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The Multiple Composed Erdélyi-Kober Fractional Integrals and Derivatives and Their Convolutions

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Abstract: This article addresses the multiple composed Erdélyi-Kober fractional derivatives and integrals that are compositions of the suitable right- and left-sided Erdélyi-Kober derivatives and integrals. These operators are important, say, in the framework of the Euler-Lagrange equations in the fractional calculus of variations. We start with a discussion of their properties including inversion formulas, compositions, and mapping properties. Then, we introduce an integral transform of the Mellin convolution type related to the multiple composed Erdélyi-Kober integrals and derive some operational relations. Finally, a one parameter family of convolutions for the multiple composed Erdélyi-Kober integrals in the sense of Dimovski is constructed.

Keywords: Multiple composed Erdélyi-Kober derivatives and integrals, Caputo-type Erdélyi-Kober derivative, Mellin-Barnes integral representations, generalized Obrechkoff-Stiltjes integral transform, convolutions in the sense of Dimovski.

1 Introduction

The Erdélyi-Kober fractional derivatives and integrals (E-K-D and E-K-I) are among the most used and useful operators of Fractional Calculus [1,2,3,4,5]. Following the standard terminology, one distinguishes between the so-called left- and right-sided E-K-D and E-K-I. The theory of the left-sided and the right-sided E-K-D and E-K-I, respectively, as well as the compositions of either the left-sided or the right-sided E-K-D or E-K-I is well developed. Because the left-sided operators can be reduced to the right-sided ones by some simple variables substitutions, the basic properties of these operators are very similar. However, only few attention was given to the case of the compositions of the left-sided and the right-sided E-K-D and E-K-I. These operators (i.e. composed Erdélyi-Kober operators) are essentially different compared to the compositions of either the left-sided or the right-sided E-K-D and E-K-I, respectively.

A general framework for treating the composed Erdélyi-Kober operators was suggested in [5], where both the H-transform with the Fox H-function in the kernel and the generalized H-transform were introduced and investigated in detail. The generalized H-transform is defined via the Parseval equality for the Mellin integral transform in form of the Mellin-Barnes integral with the quotients of products of some Gamma-functions in the kernel. Thus, it contains both the H-transforms, the inverse H-transforms, and their compositions including the E-K-D and E-K-I as its particular cases. The known asymptotic behavior of the Gamma-function allows a convenient treatment of the generalized H-transform in the special spaces of functions defined as the inverse Mellin integral transform of functions with a suitable asymptotic on the vertical lines in the complex plane [5]. In this paper, some elements of this theory are employed for derivation of the results and their proofs.

A realization of this framework for the case of a composition of a left-sided E-K derivative/integral and a right-sided E-K integral/derivative was suggested in [6]. In particular, a Mellin-Barnes representation of the composition was deduced and employed for derivation of an explicit formula for a one parameter family of convolutions for the composed Erdélyi-Kober operators in the sense of Dimovski. In this paper, we consider a far reaching generalization of the results presented in [6] and investigate the case of a composition of several left-sided E-K derivatives/integrals and several right-sided E-K integrals/derivatives.

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The main motivation for studying these operators and the fractional differential equations with these operators stems from the fractional calculus of variations (FCV) that deals with optimization of the functionals depending on the fractional derivatives [7,8]. The Euler-Lagrange equations of FCV are fractional differential equations with the left- and right-sided fractional derivatives and integrals and their compositions [7].

For analytical treatment of the fractional Euler-Lagrange equation on a finite interval, some methods were suggested in [9]. However, these methods do not work in the case of an infinite interval that is important for some applications. In this paper, we develop some basic elements and constructions for a Mikusiński type operational calculus [10] for the multiple composed E-K-D and E-K-I can be employed for analytical treatment of the corresponding fractional Euler-Lagrange equations on infinite intervals. In particular, we investigate the basic properties of the multiple composed E-K-D and E-K-I, introduce an integral transform that acts as a transmutation operator with respect to the multiple composed E-K-I, and derive a one parameter family of the convolutions for the multiple composed E-K-I in the sense of Dimovski.

The operational method was successfully used in Fractional Calculus (FC) for analytical treatment of the fractional differential equations of different types. In [11], the fractional differential equations with the Riemann-Liouville fractional derivatives were considered. The case of the fractional differential equations with the Caputo fractional derivatives was treated in [12]. In [13], the operational method was employed for solving some fractional differential equations with the Hilfer derivatives. The Mikusiński type operational calculi for the left-sided E-K-D and for the compositions of a finite number of such derivatives were developed in [14,15,5]. In [16], the case of the Caputo type modifications of the E-K-D was considered. In [17], an operational calculus for the Riemann-Liouville fractional derivatives was used for analytical treatment of the generalized Abel integral equations. In [18], a general overview of several operational calculi of Mikusiński type for different fractional differential operators was provided.

Another possible approach for analytical solution of the fractional differential equations is application of the suitable integral transforms, the so-called transmutation operators, that translate the integro-differential equations to some simple algebraic equations. As we show in this paper, the transmutations of the composed E-K fractional operators are provided in form of the generalized Obrechkoff-Stiltjes integral transform that contains both the Obrechkoff and the Stiltjes integral transforms as its particular cases. The Obrechkoff transform was introduced in [19]. In [20,21], it was used as a transmutation that translates the hyper-Bessel differential operator to a multiplication with a power function. Because the hyper-Bessel operator is a particular case of a composition of the E-K-D, these results are included in our schema. In [22], a particular case of the generalized Obrechkoff-Stiltjes integral transform was employed to derive some explicit formulas for solutions to the initial value problems for the fractional differential equations that contain both the right- and the left-sided E-K-D.

The technique employed for derivation of the results presented in this paper is based on the Mellin integral transform. For the basic elements of the Mellin integral transform and its applications we refer the reader to [23]. In [24,25], the surveys of several different applications of the Mellin transform technique in FC are provided. In particular, in [25], the Mellin integral transform is applied for calculating improper integrals containing the FC special functions and for derivation of basic properties of the E-K-D and E-K-I. The more advanced applications comprise derivation of a modified Post-Widder formula for the inversion of the Laplace transform, investigation of the completely monotone functions and their connection with the probability density functions, derivation of some subordination formulas for the multi-dimensional space-time fractional diffusion equations, and an analytical treatment of dual integral equations with the Meijer G-functions in the kernel.

The rest of this paper is organized as follows: The second section is devoted to the definitions of the multiple composed E-K-D and E-K-I in form of the compositions of the suitable left- and right-sided E-K-D and E-K-I. In connection with these definitions, some relevant properties of the left- and right-sided E-K-D and E-K-I are revisited. In the third section, an in-deep discussion of the multiple composed E-K-D and E-K-I is presented. In particular, their Mellin-Barnes representations are derived and used to show that the multiple composed E-K-D are the left-inverse operators to the multiple composed E-K-I. Moreover, we introduce a transmutation operator for the multiple composed E-K-I in form of the generalized Obrechkoff-Stiltjes integral transform and derive some operational relations. In the last section, we present a construction of a one parameter family of the convolutions for the generalized Obrechkoff-Stiltjes integral transform that can be also interpreted as the convolutions of the multiple composed E-K-I in the sense of Dimovski.

2 The multiple composed Erdélyi-Kober fractional derivatives and integrals

In this paper, we handle the multiple composed E-K-I and E-K-D. They are defined in form of certain multiple compositions of the left- and right-sided E-K-I and E-K-D as follows:

Definition 1. *Let* $\mu > 0,\ a_i > 0,\ \alpha_i \in \mathbb{R},\ i = 1,\dots,n,\ b_i > 0,\ \beta_i \in \mathbb{R},\ i = 1,\dots,m$ *and*

$$\sum_{i=1}^{n} a_i > \sum_{i=1}^{m} b_i. \tag{1}$$

The operators

$$(L_\mu f)(x) = x^\mu \left(\prod_{i=1}^{m} P_{1/b_i}^{\beta_i - b_i\mu, b_i\mu} \prod_{i=1}^{n} I_{1/a_i}^{-\alpha_i, a_i\mu} f \right)(x), \tag{2}$$

$$(D_\mu f)(x) = x^{-\mu} \left(\prod_{i=1}^{m} K_{1/b_i}^{\beta_i, b_i\mu} \prod_{i=1}^{n} D_{1/a_i}^{-\alpha_i - a_i\mu, a_i\mu} f \right)(x) \tag{3}$$

are called the multiple composed E-K-I and E-K-D of the Riemann-Liouville type, respectively.

In the definition, the operators $I_\beta^{\gamma,\delta}$, $K_\beta^{\tau,\alpha}$, $D_\beta^{\gamma,\delta}$, $P_\beta^{\tau,\alpha}$ stand for the left- and right-sided E-K-I and E-K-D of order δ or α, respectively [1,3,5]:

$$(I_\beta^{\gamma,\delta} f)(x) = \frac{\beta}{\Gamma(\delta)} x^{-\beta(\gamma+\delta)} \int_0^x (x^\beta - t^\beta)^{\delta-1} t^{\beta(\gamma+1)-1} f(t)dt,\ \delta,\beta > 0, \gamma \in \mathbb{R}, \tag{4}$$

$$(K_\beta^{\tau,\alpha} f)(x) = \frac{\beta}{\Gamma(\alpha)} x^{\beta\tau} \int_x^\infty (t^\beta - x^\beta)^{\alpha-1} t^{-\beta(\tau+\alpha-1)-1} f(t)dt,\ \alpha,\beta > 0, \tau \in \mathbb{R}, \tag{5}$$

$$(D_\beta^{\gamma,\delta} f)(x) = \prod_{j=1}^{k} \left(\gamma + j + \frac{1}{\beta} x \frac{d}{dx} \right) (I_\beta^{\gamma+\delta, k-\delta} f)(x),\ k-1 < \delta \le k,\ k \in \mathbb{N}, \tag{6}$$

$$(P_\beta^{\tau,\alpha} f)(x) = \prod_{j=0}^{l-1} \left(\tau + j - \frac{1}{\beta} x \frac{d}{dx} \right) (K_\beta^{\tau+\alpha, l-\alpha} f)(x),\ l-1 < \alpha \le l,\ l \in \mathbb{N}. \tag{7}$$

For $\delta = 0$ and $\alpha = 0$, the left- and right-sided E-K-I are defined as the identity operators:

$$(I_\beta^{\gamma,0} f)(x) = \mathrm{Id}\, f = f(x),\ (K_\beta^{\tau,0} f)(x) = \mathrm{Id}\, f = f(x).$$

Accordingly, for $\delta = k,\ k \in \mathbb{N}$ or for $\alpha = l,\ l \in \mathbb{N}$, the left- and right-sided E-K-D are reduced to the so-called hyper-Bessel differential operators:

$$(D_\beta^{\gamma,k} f)(x) = \prod_{j=1}^{k} \left(\gamma + j + \frac{1}{\beta} x \frac{d}{dx} \right) f(x),\ k \in \mathbb{N}, \tag{8}$$

$$(P_\beta^{\tau,l} f)(x) = \prod_{j=0}^{l-1} \left(\tau + j - \frac{1}{\beta} x \frac{d}{dx} \right) f(x),\ l \in \mathbb{N}. \tag{9}$$

For $\beta = 1$, the E-K-I (4), (5) are reduced to the Riemann-Liouville fractional integrals with the power functions weights [26], [5]:

$$(I_1^{\gamma,\delta} f)(x) = (x^{-\gamma-\delta} I_{0+}^{\delta} x^\gamma f)(x) := \frac{1}{\Gamma(\delta)} x^{-\gamma-\delta} \int_0^x (x-t)^{\delta-1} t^\gamma f(t)dt, \tag{10}$$

$$(K_1^{\tau,\alpha} f)(x) = (x^\tau I_-^\alpha x^{-\tau-\alpha} f)(x) := \frac{1}{\Gamma(\alpha)} x^\tau \int_x^\infty (t-x)^{\alpha-1} t^{-\tau-\alpha} f(t)dt. \tag{11}$$

The multiple composed E-K-I and E-K-D are integro-differential operators. Under the condition (1), the multiple composed E-K-I have some typical properties of the integral operators, whereas the multiple composed E-K-D behave more like differential operators.

The E-K-I and E-K-D were treated in different spaces of functions (see, e.g., [1,26,5]). Let us mention some results for the space C_α [27] that are relevant for development of the operational calculi of Mikusiński type for the E-K-D and their compositions.

The space of functions $C_\alpha,\ \alpha \in \mathbb{R}$ contains the functions $f = f(x),\ x > 0$ in the form $f(x) = x^p f_1(x)$ with $p > \alpha$ and $f_1 \in C([0,\infty))$.

On the space C_α, $\alpha \geq -\beta(\gamma+1)$, the left-sided E-K-D is a left-inverse operator to the left-sided E-K-I [3]:

$$(D_\beta^{\gamma,\delta} I_\beta^{\gamma,\delta} f)(x) = f(x),\ f \in C_\alpha. \tag{12}$$

The right-sided E-K-D is a left-inverse operator to the right-sided E-K-I on a suitable space of functions, too.

However, the E-K-D are not the right-inverse operators to the E-K-I [3]:

$$(I_\beta^{\gamma,\delta} D_\beta^{\gamma,\delta} f)(x) \;=\; f(x) - \sum_{j=0}^{k-1} c_j x^{-\beta(1+\gamma+j)}, \tag{13}$$

$$c_j = \frac{\Gamma(k-j)}{\Gamma(\delta-j)} \lim_{x\to 0} x^{\beta(1+\gamma+j)} \prod_{i=j+1}^{k-1} (1+\gamma+i+\frac{1}{\beta}x\frac{d}{dx})(I_\beta^{\gamma+\delta,k-\delta} f)(x). \tag{14}$$

In the formulas above $f \in C_\alpha^k$, $k-1 < \delta \leq k,\ k \in \mathbb{N}$ and $\alpha \geq -\beta(\gamma+1)$. The space of functions C_α^k is a subspace of C_α. It contains the functions $f = f(x),\ x > 0$ in the form $f(x) = x^p f_1(x)$ with $p > \alpha$ and $f_1 \in C^k([0,\infty))$. For the properties of C_α^k we refer to [1] and [5].

It turns out that the coefficients c_j given by the formula (14) determine the natural initial conditions for the fractional differential equations with the E-K-D. As we see, they depend on the limits of some complicated integro-differential operators that are difficult to be interpreted in the concrete applications. To avoid this situation, a Caputo-type modification of the E-K-D was introduced in [28] and analyzed in details in [3]. These fractional derivatives are similar to the E-K-D, but allow a traditional form of the initial conditions, while considering initial value problems for the fractional differential equations with these fractional derivatives.

Let $k-1 < \delta \leq k,\ k \in \mathbb{N}$, $l-1 < \alpha \leq l,\ l \in \mathbb{N}$, and $\beta > 0$. The operators

$$({}_*D_\beta^{\gamma,\delta} f)(x) = (I_\beta^{\gamma+\delta,k-\delta} \prod_{j=1}^{k} (\gamma+j+\frac{1}{\beta}t\frac{d}{dt}) f)(x),\ x > 0, \tag{15}$$

$$({}_*P_\beta^{\tau,\alpha} f)(x) = (K_\beta^{\tau+\alpha,l-\alpha} \prod_{j=0}^{l-1} (\tau+j-\frac{1}{\beta}t\frac{d}{dt}) f)(x),\ x > 0 \tag{16}$$

are called the left-sided and the right-sided Caputo type modifications of the E-K-D of order δ and α, respectively.

The Caputo type modifications of the E-K-D are the left-inverse operators to the corresponding E-K-I ([3]):

$$({}_*D_\beta^{\gamma,\delta} I_\beta^{\gamma,\delta} f)(x) = f(x), \tag{17}$$

$$({}_*P_\beta^{\tau,\alpha} K_\beta^{\tau,\alpha} f)(x) = f(x). \tag{18}$$

As in the case of the E-K-D, the Caputo type modifications of the E-K-D are not the right-inverse operators to the E-K-I. A closed form formula for the composition of the left-sided E-K-I and the left-sided Caputo type modification of the E-K-D was derived in [3]:

$$(I_\beta^{\gamma,\delta}\ {}_*D_\beta^{\gamma,\delta} f)(x) \;=\; f(x) - \sum_{j=0}^{k-1} p_j x^{-\beta(1+\gamma+j)}, \tag{19}$$

$$p_j = \lim_{x\to 0} x^{\beta(1+\gamma+j)} \prod_{i=j+1}^{k-1} (1+\gamma+i+\frac{1}{\beta}x\frac{d}{dx}) f(x), \tag{20}$$

where $k-1 < \delta \leq k,\ k \in \mathbb{N},\ \alpha \geq -\beta(\gamma+\delta+1)$ and $f \in C_\alpha^k$.

In general, the E-K-D and their Caputo type modifications are different operators. Say, for a function $f \in C_\alpha^k,\ k-1 < \delta \leq k,\ k \in \mathbb{N},\ \alpha \geq -\beta(\gamma+1)$, the left-sided E-K-D $D_\beta^{\gamma,\delta}$ coincides with its Caputo type modification ${}_*D_\beta^{\gamma,\delta}$ if and only if the conditions

$$\lim_{x\to 0} x^{\beta(1+\gamma+j)} \prod_{i=j+1}^{k-1} (1+\gamma+i+\frac{1}{\beta}x\frac{d}{dx}) f(x) =$$

$$\frac{\Gamma(k-j)}{\Gamma(\delta-j)} \lim_{x\to 0} x^{\beta(1+\gamma+j)} \prod_{i=j+1}^{k-1} (1+\gamma+i+\frac{1}{\beta}x\frac{d}{dx})(I_\beta^{\gamma+\delta,k-\delta} f)(x)$$

are fulfilled for all $j = 0,1,\ldots,k-1$ [3].

In analogy to the Caputo type modifications of the E-K-D, now we introduce the Caputo type modifications of the multiple composed E-K-I and E-K-D.

Definition 2. *Let $\mu > 0,\ a_i > 0,\ \alpha_i \in \mathbb{R},\ i = 1,\dots,n,\ b_i > 0,\ \beta_i \in \mathbb{R},\ i = 1,\dots,m$ and the condition (1) holds true. The operators*

$$({}_{*}L_\mu f)(x) = x^\mu \left(\prod_{i=1}^{m} {}_{*}P_{1/b_i}^{\beta_i - b_i\mu, b_i\mu} \prod_{i=1}^{n} I_{1/a_i}^{-\alpha_i, a_i\mu} f \right)(x), \tag{21}$$

$$({}_{*}D_\mu f)(x) = x^{-\mu} \left(\prod_{i=1}^{m} K_{1/b_i}^{\beta_i, b_i\mu} \prod_{i=1}^{n} {}_{*}D_{1/a_i}^{-\alpha_i - a_i\mu, a_i\mu} f \right)(x) \tag{22}$$

are called the multiple composed E-K-I and E-K-D of the Caputo type, respectively.

Some important properties of the multiple composed E-K-I, E-K-D, and their Caputo type modifications will be discussed in the next section.

3 Basic properties of the multiple composed Erdélyi-Kober operators

The Mellin integral transform plays a very important role in FC [24,25]. In this section, we employ some elements of the Mellin integral transform theory for investigation of the multiple composed E-K-I and E-K-D.

We start with the basic definitions and properties of the Mellin integral transform. Let a function f satisfy the inclusion $x^{\gamma-1} f \in L^1(\mathbb{R}_+)$. Its Mellin integral transform at the point $s = \gamma + i\tau$ is defined as follows:

$$F(s) = \mathfrak{M}\{f(x); s\} = f^*(s) = \int_0^{+\infty} f(x) x^{s-1}\, dx. \tag{23}$$

The inverse Mellin integral transform for a function $F = F(\gamma + i\tau) \in L^1(\mathbb{R})$ with a fixed $\gamma \in \mathbb{R}$ is defined as the following improper integral in the sense of the Cauchy principal value:

$$f(x) = \mathfrak{M}^{-1}\{F(s); x\} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(s) x^{-s} ds, \ \ x > 0,\ s = \gamma + i\tau. \tag{24}$$

The convolution of the Mellin integral transform (Mellin convolution) is given by the formula

$$(f \overset{\mathfrak{M}}{*} g)(x) = \int_0^{+\infty} f\left(\frac{x}{t}\right) g(t) \frac{dt}{t}. \tag{25}$$

The Mellin convolution $h = (f \overset{\mathfrak{M}}{*} g)$ is well defined for the functions $f(x)x^{\gamma-1} \in L^1(\mathbb{R}_+)$ and $g(x)x^{\gamma-1} \in L^1(\mathbb{R}_+)$. It fulfills the inclusion $h(x)x^{\gamma-1} \in L^1(\mathbb{R}_+)$ and the convolution property

$$\mathfrak{M}\{(f \overset{\mathfrak{M}}{*} g)(x); s\} = \mathfrak{M}\{f(x); s\} \cdot \mathfrak{M}\{g(x); s\}. \tag{26}$$

The Parseval equality for the Mellin integral transform in form

$$\int_0^{+\infty} f\left(\frac{x}{t}\right) g(t) \frac{dt}{t} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} f^*(s) g^*(s) x^{-s} ds \tag{27}$$

follows from the convolution property (26) and the formula (24).

Denoting by $\overset{\mathfrak{M}}{\to}$ the juxtaposition of a function f with its Mellin transform f^*, the operational rules that we need in the further discussions are as follows ([24,23,5]):

$$t^\alpha f(t) \ \overset{\mathfrak{M}}{\to}\ f^*(s + \alpha), \tag{28}$$

$$f(t^\alpha) \ \overset{\mathfrak{M}}{\to}\ \frac{1}{|\alpha|} f^*(s/\alpha),\ \alpha \neq 0. \tag{29}$$

We also need the Mellin transform formulas of some elementary functions (see [23] or [29]) for these and other formulas):

$$e^{-t^\alpha} \ \overset{\mathfrak{M}}{\to}\ \frac{1}{|\alpha|} \Gamma(s/\alpha) \ \text{ if } \Re(s/\alpha) > 0, \tag{30}$$

$$\frac{(1-t^{\beta})_{+}^{\alpha-1}}{\Gamma(\alpha)} \overset{\mathfrak{M}}{\to} \frac{\Gamma(s/\beta)}{|\beta|\Gamma(s/\beta+\alpha)} \text{ if } \Re(\alpha)>0,\ \Re(s/\beta)>0, \tag{31}$$

$$\frac{(t^{\beta}-1)_{+}^{\alpha-1}}{\Gamma(\alpha)} \overset{\mathfrak{M}}{\to} \frac{\Gamma(1-\alpha-s/\beta)}{|\beta|\Gamma(1-s/\beta)} \text{ if } 1-\Re(s/\beta)>\Re(\alpha)>0. \tag{32}$$

In the formulas (31), (32), we used the notation

$$t_{+}^{\alpha}=\begin{cases} t^{\alpha},\ t>0,\\ 0,\ \ t\le 0\end{cases}$$

for the truncated power function.

It is easy to see that the E-K-I (4), (5) can be represented as the Mellin convolutions as follows:

$$(I_{\beta}^{\gamma,\delta}f)(x)=(k_1 \overset{\mathfrak{M}}{*} f)(x),\ (K_{\beta}^{\tau,\alpha}f)(x)=(k_2 \overset{\mathfrak{M}}{*} f)(x), \tag{33}$$

where

$$k_1(x)=\frac{\beta}{\Gamma(\delta)}x^{-\beta(\gamma+\delta)}(x^{\beta}-1)_{+}^{\delta-1},\ k_2(x)=\frac{\beta}{\Gamma(\alpha)}x^{\beta\tau}(1-x^{\beta})_{+}^{\alpha-1}. \tag{34}$$

The Parseval equality (27) and the Mellin transform formulas (31), (32) lead to the representations of the E-K-I as the Mellin-Barnes integrals ([1,5]):

$$(I_{\beta}^{\gamma,\delta}f)(x)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(1+\gamma-s/\beta)}{\Gamma(1+\gamma+\delta-s/\beta)}f^{*}(s)x^{-s}ds, \tag{35}$$

$$(K_{\beta}^{\tau,\alpha})f(x)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(\tau+s/\beta)}{\Gamma(\tau+\alpha+s/\beta)}f^{*}(s)x^{-s}ds. \tag{36}$$

The representations (35) and (36) hold true, in particular, for the functions $f\in\mathscr{M}_{c,\rho}^{-1}(L)$ with some restrictions posed on c and ρ. For a discussion of the space of functions $\mathscr{M}_{c,\rho}^{-1}(L)$ in connection with the Mellin convolution type integral transforms we refer the readers to [5]. In this part of the section, we provide its definition and some properties.

Definition 3.*Let the inequality*

$$2sgn(c)+sgn(\rho)\ge 0 \tag{37}$$

hold true for some $c,\ \rho\in\mathbb{R}$. *By* $\mathscr{M}_{c,\rho}^{-1}(L)$ *we denote the space of functions* $f=f(x),\ x>0$ *that can be represented as the inverse Mellin integral transforms*

$$f(x)=\mathfrak{M}^{-1}\{F(s);x\}=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}F(s)x^{-s}ds,\ \ x>0,\ s=\gamma+i\tau \tag{38}$$

of the functions $F(s)=F(\gamma+i\tau),\ \tau\in\mathbb{R}$ *that satisfy the inclusion*

$$F(s)|s|^{\rho}e^{\pi c|\Im(s)|}\in L(\sigma),\ \sigma=\{s\in\mathbb{C}:\Re(s)=\gamma\}.$$

Evidently, the family of spaces $\mathscr{M}_{c,\gamma}^{-1}(L)$ is partially ordered:

$$\mathscr{M}_{c_1,\rho_1}^{-1}(L)\subset\mathscr{M}_{c_2,\rho_2}^{-1}(L) \tag{39}$$

if and only if

$$2\text{sgn}(c_1-c_2)+\ \text{sgn}(\rho_1-\rho_2)\ge 0. \tag{40}$$

To make the notations shorter, the space $\mathscr{M}_{0,0}^{-1}(L)$ will be denoted by $\mathscr{M}^{-1}(L)$ and the space $\mathscr{M}_{0,\rho}^{-1}(L)$ by $\mathscr{M}_{\rho}^{-1}(L)$.

The space of functions $\mathscr{M}_{c,\rho}^{-1}(L)$ is a Banach space with the norm

$$\|f\|_{\mathscr{M}_{c,\rho}^{-1}(L)}=\frac{1}{2\pi}\int_{\sigma}e^{\pi c|\Im(s)|}\ |\ s^{\rho}f^{*}(s)ds\ |\ . \tag{41}$$

Other properties of $\mathcal{M}_{c,\rho}^{-1}(L)$ are discussed in [5].

Because the E-K-D are the left-inverse operators to the E-K-I, their Mellin-Barnes representations can be obtained from the Mellin-Barnes representations of the E-K-I [5]:

$$(D_\beta^{\gamma,\delta} f)(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(1+\gamma+\delta-s/\beta)}{\Gamma(1+\gamma-s/\beta)} f^*(s)x^{-s}ds, \tag{42}$$

$$(P_\beta^{\tau,\alpha}) f(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(\tau+\alpha+s/\beta)}{\Gamma(\tau+s/\beta)} f^*(s)x^{-s}ds. \tag{43}$$

The mapping properties of the integral transforms represented as the Mellin-Barnes integrals immediately follow from the well known asymptotic Stirling formula for the Euler Gamma-function

$$\Gamma(z) = \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z}(1+O(1/z)),\ |arg(z)| < \pi,\ |z| \to \infty \tag{44}$$

and its corollary

$$|\Gamma(x+iy)| = \sqrt{2\pi}|y|^{x-\frac{1}{2}} e^{-\pi|y|/2}(1+O(1/y)),\ |y| \to \infty. \tag{45}$$

In particular, the left-sided E-K-I exists on the space $\mathcal{M}_\rho^{-1}(L),\ \rho \ge 0$ and $g(x) = (I_\beta^{\gamma,\delta} f)(x) \in \mathcal{M}_{\rho+\delta}^{-1}(L)$. Analogously, for $\rho \ge \delta$, the left-sided E-K-D maps the space $\mathcal{M}_\rho^{-1}(L)$ into the space $\mathcal{M}_{\rho-\delta}^{-1}(L)$. The right-sided E-K-I is a one-to-one mapping from the space $\mathcal{M}_\rho^{-1}(L),\ \rho \ge 0$ onto the space $\mathcal{M}_{\rho+\alpha}^{-1}(L)$. For $\rho \ge \alpha$, the right-sided E-K-D maps the space $\mathcal{M}_\rho^{-1}(L)$ into the space $\mathcal{M}_{\rho-\alpha}^{-1}(L)$.

Now let us consider the representations of the multiple composed E-K-I and E-K-D as the Mellin-Barnes integrals.

Theorem 1. *The multiple composed E-K-I of the Riemann-Liouville type maps the space $\mathcal{M}_\rho^{-1}(L),\ \rho \ge 0$ onto the space $\mathcal{M}_{\rho+\mu(\sum_{i=1}^n a_i - \sum_{i=1}^m b_i)}^{-1}(L)$. It can be represented in form of the Mellin-Barnes integral as follows:*

$$(L_\mu f)(x) = \frac{x^\mu}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \prod_{i=1}^m \frac{\Gamma(\beta_i+b_is)}{\Gamma(\beta_i-b_i\mu+b_is)} \prod_{i=1}^n \frac{\Gamma(1-\alpha_i-a_is)}{\Gamma(1-\alpha_i+a_i\mu-a_is)} f^*(s)x^{-s}ds. \tag{46}$$

For $\rho-\mu(\sum_{i=1}^n a_i - \sum_{i=1}^m b_i) \ge 0$, the multiple composed E-K-D of the Riemann-Liouville type maps the space $\mathcal{M}_\rho^{-1}(L)$ into the space $\mathcal{M}_{\rho-\mu(\sum_{i=1}^n a_i - \sum_{i=1}^m b_i)}^{-1}(L)$ and can be represented as the Mellin-Barnes integral

$$(D_\mu f)(x) = \frac{x^{-\mu}}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \prod_{i=1}^m \frac{\Gamma(\beta_i+b_is)}{\Gamma(\beta_i+b_i\mu+b_is)} \prod_{i=1}^n \frac{\Gamma(1-\alpha_i-a_is)}{\Gamma(1-\alpha_i-a_i\mu-a_is)} f^*(s)x^{-s}ds. \tag{47}$$

The Mellin-Barnes representations (46) and (47) immediately follow from the definitions (2) and (3) of the multiple composed E-K-I and E-K-D of the Riemann-Liouville type, respectively, the Mellin-Barnes representations (35), (36), (42), (43) of the E-K-I and E-K-D, and the Parseval equality for the Mellin integral transform that ensures the representation

$$(K_2 \circ K_1)(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} k_2^*(s)k_1^*(s)f^*(s)x^{-s}ds \tag{48}$$

for a composition of any integral transforms defined in form of the Mellin convolution

$$(K_1 f)(x) = \int_0^{+\infty} k_1(x/t)g(t)\frac{dt}{t},\ (K_2 f)(x) = \int_0^{+\infty} k_2(x/t)g(t)\frac{dt}{t}.$$

The mapping properties of the multiple composed E-K-I and E-K-D of the Riemann-Liouville type on the space $\mathcal{M}_\rho^{-1}(L)$ can be derived either from the mapping properties of the E-K-I and E-K-D mentioned above or directly from the asymptotic Stirling formula (45).

Remark. Because the E-K-D of the Caputo type (15) and (16) are also the left-inverse operators to the corresponding E-K-I on the space of functions $\mathcal{M}_\rho^{-1}(L),\ \rho \ge 0$, on this space of functions, they have the following Mellin-Barnes integral representations:

$$({}_*D_\beta^{\gamma,\delta} f)(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(1+\gamma+\delta-s/\beta)}{\Gamma(1+\gamma-s/\beta)} f^*(s)x^{-s}ds, \tag{49}$$

$$({}_*P_\beta^{\tau,\alpha})f(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(\tau+\alpha+s/\beta)}{\Gamma(\tau+s/\beta)} f^*(s)x^{-s}ds. \tag{50}$$

Thus, on the space $\mathscr{M}_\rho^{-1}(L)$, the E-K-D of the Caputo type coincides with the corresponding E-K-D of the Riemann-Liouville type.

As a consequence, on the space $\mathscr{M}_\rho^{-1}(L),\ \rho \geq 0$, the multiple composed E-K-I (21) and E-K-D (22) of the Caputo type have the same mapping properties and the same Mellin-Barnes integral representations as the multiple composed E-K-I (2) and E-K-D (3) of the Riemann-Liouville type, respectively (see Theorem 1).

In what follows, we refer to the multiple composed E-K-I and E-K-D of both the Riemann-Liouville type and of the Caputo type as to the multiple composed E-K-I and E-K-D.

*Remark.*The representations (46) and (47) can be rewritten in the form

$$(L_\mu f)(x) = \frac{x^\mu}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Phi(s)}{\Phi(s-\mu)} f^*(s)x^{-s}ds, \tag{51}$$

$$(D_\mu f)(x) = \frac{x^{-\mu}}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Phi(s)}{\Phi(s+\mu)} f^*(s)f^*(s)x^{-s}ds \tag{52}$$

with the kernel function Φ given by the expression

$$\Phi(s) = \prod_{i=1}^m \Gamma(\beta_i + b_i s)\prod_{i=1}^n \Gamma(1-\alpha_i - a_i s). \tag{53}$$

The representations (51) and (52) express the essence of the multiple composed E-K-I and E-K-D and are employed for derivation of their further properties.

The representations given in the last remark easily result in the following important statement.

Theorem 2.*On the space $\mathscr{M}^{-1}(L)$, the multiple composed E-K-D is a left-inverse operator to the multiple composed E-K-I:*

$$(D_\mu\, L_\mu\, f)(x) = f(x),\ \forall f \in \mathscr{M}^{-1}(L). \tag{54}$$

The statement of the theorem immediately follows from the Mellin-Barnes representations (51) and (52) of the multiple composed E-K-I and E-K-D and the shift property (29) of the Mellin integral transform:

$$(D_\mu\, L_\mu\, f)(x) = x^{-\mu}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Phi(s)}{\Phi(s+\mu)}(L_\mu\, f)^*(s)x^{-s}ds =$$

$$x^{-\mu}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Phi(s)}{\Phi(s+\mu)}\frac{\Phi(s+\mu)}{\Phi(s)} f^*(s+\mu)x^{-s}ds =$$

$$x^{-\mu}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s+\mu)x^{-s}ds = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s)x^{-s}ds = f(x).$$

Another useful result is a compact formula for a composition of several multiple composed E-K-I or E-K-D.

Theorem 3.*The representation*

$$(L_\mu^n\, f)(x) := (L_\mu \ldots L_\mu f)(x) = (L_{n\mu} f)(x) \tag{55}$$

holds true on the space $\mathscr{M}^{-1}(L)$.

If $\rho - n\mu\,(\sum_{i=1}^n a_i - \sum_{i=1}^m b_i) \geq 0$, the representation

$$(D_\mu^n\, f)(x) := (D_\mu \ldots D_\mu f)(x) = (D_{n\mu} f)(x) \tag{56}$$

hold true on the space $\mathscr{M}_\rho^{-1}(L)$.

We start with a proof of the formula (55) for a composition of two multiple composed E-K-I($n = 2$) that follows from the Mellin-Barnes representation (46) of the multiple composed E-K-I and the shift property (29) of the Mellin transform:

$$(L_\mu L_\mu f)(x) = x^\mu \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Phi(s)}{\Phi(s-\mu)}(L_\mu f)^*(s)x^{-s}ds = x^\mu \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Phi(s)}{\Phi(s-\mu)}\frac{\Phi(s+\mu)}{\Phi(s)} f^*(s+\mu)x^{-s}ds =$$

$$x^\mu \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Phi(s+\mu)}{\Phi(s-\mu)} f^*(s+\mu)x^{-s}ds = x^{2\mu}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Phi(s)}{\Phi(s-2\mu)} f^*(s)x^{-s}ds = (L_{2\mu}f)(x).$$

Evidently, in the general case, the formula (55) can be proved by the same method. The formula (56) for the compositions of the multiple composed E-K-D follows from the Mellin-Barnes representation (52) and by applying the same method as the one used to prove the formula (55). The mapping properties of the compositions of the multiple composed E-K-I and E-K-D follow from Theorem 1.

In the rest of this section, we introduce an integral transform of the Mellin convolution type associated with the multiple composed E-K-I and derive some operational relations.

Definition 4. *The generalized Obrechkoff-Stieltjes transform is defined as the following Mellin-Barnes integral:*

$$(\mathscr{O}\mathscr{S} f)(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \prod_{i=1}^{m}\Gamma(\beta_i+b_i s)\prod_{i=1}^{n}\Gamma(1-\alpha_i-a_i s) f^*(s)x^{-s}ds,\ f\in \mathscr{M}^{-1}(L). \tag{57}$$

The generalized Obrechkoff-Stieltjes transform is a particular case of the generalized H-transform introduced and investigated in [5]. In its turn, both the Obrechkoff transform and the Stieltjes transform are particular cases of the generalized Obrechkoff-Stieltjes transform (57).

Indeed, the generalized Obrechkoff transform [21,19,5] can be represented as the Mellin convolution type integral

$$(\mathscr{O} f)(x) = \int_0^\infty H_{n,0}^{0,n}\left(\frac{x}{t}\middle|\begin{matrix}(\alpha,a)_{1,n}\\ -\end{matrix}\right) f(t)\frac{dt}{t},\ x>0, \tag{58}$$

where $H_{n,0}^{0,n}$ is a particular case of the Fox H-function with the Mellin integral transform given by the formula ([1,5])

$$H_{n,0}^{0,n}\left(x\middle|\begin{matrix}(\alpha,a)_{1,n}\\ -\end{matrix}\right) \stackrel{\mathfrak{M}}{\rightarrow} \prod_{i=1}^{n}\Gamma(1-\alpha_i-a_i s). \tag{59}$$

Because the right-hand side of (58) is a Mellin convolution, the Parseval formula for the Mellin integral transform along with the formula (59) immediately lead to the Mellin-Barnes representation of the generalized Obrechkoff transform on the space of functions $\mathscr{M}^{-1}(L)$

$$(\mathscr{O} f)(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\prod_{i=1}^{n}\Gamma(1-\alpha_i-a_i s) f^*(s)x^{-s}ds \tag{60}$$

that is a particular case of the formula (57).

In the literature, another form of the generalized Obrechkoff transform was often employed [21,19,5]:

$$(\mathscr{O} f)(x) = \int_0^\infty \Phi_n\left(\frac{x}{t}\mid(\alpha_i,a_i)_{1,n}\right) f(t)\frac{dt}{t} \tag{61}$$

with the kernel function

$$\Phi_n(z|(\alpha_i,a_i)_{1,n}) = \frac{z^{\frac{\alpha_n-1}{a_n}}}{a_n}\int_0^\infty\cdots\int_0^\infty \exp\left(-\sum_{i=1}^{n-1}t_i - z^{-\frac{1}{a_n}}\prod_{i=1}^{n-1}t_i^{-\frac{a_i}{a_n}}\right)\times \tag{62}$$

$$\prod_{i=1}^{n-1}t_i^{-a_i\frac{1-\alpha_n}{a_n}-\alpha_i}dt_1\dots dt_{n-1}.$$

Another important particular case of the generalized Obrechkoff-Stieltjes transform (57) is the generalized Stieltjes integral transform [5]:

$$(\mathscr{S}_\beta^\alpha f)(x) = \frac{x^{\frac{\alpha}{\beta}}}{\beta}\int_0^\infty \frac{f(t)t^{\frac{1-\alpha}{\beta}}}{x^{\frac{1}{\beta}}+t^{\frac{1}{\beta}}}\frac{dt}{t},\ x>0,\ \beta>0. \tag{63}$$

The conventional Stieltjes transform is a particular case of (63) for the parameter values $\alpha = 0$ and $\beta = 1$. On the space of functions $\mathscr{M}^{-1}(L)$, the generalized Stieltjes integral transform (63) has the Mellin-Barnes integral representation ([5])

$$(\mathscr{S}_\beta^\alpha f)(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(\alpha+\beta s)\Gamma(1-\alpha-\beta s) f^*(s) x^{-s} ds. \tag{64}$$

Thus, this integral transform is a particular case of the generalized Obrechkoff-Stieltjes transform (57).

Now, let us derive some other representations of the generalized Obrechkoff-Stieltjes transform (57) and investigate its connection to the multiple composed E-K-I.

The known formula for the Mellin integral transform of the Fox H-function [1,5], the Parseval formula for the Mellin integral transform, and the asymptotic behavior of the H-function [1,5] immediately lead to a convolution form of the generalized Obrechkoff-Stieltjes transform with the Fox H-function in the kernel:

$$(\mathscr{O}\mathscr{S} f)(x) = \int_0^\infty H^{m,n}_{n,m}\left(\frac{x}{t}\left|\begin{matrix}(\alpha,a)_{1,n}\\(\beta,b)_{1,m}\end{matrix}\right.\right) f(t)\frac{dt}{t},\ x>0. \tag{65}$$

Otherwise, in the case $\min\limits_{1\le i\le n}(1-\alpha_i)/a_i > \max\limits_{1\le i\le m} -\beta_j/b_i$, we can substitute the well known integral representation of the Euler Gamma-function

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\,dt,\ \ \Re(s)>0 \tag{66}$$

for the functions $\Gamma(\beta_i+b_i s),\ i=1,\dots,m$ and $\Gamma(1-\alpha_i-a_i s),\ i=1,\dots,n$ into the right-hand side of the formula (57) and get the following chain of equalities:

$$(\mathscr{O}\mathscr{S} f)(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\prod_{i=1}^m\Gamma(\beta_i+b_i s)\prod_{i=1}^n\Gamma(1-\alpha_i-a_i s) f^*(s)x^{-s}ds = \tag{67}$$

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\int_0^\infty\cdots\int_0^\infty \exp\left(-\sum_{i=1}^m u_i\right)\prod_{i=1}^m u_i^{\beta_i+b_i s-1}du_1\dots du_m\times$$

$$\int_0^\infty\cdots\int_0^\infty \exp\left(-\sum_{i=1}^n v_i\right)\prod_{i=1}^n v_i^{-\alpha_i-a_i s}dv_1\dots dv_n\, f^*(s)x^{-s}ds =$$

$$\int_0^\infty\cdots\int_0^\infty \exp\left(-\sum_{i=1}^m u_i-\sum_{i=1}^n v_i\right)\prod_{i=1}^m u_i^{\beta_i-1}\prod_{i=1}^n v_i^{-\alpha_i}du_1\dots du_m dv_1\dots dv_n \frac{1}{2\pi i}\int_\sigma f^*(s)\left(x\prod_{i=1}^m u_i^{-b_i}\prod_{i=1}^n v_i^{a_i}\right)^{-s}ds =$$

$$\int_0^\infty\cdots\int_0^\infty \exp\left(-\sum_{i=1}^m u_i-\sum_{i=1}^n v_i\right)\prod_{i=1}^m u_i^{\beta_i-1}\prod_{i=1}^n v_i^{-\alpha_i} f\left(x\prod_{i=1}^m u_i^{-b_i}\prod_{i=1}^n v_i^{a_i}\right)du_1\dots du_m dv_1\dots dv_n.$$

The multiple composed E-K-I and the generalized Obrechkoff-Stieltjes transform are connected each to other as formulated in the following theorem.

Theorem 4. *For the generalized Obrechkoff-Stieltjes transform and the multiple composed E-K-I the following operational relation holds true:*

$$(\mathscr{O}\mathscr{S} L_\mu f)(x) = x^\mu(\mathscr{O}\mathscr{S} f)(x),\ f\in\mathscr{M}^{-1}(L). \tag{68}$$

To prove the theorem, the Mellin-Barnes representations of the generalized Obrechkoff-Stieltjes transform and the multiple composed E-K-I are employed:

$$(\mathscr{O}\mathscr{S} L_\mu f)(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Phi(s)\,(L_\mu f)^*(s)x^{-s}ds = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Phi(s)\,\frac{\Phi(s+\mu)}{\Phi(s)}f^*(s+\mu)x^{-s}ds =$$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Phi(s+\mu)f^*(s+\mu)x^{-s}ds = x^\mu\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Phi(s)f^*(s)x^{-s}ds = x^\mu(\mathscr{O}\mathscr{S} f)(x).$$

4 Convolutions of the multiple composed Erdélyi-Kober fractional integrals

For development of a Mikusiński type operational calculus for a certain differential operator, a convolution for its right-inverse integral operator in the sense of Dimovski is required [30]. In general, derivation of these convolutions is a difficult problem. However, in [5] a construction of the convolutions for the generalized H-transforms in form of the double Mellin-Barnes integrals was suggested. Of course, this form of convolutions is not immediately appropriate for development of operational calculi. It is still a good starting point and we employ it in this paper for construction of a one-parameter family of convolutions for the multiple composed E-K-I in the sense of Dimovski.

As already mentioned in the previous section, the generalized Obrechkoff-Stieltjes transform (57) is a particular case of the generalized H-transform addressed in [5]. In this section, we first derive a family of convolutions for its modification in the form

$$(\mathscr{O}\mathscr{S}_\lambda f)(x) = x^\lambda (\mathscr{O}\mathscr{S} f)(x), \lambda \in \mathbb{R}. \tag{69}$$

Theorem 5.*Let the condition*

$$\min_{i=1,\dots,n} \frac{\alpha_i - 1}{a_i} > \max_{i=1,\dots,m} -\frac{\beta_i}{b_i} \tag{70}$$

hold true and $f, g \in \mathscr{M}_{c,\rho}^{-1}(L)$. *Then, the binary operations*

$$(f \overset{\lambda}{*} g)(x) = x^\lambda (\prod_{i=1}^m \mathscr{U}_i \prod_{i=1}^n \mathscr{V}_i f \circ g)(x) \tag{71}$$

build a one-parameter family of convolutions for the generalized Obrechkoff-Stieltjes transform (69) with the power function weight, i.e., the convolution property

$$(\mathscr{O}\mathscr{S}_\lambda (f \overset{\lambda}{*} g))(x) = (\mathscr{O}\mathscr{S}_\lambda f)(x)(\mathscr{O}\mathscr{S}_\lambda g)(x) \tag{72}$$

holds true.

In the formula (71), the operators $\mathscr{U}_i,\ i = 1,\dots m$ *and* $\mathscr{V}_i,\ i = 1,\dots n$ *and the binary operation* $f \circ g$ *are defined as follows:*

$$(\mathscr{U}_i \phi)(x) = \begin{cases} (K_{1/b_i}^{2\beta_i, -\lambda b_i - \beta_i} \phi)(x) & \text{for } -\lambda b_i - \beta_i \geq 0, \\ \phi(x), & \text{for } -\lambda b_i - \beta_i = 0, \\ (P_{1/b_i}^{\beta_i - \lambda b_i, \lambda b_i + \beta_i} \phi)(x) & \text{for } -\lambda b_i - \beta_i < 0, \end{cases} \tag{73}$$

$$(\mathscr{V}_i \phi)(x) = \begin{cases} (I_{1/a_i}^{1-2\alpha_i, \lambda a_i + \alpha_i - 1} \phi)(x) & \text{for } \lambda a_i + \alpha_i - 1 \geq 0, \\ \phi(x) & \text{for } \lambda a_i + \alpha_i - 1 = 0, \\ (D_{1/a_i}^{-\alpha_i + \lambda a_i, -\lambda a_i - \alpha_+ 1} \phi)(x) & \text{for } \lambda a_i + \alpha_i - 1 < 0, \end{cases} \tag{74}$$

$$(f \circ g)(x) = \int_0^1 \cdots \int_0^1 \prod_{i=1}^m u_i^{\beta_i - 1}(1-u_i)^{\beta_i - 1} \prod_{i=1}^n v_i^{-\alpha_i}(1-v_i)^{-\alpha_i} \times \tag{75}$$

$$f\left(x \prod_{i=1}^m u_i^{-b_i} \prod_{i=1}^n v_i^{a_i}\right) g\left(x \prod_{i=1}^m (1-u_i)^{-b_i} \prod_{i=1}^n (1-v_i)^{a_i}\right) du_1 \dots du_m dv_1 \dots dv_n.$$

To prove the theorem, we first mention that the integral transform (69) can be interpreted as the H-transform with a power function weight. In [5], a one parameter convolution family for the H-transform with a power function weight was suggested in form of a double Mellin-Barnes integral. Specifying these results for the case of the integral transform (69), we can write down its convolutions as follows:

$$(f \overset{\lambda}{*} g)(x) = \frac{x^\lambda}{(2\pi i)^2} \int_{\gamma - i\infty}^{\gamma + i\infty} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Phi(s)\Phi(t)}{\Phi(s+t-\lambda)} f^*(s) g^*(t) x^{-s-t} \, ds dt, \tag{76}$$

where the kernel function Φ is given by (53). The general theory of convolutions for the generalized H-transform presented in [5] ensures that the operation $\overset{\lambda}{*}$ possesses the properties of bi-linearity, commutativity, and associativity on some subspaces of $\mathscr{M}_{c,\rho}^{-1}(L)$. Moreover, the convolution property (72) for the generalized Obrechkoff-Stieltjes transform with the power function weight holds true.

Now, let us bring (76) to the form (71). To do this, we first transform the kernel of the convolution (76) as follows:

$$\frac{\Phi(s)\Phi(t)}{\Phi(s+t-\lambda)} = \frac{\prod_{i=1}^{m}(\Gamma(\beta_i+b_is)\Gamma(\beta_i+b_it))\prod_{i=1}^{n}(\Gamma(1-\alpha_i-a_is)\Gamma(1-\alpha_i-a_it))}{\prod_{i=1}^{m}\Gamma(\beta_i+b_i(s+t-\lambda))\prod_{i=1}^{n}\Gamma(1-\alpha_i-a_i(s+t-\lambda))} =$$

$$\prod_{i=1}^{m}\frac{\Gamma(2\beta_i+b_i(s+t))}{\Gamma(\beta_i+b_i(s+t-\lambda))}\prod_{i=1}^{n}\frac{\Gamma(2-2\alpha_i-a_i(s+t))}{\Gamma(1-\alpha_i-a_i(s+t-\lambda))}\times$$

$$\prod_{i=1}^{m}\frac{\Gamma(\beta_i+b_is)\Gamma(\beta_i+b_it)}{\Gamma(2\beta_i+b_i(s+t))}\prod_{i=1}^{n}\frac{\Gamma(1-\alpha_i-a_is)\Gamma(1-\alpha_i-a_it)}{\Gamma(2-2\alpha_i-a_i(s+t))} =$$

$$\prod_{i=1}^{m}\frac{\Gamma(2\beta_i+\lambda b_i+b_i(s+t-\lambda))}{\Gamma(\beta_i+b_i(s+t-\lambda))}\prod_{i=1}^{n}\frac{\Gamma(2-2\alpha_i-\lambda a_i-a_i(s+t-\lambda))}{\Gamma(1-\alpha_i-a_i(s+t-\lambda))}\times$$

$$\prod_{i=1}^{m}\frac{\Gamma(\beta_i+b_is)\Gamma(\beta_i+b_it)}{\Gamma(2\beta_i+b_i(s+t))}\prod_{i=1}^{n}\frac{\Gamma(1-\alpha_i-a_is)\Gamma(1-\alpha_i-a_it)}{\Gamma(2-2\alpha_i-a_i(s+t))} =$$

$$\prod_{i=1}^{m}\Phi_i(s+t-\lambda)\prod_{i=1}^{n}\Psi_i(s+t-\lambda)\prod_{i=1}^{m}B(\beta_i+b_is,\beta_i+b_it)\prod_{i=1}^{n}B(1-\alpha_i-a_is,1-\alpha_i-a_it),$$

where we used the known relation

$$B(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \tag{77}$$

between the Euler Gamma-function and the Euler Beta-function and the notations

$$\Phi_i(\tau) = \frac{\Gamma(2\beta_i+\lambda b_i+b_i\tau)}{\Gamma(\beta_i+b_i\tau)}, \tag{78}$$

$$\Psi_i(\tau) = \frac{\Gamma(2-2\alpha_i-\lambda a_i-a_i\tau)}{\Gamma(1-\alpha_i-a_i\tau)}. \tag{79}$$

The function Φ_i can be interpreted as the kernel in the Mellin-Barnes representation of the right-sided E-K-I (36) (for $-\lambda b_i-\beta_i \geq 0$) or the right-sided E-K-D (43) (for $-\lambda b_i-\beta_i < 0$), respectively, whereas the function Ψ_i is the kernel in the Mellin-Barnes representation of the left-sided E-K-I (35) (for $\lambda a_i+\alpha_i-1 \geq 0$) or the left-sided E-K-D (42) (for $\lambda a_i+\alpha_i-1<0$), respectively.

The shift property (29) of the Mellin integral transform allows to represent (76) in the form

$$(f \overset{\lambda}{*} g)(x) = x^{\lambda}(\prod_{i=1}^{m}\mathscr{U}_i\prod_{i=1}^{n}\mathscr{V}_i f\circ g)(x), \tag{80}$$

where the operators $\mathscr{U}_i$ are defined by (73), the operators $\mathscr{V}_i$ by (74), and the binary operation $f\circ g$ has the following Mellin-Barnes representation:

$$(f\circ g)(x) = \frac{1}{(2\pi i)^2}\int_{\gamma-i\infty}^{\gamma+i\infty}\int_{\gamma-i\infty}^{\gamma+i\infty}\prod_{i=1}^{m}B(\beta_i+b_is,\beta_i+b_it)\prod_{i=1}^{n}B(1-\alpha_i-a_is,1-\alpha_i-a_it)\,f^*(s)g^*(t)x^{-s-t}\,dsdt. \tag{81}$$

The condition (70) ensures that there exists a $\gamma = \Re(s) = \Re(t)$ such that

$$\beta_i+b_i\gamma>0,\ i=1,\ldots,m,\ \ 1-\alpha_i-a_i\gamma>0,\ i=1,\ldots,n,$$

so for this γ, we can employ the representation

$$B(s,t) = \int_0^1 \tau^{s-1}(1-\tau)^{t-1}\,d\tau,\ \Re(s)>0,\ \Re(t)>0 \tag{82}$$

for all functions $B(\beta_i+b_is,\beta_i+b_it)$ and $B(1-\alpha_i-a_is,1-\alpha_i-a_it)$:

$$B(\beta_i+b_is,\beta_i+b_it) = \int_0^1 u_i^{\beta_i+b_is-1}(1-u_i)^{\beta_i+b_it-1}\,du_i,\ i=1,\ldots,m, \tag{83}$$

$$B(1-\alpha_i-a_is,1-\alpha_i-a_it)=\int_0^1 v_i^{-\alpha_i-a_is}(1-v_i)^{-\alpha_i-a_t}\,dv_i,\ i=1,\dots,n. \tag{84}$$

The integral representations (83) and (84) are now substituted into the right-hand side of (81). In the obtained multiple integral, the orders of integration can be interchanged (the corresponding integrals absolutely converge for $f,g\in\mathscr{M}_{c,\rho}^{-1}(L)$ under some restrictions on c and ρ). Then we get:

$$(f\circ g)(x)=\frac{1}{(2\pi i)^2}\int_{\gamma-i\infty}^{\gamma+i\infty}\int_{\gamma-i\infty}^{\gamma+i\infty}\int_0^1\cdots\int_0^1\prod_{i=1}^m u_i^{\beta_i+b_is-1}(1-u_i)^{\beta_i+b_it-1}\,du_1\dots du_m\times$$

$$\int_0^1\cdots\int_0^1\prod_{i=1}^n v_i^{-\alpha_i-a_is}(1-v_i)^{-\alpha_i-a_t}\,dv_1\dots dv_n\,f^*(s)g^*(t)x^{-s-t}\,dsdt=$$

$$\int_0^1\cdots\int_0^1\prod_{i=1}^m(u_i(1-u_i))^{\beta_i-1}\prod_{i=1}^n(v_i(1-v_i))^{-\alpha_i}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}f^*(s)\left(x\prod_{i=1}^m u_i^{-b_i}\prod_{i=1}^n v_i^{a_i}\right)^{-s}ds\times$$

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}g^*(t)\left(x\prod_{i=1}^m(1-u_i)^{-b_i}\prod_{i=1}^n(1-v_i)^{a_i}\right)^{-t}dt\,du_1\dots du_m dv_1\dots dv_n=$$

$$\int_0^1\cdots\int_0^1\prod_{i=1}^m u_i^{\beta_i-1}(1-u_i)^{\beta_i-1}\prod_{i=1}^n v_i^{-\alpha_i}(1-v_i)^{-\alpha_i}f\left(x\prod_{i=1}^m u_i^{-b_i}\prod_{i=1}^n v_i^{a_i}\right)\times$$

$$g\left(x\prod_{i=1}^m(1-u_i)^{-b_i}\prod_{i=1}^n(1-v_i)^{a_i}\right)du_1\dots du_m dv_1\dots dv_n.$$

Combining the last formula with the representation (80) completes the proof of the theorem.

In the rest of this section, we discuss an important relation between the convolutions (71) for the generalized Obrechkoff-Stieltjes transform $\mathscr{OS}_\lambda$ with the power function weight and the multiple composed E-K-I L_μ. According to Theorem 4, these operators are connected through the operational relation

$$(\mathscr{OS}_\lambda L_\mu f)(x)=x^\mu(\mathscr{OS}_\lambda f)(x), \tag{85}$$

so the multiple composed E-K-I L_μ is a generating operator for the generalized Obrechkoff-Stieltjes transform $\mathscr{OS}_\lambda$ with the power function weight [31,5]. Now, we formulate the following important theorem:

Theorem 6. *The binary operations (71) can be interpreted as convolutions of the multiple composed E-K-I (2) or (21) in the sense of Dimovski for the functions from $\mathscr{M}_{c,\rho}^{-1}(L)$ with some restrictions on c and ρ and under the condition (70).*

To prove the theorem, we refer to the Theorem 14.2 from [5] that states that a convolution for an integral transform is always a convolution for its generating operator in the sense of Dimovski. According to the formula (85), the multiple composed E-K-I L_μ is a generating operator for the generalized Obrechkoff-Stieltjes transform $\mathscr{OS}_\lambda$ with the power function weight that immediately leads to the statement of the theorem.

For the readers' convenience, we recall the definition of a convolution in the sense of Dimovski: Let $\mathscr{V}$ be a linear vector space and let $L:\mathscr{V}\to\mathscr{V}$ be a linear operator. A bilinear, commutative, and associative operation $*:\mathscr{V}\times\mathscr{V}\to\mathscr{V}$ is called a convolution of the operator L in the sense of Dimovski if the relation

$$L(x*y)=(Lx)*y \tag{86}$$

holds true $\forall x,y\in\mathscr{V}$.

Say, the convolution

$$(f*g)(x)=\int_0^x f(t)g(x-t)\,dt$$

for the Laplace transform

$$(Lf)(x)=\int_0^{+\infty}f(t)e^{-xt}\,dt$$

is also a convolution of the Volterra integration operator

$$(I^1_{0+}f)(x) = \int_0^x f(t)\,dt$$

in the sense of Dimovski because of the operational relation

$$(LI^1_{0+}f)(x) = x^{-1}(Lf)(x).$$

In [30], a general schema for construction of an operational calculus of Mikusiński type for a left-inverse (differential) operator to an (integral) operator based an a convolution in the sense of Dimovski was presented. Thus, the convolutions (71) can be used to develop a Mikusiński type operational calculus for the multiple composed E-K-D (3) and its Caputo type modification (22).

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