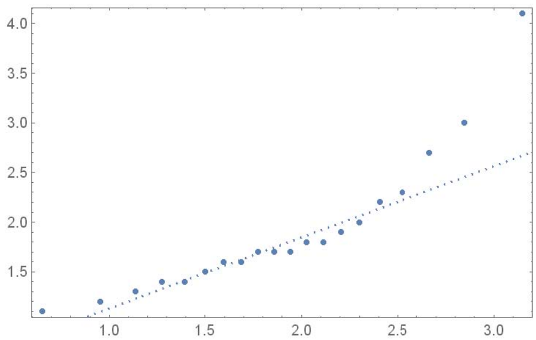

Fig. 2: Q-Q plot compare data to a specific distribution.

The unknown parameters using the ML estimates and the Bayes procedure are estimated. For computing the ML estimates, use the numerical method and also compute the 95% ACIs using the observed Fisher information matrix. For computing the Bayes estimates we consider the SE loss function and LINEX loss function are presented in Table 16 - 19. For comparison purpose, the informative and the non-informative priors were assumed. The Bayes estimates are obtained by using samples of size N=11000. In all cases informative and the non-informative priors $\alpha = 2.78703, \theta$ =0.012 , $\lambda = 0.469487, c = 2, -2, 0.001$.

First, the informative priors are considered with $w_1 = 1.2, w_2 = 0.01, w_3 = 1, w_4 = 1.2, w_5 = 0.01$ and $w_6 = 1$.

Table 16: Estimates of the parameters α, θ , λ for different methods under the informative priors

Parameters	MLEs	BAYES				MCMC			
		SEL	LINEX			SEL	LINEX		
			c=-2	c=0.001	c=2		c=-2	c=0.001	c=2
α	1.9156	1.747	1.8113	1.747	1.7175	1.8299	1.8972	1.8299	1.7536
θ	0.1833	0.1719	0.1738	0.1719	0.1701	0.1854	0.1854	0.1854	0.1854
λ	0.2667	0.2608	0.2642	0.2608	0.2576	0.2421	0.244	0.2421	0.2401

Table 17: The 95% ACI and the HPD credible interval of the parameters α, θ , λ under the informative priors.

	α	Lenght	θ	Lenght	λ	Lenght
ACI	(1.6064,2.2248)	0.618437	(0.1325,0.234)	0.10157	(0.2091,0.3243)	0.1152
HPD	(1.2477,2.2774)	1.02967	(0.1803,0.1912)	0.0108495	(0.1578,0.3256)	0.167789

Table18: Estimates of the parameters α, θ , λ for different methods under the non- informative priors

Parameters	MLEs	BAYES				MCMC			
		SEL	LINEX			SEL	LINEX		
			c=-2	c=0.001	c=2		c=-2	c=0.001	c=2
α	1.9156	1.7897	1.8613	1.7897	1.7471	1.7865	1.8694	1.8361	1.708
θ	0.1833	0.1718	0.1737	0.1718	0.1701	0.1882	0.1883	0.1824	0.1882
λ	0.2667	0.2562	0.2595	0.2562	0.2531	0.2284	0.2344	0.2161	0.2227

Table 19: The 95% ACI and the HPD credible interval of the parameters α, θ , λ under the non-informative priors

	α	Lenght	θ	Lenght	λ	Lenght
ACI	(1.6064, 2.2248)	0.618437	(0.1325,0.234)	0.10157	(0.2091, 0.3243)	0.11589
HPD	(1.2611, 2.349)	1.08784	(0.181,0.1998)	0.0188514	(0.1372 ,0.3651)	0.227894

7 Conclusion

In this paper, MLE and Bayes estimation of the unknown parameter for the Type-II hybrid censored GLE distribution are considered. We provide the maximum likelihood estimators and it is observed that the maximum likelihood estimators of the unknown parameters can not obtained in the closed form and we use the numerical to compute them. Also, we find the Bayes estimators of the unknown parameters and show that they can not be obtained in explicit forms, and we have proposed two approximation methods to compute them. Lindley approximations and the MCMC method are used. We have compared the performance of the different methods by Monte Carlo simulations, and it is observed that the performance of quite satisfactory.

Under the simulation study:

1- Tables (1-9) showed that the mean square errors decrease by increasing n and r.

2- The Bayes estimates of the three parameters α, θ and λ using MCMC method are generally quite to their ML estimates and Bayes estimates using Lindley approximation , based on MSEs.

Furthermore, we observe the following from real data analysis, it is observed that: from Table 17 and 19, the length of the HPD credible intervals of α, θ and λ based on informative priors are al almost smaller than the corresponding length of the HPD credible intervals based on non-informative priors.

Acknowledgment

The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

Conflict of Interest

The authors declare that they have no conflict of interest.

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