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Marshall-Olkin Alpha Power Inverse Weibull Distribution:Non Bayesian and Bayesian Estimations

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Abstract: The present paper aims to introduce an extension of the inverse Weibull (IW) distribution which offers a more flexible distribution for modeling lifetime data. We extend the inverse Weibull distribution using Marshall Olkin alpha power (MOAP) method. Its characterization and statistical properties, such as reliability, moments, entropy and order statistics are obtained. Moreover, the estimation of the MOAPIW distribution parameters is discussed using the non-Bayesian and Bayesian estimations method. Finally, a real data application illustrates the performance of the distribution. In addition, comparisons to other distributions are carried out to illustrate the flexibility of the proposed distribution.

Keywords: Marshall Olkin transformation, Alpha power transformation, Inverse Weibull distribution, Reliability, Non-Bayesian estimation, Bayesian estimation, Simulation.

1 Introduction

Marshall and Olkin [1] have suggested a new family of distributions which count on adding a parameter to a family of distributions and called them extended distributions. Using the cumulative function of any distribution, $F(x)$, the cumulative function of the new family of distributions is given by

$$G(x;\theta) = \frac{F(x)}{\theta - (\theta - 1)F(x)}, \qquad -\infty < x < \infty, \theta > 0. \tag{1}$$

The M-O extended distribution offers a wide range of behaviors from which basic distributions are derived. For more details, see Ghitany [2], Ghitany et al. [3], Alice and Jose [4], Okasha and Kayid [5] and Okasha et al. [6,7,8].

On the other hand, Mahdavi and Kundu [9] proposed a transformation of the baseline (CDF) by adding a new parameter to obtain a family of distributions. The proposed method is called alpha power transformation (APT). If $F(x)$ is a cumulative density function of any distribution, then the $G_{APT}(x)$ is cumulative density function distribution

$$G_{APT}(x) = \begin{cases} \frac{\alpha^{F(x)}-1}{\alpha-1}, & \alpha > 0, \alpha \neq 1 \\ F(x), & \alpha = 1. \end{cases} \tag{2}$$

A lot of work has been done on APT distribution, for example, Basheer [10] has investigated the alpha power inverse Weibull distribution with reliability application and Nassar et al. [11] have explored alpha power Weibull distribution.

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Recently, Nassar et al. [12] proposed a new method to introduce an extra parameter to a family of distributions for more flexibility. The proposed method is called Marshall Olkin alpha power (MOAP) family. Substituting the CDF given by (2) in CDF of (1) we get a new distribution denoted as MOAP distribution with CDF given by

$$G_{MOAP}(x) = \begin{cases} \frac{\alpha^{F(x)}-1}{(\alpha-1)\theta-(\theta-1)(\alpha^{F(x)}-1)}, & \alpha>0, \alpha\neq 1 \\ F(x), & \alpha=1, \end{cases} \tag{3}$$

its corresponding probability density function (PDF) is given by

$$g_{MOAP}(x) = \begin{cases} \frac{(\alpha-1)\theta\log(\alpha)\alpha^{F(x)}f(x)}{[(\alpha-1)\theta-(\theta-1)(\alpha^{F(x)}-1)]^2}, & \alpha>0, \alpha\neq 1 \\ f(x), & \alpha=1. \end{cases} \tag{4}$$

Basheer [13] has studied the Marshall Olkin alpha power inverse exponential distribution.

The present paper aims to propose a new distribution model called MOAPIW distribution based on the method of MOAPT. The rest of the paper is organized, as follows: In Section 2, we define our proposed model and its special cases are presented. In Section 3, its reliability analysis is presented. In Section 4, its statistical properties are given. The parameters of this distribution are estimated by non-Bayesian and Bayesian estimations method in Section 5. Finally, the proposed model is applied on real data and the results are presented in Section 6.

2 New model

In this section, we present the MOAPIW distribution and some of its sub-models.

2.1 MOAPIW specification

We apply the MOAP family to the inverse Weibull distribution. The generated distribution is called the MOAPIW distribution and it is represented by the random variable X $\sim$ MOAPIW$(\alpha,\lambda,\beta,\theta)$. The CDF and PDF of the inverse Weibull distribution are (for $x>0,\lambda>0,\beta>0$) $F(x)=e^{-\lambda x^{-\beta}}$ and $f(x)=\lambda\beta x^{-(\beta+1)}e^{-\lambda x^{-\beta}}$, respectively. The CDF of X follows from (3) as

$$G_{MOAPIW}(x) = \begin{cases} \frac{\alpha^{e^{-\lambda x^{-\beta}}}-1}{(\alpha-1)\theta-(\theta-1)(\alpha^{e^{-\lambda x^{-\beta}}}-1)}, & \alpha>0, \alpha\neq 1 \\ e^{-\lambda x^{-\beta}}, & \alpha=1, \end{cases} \tag{5}$$

its corresponding PDF is given by

$$g_{MOAPIW}(x) = \begin{cases} \frac{(\alpha-1)\lambda\beta\theta\log(\alpha)x^{-(\beta+1)}e^{-\lambda x^{-\beta}}\alpha^{e^{-\lambda x^{-\beta}}}}{[(\alpha-1)\theta-(\theta-1)(\alpha^{e^{-\lambda x^{-\beta}}}-1)]^2}, & \alpha>0, \alpha\neq 1 \\ \lambda\beta x^{-(\beta+1)}e^{-\lambda x^{-\beta}}, & \alpha=1, \end{cases} \tag{6}$$

where $\lambda,\beta,\theta>0$.

Using the generalized binomial expansion and the power series,

$$(1-z)^{-2}=\sum_{k=0}^{\infty}(k+1)z^k, |z|<1, \qquad \alpha^z=\sum_{m=0}^{\infty}\frac{(\log(\alpha))^m z^m}{m!},$$

we obtain a useful linear representation for the PDF (6) (if α ¿ 0, $\alpha\neq 1$) as

$$g_{MOAPIW}(x) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{\infty} W_{k,j,m}\lambda\beta(m+1)x^{-(\beta+1)}e^{-(m+1)\lambda x^{-\beta}}, \tag{7}$$

where

$$W_{k,j,m} = \begin{cases} (-1)^j\binom{k}{j}(k+1)\frac{(\theta-1)^k(k-j+1)^m(\log(\alpha))^{m+1}}{\theta^{k+1}(\alpha-1)^{k+1}(m+1)!}, & \theta > 1 \\ (-1)^j\binom{k}{j}(k+1)\frac{(1-\theta)^k(j+1)^m(\log(\alpha))^{m+1}}{\theta^{k+1}(\alpha-1)^{k+1}(m+1)!}, & 0 < \theta < 1. \end{cases}$$

Figure 1 presents graphical representation of PDF for different values of α, λ , β and θ.

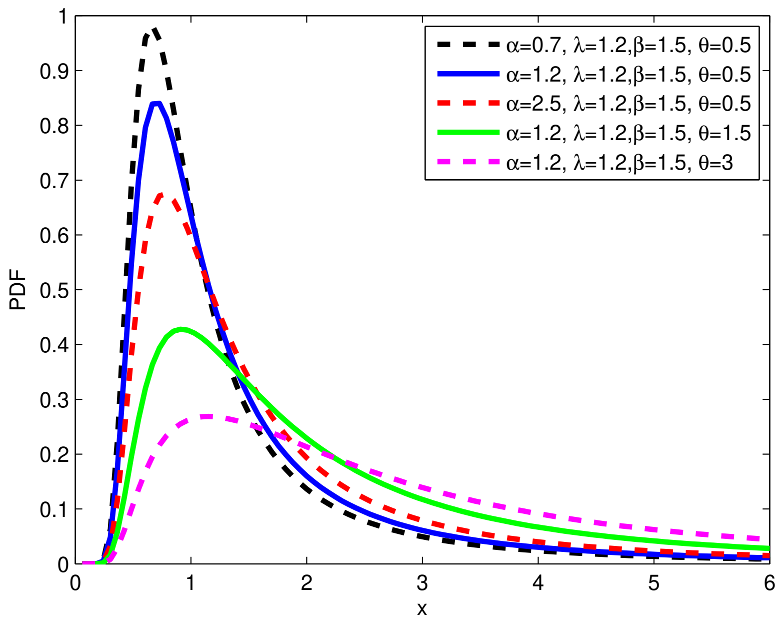

Fig. 1: PDF of the MOAPIW distribution.

2.2 MOAPIW sub-models

From Eq. (5), we study the sub-models of MOAPIW distribution.
Table 1 summarizes the sub-models of the MOAPIW distribution.

Table 1: Sub-models of the MOAPIW distribution.

Distributions	α	λ	β	θ
Marshall Olkin inverse Weibull (MOIW)	1	λ	β	θ
Marshall Olkin Frechet (MOF)	1	1	β	θ
Marshall Olkin inverse Rayleigh (MOIR)	1	λ	2	θ
Marshall Olkin inverse exponential (MOIE)	1	λ	1	θ
Alpha power inverse Weibull (APIW)	α	λ	β	1
Alpha power Frechet (APF)	α	1	β	1
Alpha power inverse Rayleigh (APIR)	α	λ	2	1
Alpha power inverse exponential (APIE)	α	λ	1	1
Inverse Weibull (IW)	1	λ	β	1
Frechet (F)	1	1	β	1
Inverse Rayleigh (IR)	1	λ	2	1
Inverse exponential (IE)	1	λ	1	1

3 Reliability analysis

The reliability function (survival function) of MOAPIW distribution is given by

$$R_{MOAPIW}(x) = \begin{cases} \frac{\alpha\theta(1-\alpha^{e^{-\lambda x^{-\beta}}-1})}{(\alpha-1)\theta-(\theta-1)(\alpha^{e^{-\lambda x^{-\beta}}}-1)}, & \alpha > 0, \alpha \neq 1 \\ 1-e^{-\lambda x^{-\beta}}, & \alpha = 1. \end{cases} \tag{8}$$

3.1 Hazard rate function

The hazard rate (HR) function (failure rate) of a lifetime random variable X with MOAPIW distribution is given by

$$h_{MOAPIW}(x) = \begin{cases} \frac{(\alpha-1)\lambda\beta\log(\alpha)x^{-(\beta+1)}e^{-\lambda x^{-\beta}}\alpha^{e^{-\lambda x^{-\beta}}-1}}{[(\alpha-1)\theta-(\theta-1)(\alpha^{e^{-\lambda x^{-\beta}}}-1)](1-\alpha^{e^{-\lambda x^{-\beta}}-1})}, & \alpha > 0, \alpha \neq 1 \\ \frac{\lambda\beta x^{-(\beta+1)}e^{-\lambda x^{-\beta}}}{1-e^{-\lambda x^{-\beta}}}, & \alpha = 1. \end{cases} \tag{9}$$

Figure 2 presents graphical representation of HRF for different values of α, λ , β and θ.

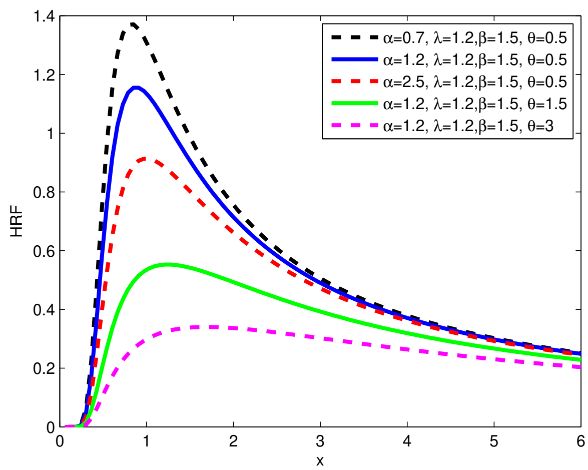

Fig. 2: HRF of the MOAPIW distribution.

3.2 Reversed hazard rate function

The reversed hazard rate (RHR) function of a lifetime random variable X with MOAPIW distribution is given by

$$r_{MOAPIW}(x) = \begin{cases} \frac{(\alpha-1)\theta\lambda\beta\log(\alpha)x^{-(\beta+1)}e^{-\lambda x^{-\beta}}\alpha^{e^{-\lambda x^{-\beta}}}}{[(\alpha-1)\theta-(\theta-1)(\alpha^{e^{-\lambda x^{-\beta}}}-1)](\alpha^{e^{-\lambda x^{-\beta}}}-1)}, & \alpha>0, \alpha\neq 1 \\ \lambda\beta x^{-(\beta+1)}, & \alpha=1. \end{cases} \quad (10)$$

Figure 3 presents graphical representation of RHRF for λ=1.2 and different values of α, β and θ.

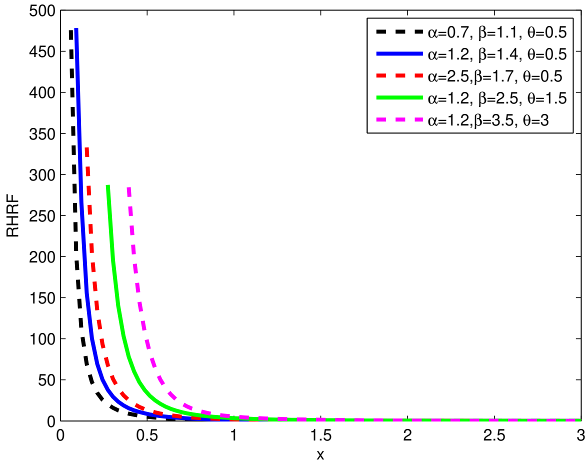

Fig. 3: RHRF of the MOAPIW distribution.

3.3 Mean residual life

The mean residual life (MRL) function describes the aging process, so it is very important in reliability and survival analysis. The MRL function of a lifetime random variable X is given by

$$\mu(x) = \frac{1}{R(x)}\int_x^\infty tg(t)dt - x, \quad x>0.$$

Proposition 1.*The MRL function of a lifetime random variable X with MOAPIW distribution is given by*

$$\mu(x) = \frac{1}{R(x)}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{\infty} W_{k,j,m}((m+1)\lambda)^{\frac{1}{\beta}}\gamma(1-\frac{1}{\beta},(m+1)\lambda x^{-\beta}) - x, \quad \beta>1. \quad (11)$$

proof: From equation (11) of MRL, we get

$$\begin{aligned}\mu(x) &= \frac{1}{R(x)}\int_x^\infty tg(t)dt - x \\ &= \frac{1}{R(x)}\int_x^\infty t\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{\infty} W_{k,j,m}\lambda\beta(m+1)t^{-(\beta+1)}e^{-(m+1)\lambda t^{-\beta}}dt - x \\ &= \frac{1}{R(x)}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{\infty} W_{k,j,m}\int_x^\infty \lambda\beta(m+1)t^{-\beta}e^{-(m+1)\lambda t^{-\beta}}dt - x\end{aligned}$$

$$I=\int_x^{\infty}\lambda\beta(m+1)t^{-\beta}e^{-\lambda(m+1)t^{-\beta}}dt$$

put

$$y=\lambda(m+1)t^{-\beta},\qquad dy=-\lambda\beta(m+1)t^{-\beta-1}dt$$

$$(t=x,t=\infty)\Rightarrow(y=\lambda(m+1)x^{-\beta},y=0).$$

Thus

$$I=\int_0^{\lambda(m+1)x^{-\beta}}(\lambda(m+1))^{\frac{1}{\beta}}y^{-\frac{1}{\beta}}e^{-y}dy=(\lambda(m+1))^{\frac{1}{\beta}}\int_0^{\lambda(m+1)x^{-\beta}}y^{-\frac{1}{\beta}}e^{-y}dy$$
$$=(\lambda(m+1))^{\frac{1}{\beta}}\int_0^{\lambda(m+1)x^{-\beta}}y^{(1-\frac{1}{\beta})-1}e^{-y}dy=(\lambda(m+1))^{\frac{1}{\beta}}\gamma(1-\frac{1}{\beta},\lambda(m+1)x^{-\beta}),$$

$$\mu(x)=\frac{1}{R(x)}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{\infty}W_{k,j,m}((m+1)\lambda)^{\frac{1}{\beta}}\gamma(1-\frac{1}{\beta},(m+1)\lambda x^{-\beta})-x,\quad \beta>1,$$

where $\gamma(c,x)=\int_0^x t^{c-1}e^{-t}dt$, $c>0$.

Figure 4 presents graphical representations of MRL for different values of α and θ.

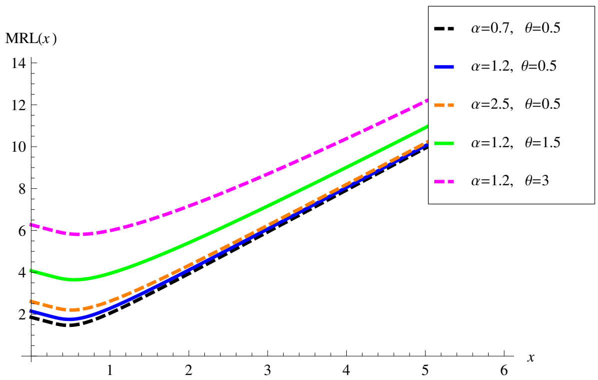

Fig. 4: MRL of the MOAPIW distribution where $\lambda=1.2,\beta=1.5$.

3.4 Mean inactivity time

The mean inactivity time (MIT) function is a recognized reliability measure which has applications, such as forensic science, reliability theory and survival analysis. The MIT function of a lifetime random variable X is given by

$$m(x)=x-\frac{1}{G(x)}\int_0^x tg(t)dt,\qquad x>0.$$

Proposition 2. *The MIT function of a lifetime random variable X with MOAPIW distribution is given by*

$$m(x)=x-\frac{1}{G(x)}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{\infty}W_{k,j,m}((m+1)\lambda)^{\frac{1}{\beta}}\Gamma(1-\frac{1}{\beta},(m+1)\lambda x^{-\beta}).\tag{12}$$

proof: Using the proof of MRL and relation $\Gamma(c,x) = \int_x^\infty t^{c-1}e^{-t}dt$, we get the above-mentioned result.

Figure 5 presents graphical representations of MIT for different values of α.

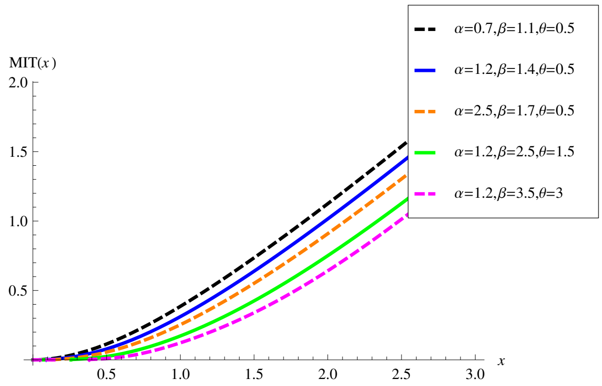

Fig. 5: MIT of the MOAPIW distribution where $\lambda = 1.2$.

Table 2 gives the values of HR, MRL, RHR and MIT for the selected values of $\lambda = 0.6$, $\beta = 1.2$ and x=0.6 and for different values of the parameters α and θ. One can observe that the values of HR are decreasing and values of MRL are increasing and the same for RHR and MIT.

Table 2: Some reliability of MOAPIW for selected values of λ=0.6 and β=1.2 at x=0.6.

α	θ	HR	MRL	RHR	MIT
0.5	0.7	1.66018	3.19571	1.6769	0.221683
	1.2	1.22167	4.14797	2.1155	0.205925
	2.5	0.72427	6.46056	2.6129	0.191684
	3.4	0.56501	7.96925	2.7721	0.187723
1.5	0.7	1.21014	4.22072	2.1078	0.206273
	1.2	0.83241	5.79752	2.4855	0.195139
	2.5	0.45950	9.58039	2.8584	0.185769
	3.4	0.35072	12.0304	2.9672	0.183282

4 Statistical properties

In this section, we study the statistical properties of the MOAPIW distribution, specially quantiles, moments, moment generating function, entropy and order statistics.

4.1 Quantiles

The quantile of any distribution is given by solving the equation

$$G(x_q) = q, \qquad 0 < q < 1, \tag{13}$$

the following proposition gives the quantile of MOAPIW distribution.

Proposition 3. *If X has MOAPIW $(\alpha, \lambda, \beta, \theta)$ distribution, then the quantile of a random variable X is given by*

$$x_q = G^{-1}(q) = \left[\frac{1}{\lambda}\log\left(\frac{\log(\alpha)}{\log(\frac{1+(\alpha\theta-1)q}{1+(\theta-1)q})}\right)\right]^{-\frac{1}{\beta}}. \tag{14}$$

proof: By assuming $z=\alpha^{e^{-\lambda x^{-\beta}}}$, the CDF of MOAPIW distribution can be written as $G(x)=\frac{z-1}{(\alpha-1)\theta-(\theta-1)(z-1)}$.
The q^{th} quantile function is obtained by solving $G(x)=q$. Again, solving for x by using $z=\alpha^{e^{-\lambda x^{-\beta}}}$, we get
$x_q=G^{-1}(q)=\left[\frac{1}{\lambda}\log\left(\frac{\log(\alpha)}{\log(\frac{1+(\alpha\theta-1)q}{1+(\theta-1)q})}\right)\right]^{-\frac{1}{\beta}}$.
Table 3 gives the quantiles for the selected values of λ=0.6, β=1.2 and for different values of the parameters α and θ.

Table 3: Quantiles of MOAPIW for selected values of λ=0.6 and β=1.2.

α	θ	Q_1	Q_2(Median)	Q_3
0.5	0.7	0.37571	0.60303	1.13313
	1.2	0.46514	0.80553	1.63018
	2.5	0.65939	1.27018	2.79332
	3.4	0.77973	1.56624	3.53873
1.5	0.7	0.46574	1.13313	1.64872
	1.2	0.59859	1.63018	2.44508
	2.5	0.89762	2.79332	4.31209
	3.4	1.08667	3.53873	5.50786

4.2 Moments

In this subsection we will present the r^{th} moments of MOAPIW distribution. Moments are important in any statistical analysis.

Proposition 4. *If X has MOAPIW $(\alpha,\lambda,\beta,\theta)$ distribution, then the r^{th} moments of a random variable X is given by*

$$E(X^r)=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{\infty}W_{k,j,m}((m+1)\lambda)^{\frac{1}{\beta}}\Gamma(1-\frac{1}{\beta}),\qquad \beta>r. \tag{15}$$

Table 4 gives the moments of MOAPIW distribution for selected values of $\lambda=0.6$ and $\beta=5$ and for different values of parameters α and θ.

Table 4: Moments of MOAPIW for selected values of λ=0.6 and β=5.

α	θ	Mean	Variance	Skewness	Kurtosis
0.5	0.7	0.95221	0.07256	3.93708	58.228
	1.2	1.02536	0.09829	3.63257	50.485
	2.5	1.14656	0.14550	3.29198	42.912
	3.4	1.20555	0.17026	3.17371	40.541
1.5	0.7	1.02739	0.09976	3.61848	50.067
	1.2	1.11448	0.13333	3.35999	44.282
	2.5	1.25725	0.19376	3.08055	38.737
	3.4	1.32612	0.22508	2.98628	37.032

4.3 Moment generating function

The moment generating function of a random variable X provides the basis of an alternative route to analytic results compared with working directly with the cumulative distribution function or probability density function of X.

Proposition 5. *If X has MOAPIW $(\alpha,\lambda,\beta,\theta)$ distribution, then the moment generating function of a random variable X is given by*

$$M_X(t)=\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{\infty}W_{k,j,m}\frac{t^r}{r!}((m+1)\lambda)^{\frac{1}{\beta}}\Gamma(1-\frac{1}{\beta}),\qquad \beta>r. \tag{16}$$

4.4 Rényi entropy

Rényi entropy of order δ is given by

$$H_\delta = \frac{1}{1-\delta}\log\left(\int_{-\infty}^{\infty}(g(x))^\delta dx\right), \quad \delta \geq 0, \quad \delta \neq 1.$$

Proposition 6. *If X has MOAPIW $(\alpha,\lambda,\beta,\theta)$ distribution, then the Rényi entropy of a random variable X is given by*

$$\begin{aligned} H_\delta &= \frac{1}{1-\delta}\log\left(\int_0^\infty\left(\frac{(\alpha-1)\theta\lambda\beta\log(\alpha)x^{-(\beta+1)}e^{-\lambda x^{-\beta}}\alpha^{e^{-\lambda x^{-\beta}}}}{[(\alpha-1)\theta-(\theta-1)(\alpha^{e^{-\lambda x^{-\beta}}}-1)]^2}\right)^\delta dx\right) \\ &= \frac{1}{1-\delta}\log\left(\frac{1}{\beta}\sum_{k_1=0}^{\infty}\sum_{j_1=0}^{k_1}\sum_{m_1=0}^{\infty}W_{k_1,j_1,m_1}(\lambda(\delta+m_1))^{-\frac{\delta\beta+\delta-1}{\beta}}\Gamma\left(\frac{\delta\beta+\delta-1}{\beta}\right)\right). \end{aligned} \quad (17)$$

Table 5 gives the moments of MOAPIW distribution for selected values of $\lambda = 0.6$ and $\beta = 5$ and for different values of parameters α and θ.

Table 5: Rényi entropy of MOAPIW for selected values of λ=0.6 and β=5.

δ	α	θ	Rényi entropy
0.4	0.5	0.7	0.46357
		1.2	0.61309
		2.5	0.81034
		3.4	0.89057
0.8	1.5	0.7	0.04763
		1.2	0.21799
		2.5	0.43723
		3.4	0.52432

4.5 Order statistics

The order statistics of a random sample $X_1,\dots,X_n$ are the sample values placed in ascending order. They are denoted by $X_{1:n},\dots,X_{n:n}$. The PDF of the i^{th} order statistics $X_{i:n}$ is given by

$$g_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}g(x)[G(x)]^{i-1}[1-G(x)]^{n-i}.$$

Proposition 7. *The PDF of the i^{th} order statistics $X_{i:n}$ of MOAPIW distribution is given by*

$$g_{i:n}(x) = \frac{\lambda\beta}{(i-1)!(n-i)!}\sum_{k_2=0}^{\infty}\sum_{j_3=0}^{n-i}\sum_{j_4=0}^{i+k_2-1}W_{k_2,j_3,j_4}x^{-(\beta+1)}e^{\lambda x^{-\beta}}\alpha^{(n-i-j_3)+(i+k_2-j_4)e^{\lambda x^{-\beta}}}. \quad (18)$$

5 Methods of estimation

For the considered distribution, we use two methods of estimation: non-Bayesian estimation (maximum likelihood, least square, weighted least square and maximum product spacing) and Bayesian estimation (squared error and LINEX) for estimating the parameters of MOAPIW distribution.

5.1 Maximum likelihood estimation

Let $x_1,...,x_n$ be a random sample from MOAPIW $\Theta = (\alpha, \lambda, \beta, \theta)$ distribution; then the likelihood function is given by:

$$\ell(x_1,...,x_n|\Theta) = \prod_{i=1}^{n} g(x_i) = \frac{(\alpha-1)^n(\log(\alpha))^n\lambda^n\beta^n\theta^n e^{-\lambda\sum_{i=1}^{n}x_i^{-\beta}}\alpha^{\sum_{i=1}^{n}e^{-\lambda x_i^{-\beta}}}(\prod_{i=1}^{n}x_i^{-(\beta+1)})}{\prod_{i=1}^{n}[(\alpha-1)\theta-(\theta-1)(\alpha^{e^{-\lambda x_i^{-\beta}}}-1)]^2}. \tag{19}$$

Then, the logarithm of the likelihood function is

$$L(x_1,...,x_n|\Theta) = n\log((\alpha-1)\log(\alpha)\lambda\beta\theta) - \lambda\sum_{i=1}^{n}x_i^{-\beta} + \log(\alpha)\sum_{i=1}^{n}e^{-\lambda x_i^{-\beta}} - (\beta+1)\sum_{i=1}^{n}\log(x_i) - 2\sum_{i=1}^{n}\log[(\alpha-1)\theta-(\theta-1)(\alpha^{e^{-\lambda x_i^{-\beta}}}-1)]. \tag{20}$$

By taking the first partial derivatives of the log-likelihood function with respect to the four parameters in Θ

$$\frac{\partial L}{\partial\alpha} = \frac{n}{\alpha-1} + \frac{n}{\alpha\log(\alpha)} + \frac{1}{\alpha}\sum_{i=1}^{n}e^{-\lambda x_i^{-\beta}} - 2\sum_{i=1}^{n}\frac{\theta-(\theta-1)e^{-\lambda x_i^{-\beta}}\alpha^{e^{-\lambda x_i^{-\beta}}-1}}{(\alpha-1)\theta-(\theta-1)(\alpha^{e^{-\lambda x_i^{-\beta}}}-1)}, \tag{21}$$

$$\frac{\partial L}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}x_i^{-\beta} - \log(\alpha)\sum_{i=1}^{n}x_i^{-\beta}e^{-\lambda x_i^{-\beta}} - 2\sum_{i=1}^{n}\frac{(\theta-1)\log(\alpha)x_i^{-\beta}e^{-\lambda x_i^{-\beta}}\alpha^{e^{-\lambda x_i^{-\beta}}}}{(\alpha-1)\theta-(\theta-1)(\alpha^{e^{-\lambda x_i^{-\beta}}}-1)}, \tag{22}$$

$$\frac{\partial L}{\partial\beta} = \frac{n}{\beta} + \lambda\sum_{i=1}^{n}x_i^{-\beta}\log(x_i) + \log(\alpha)\lambda\sum_{i=1}^{n}x_i^{-\beta}\log(x_i)e^{-\lambda x_i^{-\beta}} - \sum_{i=0}^{n}\log(x_i) + 2\sum_{i=1}^{n}\frac{(\theta-1)\lambda\log(\alpha)x_i^{-\beta}\log(x_i)e^{-\lambda x_i^{-\beta}}\alpha^{e^{-\lambda x_i^{-\beta}}}}{(\alpha-1)\theta-(\theta-1)(\alpha^{e^{-\lambda x_i^{-\beta}}}-1)}, \tag{23}$$

and

$$\frac{\partial L}{\partial\theta} = \frac{n}{\theta} - 2\sum_{i=1}^{n}\frac{\alpha-\alpha^{e^{-\lambda x_i^{-\beta}}}}{(\alpha-1)\theta-(\theta-1)(\alpha^{e^{-\lambda x_i^{-\beta}}}-1)}. \tag{24}$$

The above-mentioned equations are not closely form, so we can use any iterative procedure techniques, such as conjugate-gradient algorithms, to get the solution.

5.2 Least square estimation

Swain et al. [14] introduced the least square estimators. We use the least square procedure for estimating the parameters α, λ, β and θ of the MOAPIW distribution. The least square estimation is obtained by minimizing

$$P(\alpha,\lambda,\beta,\theta) = \sum_{i=1}^{n}\left(G(X_i,\Theta) - \frac{i}{n+1}\right)^2. \tag{25}$$

After differentiating the equation (25) with respect to parameters Θ and then equating to zero

$$\frac{\partial P_{LS}}{\partial \alpha} = 2\sum_{i=1}^{n}(G(x_i,\Theta) - \frac{i}{n+1})\left[\frac{\alpha^{e^{-\lambda x_i^{-\beta}}-1}e^{-\lambda x_i^{-\beta}}}{(\alpha-1)\theta-(\theta-1)(\alpha^{e^{-\lambda x_i^{-\beta}}}-1)} - \frac{G(x_i,\Theta)(\theta-\alpha^{e^{-\lambda x_i^{-\beta}}}e^{-\lambda x_i^{-\beta}}(\theta-1))}{(\alpha-1)\theta-(\theta-1)(\alpha^{e^{-\lambda x_i^{-\beta}}}-1)}\right], \tag{26}$$

$$\frac{\partial P_{LS}}{\partial \lambda} = 2\sum_{i=1}^{n}(G(x_i,\Theta) - \frac{i}{n+1})\left[\frac{-\alpha^{e^{-\lambda x_i^{-\beta}}}e^{-\lambda x_i^{-\beta}}x_i^{-\beta}\log(\alpha)(1+(\theta-1)G(x_i,\Theta))}{(\alpha-1)\theta-(\theta-1)(\alpha^{e^{-\lambda x_i^{-\beta}}}-1)}\right], \tag{27}$$

$$\frac{\partial P_{LS}}{\partial \beta} = 2\sum_{i=1}^{n}(G(x_i,\Theta) - \frac{i}{n+1})\left[\frac{\lambda\alpha^{e^{-\lambda x_i^{-\beta}}}e^{-\lambda x_i^{-\beta}}x_i^{-\beta}\log(\alpha)\log(x_i)(1+(\theta-1)G(x_i,\Theta))}{(\alpha-1)\theta-(\theta-1)(\alpha^{e^{-\lambda x_i^{-\beta}}}-1)}\right], \tag{28}$$

and

$$\frac{\partial P_{LS}}{\partial \theta} = 2\sum_{i=1}^{n}(G(x_i,\Theta) - \frac{i}{n+1})G(x_i,\Theta)(-\alpha+\alpha^{e^{-\lambda x_i^{-\beta}}}).$$

The above-mentioned nonlinear equations cannot be solved analytically, so the $\hat{\Theta}_{LS}$ of Θ can use any iterative procedure techniques, such as conjugate-gradient algorithms, to obtain the numerical solution.

5.3 Weighted least square estimation

Swain et al. [14] introduced the weighted least square estimators. We use the WLS procedure for estimating the parameters α,λ,β and θ of the MOAPIW distribution. The weighted least square estimation is obtained by minimizing

$$W(\alpha,\lambda,\beta,\theta) = \sum_{i=1}^{n} W_i\left(G(X_i,\Theta) - \frac{i}{n+1}\right)^2, \tag{29}$$

where

$$W_i = \frac{(n+1)^2(n+2)}{i(n-i+1)}.$$

After differentiating the equation (29) with respect to parameters Θ and then equating to zero

$$\frac{\partial W}{\partial \alpha} = 2\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}(G(x_i,\Theta) - \frac{i}{n+1})\left[\frac{\alpha^{e^{-\lambda x_i^{-\beta}}-1}e^{-\lambda x_i^{-\beta}}}{(\alpha-1)\theta-(\theta-1)(\alpha^{e^{-\lambda x_i^{-\beta}}}-1)} - \frac{G(x_i,\Theta)(\theta-\alpha^{e^{-\lambda x_i^{-\beta}}}e^{-\lambda x_i^{-\beta}}(\theta-1))}{(\alpha-1)\theta-(\theta-1)(\alpha^{e^{-\lambda x_i^{-\beta}}}-1)}\right], \tag{30}$$

$$\frac{\partial W}{\partial \lambda} = 2\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}(G(x_i,\Theta) - \frac{i}{n+1})\left[\frac{-\alpha^{e^{-\lambda x_i^{-\beta}}}e^{-\lambda x_i^{-\beta}}x_i^{-\beta}\log(\alpha)(1+(\theta-1)G(x_i,\Theta))}{(\alpha-1)\theta-(\theta-1)(\alpha^{e^{-\lambda x_i^{-\beta}}}-1)}\right], \tag{31}$$

$$\frac{\partial W}{\partial \beta} = 2\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}(G(x_i,\Theta) - \frac{i}{n+1}) \times\left[\frac{\lambda\alpha^{e^{-\lambda x_i^{-\beta}}}e^{-\lambda x_i^{-\beta}}x_i^{-\beta}\log(\alpha)\log(x_i)(1+(\theta-1)G(x_i,\Theta))}{(\alpha-1)\theta-(\theta-1)(\alpha^{e^{-\lambda x_i^{-\beta}}}-1)}\right], \tag{32}$$

and

$$\frac{\partial W}{\partial \theta} = 2\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} (G(x_i,\Theta) - \frac{i}{n+1}) G(x_i,\Theta)(-\alpha + \alpha^{e^{-\lambda x_i^{-\beta}}}).$$

The above-mentioned nonlinear equations cannot be solved analytically, so the $\hat{\Theta}_{WLS}$ of Θ can use any iterative procedure techniques, such as conjugate-gradient algorithms, to obtain the numerical solution.

5.4 Maximum product spacing estimation

Cheng and Amin [15] introduced maximum product spacing, as follows:

$$F = \left(\prod_{i=1}^{n+1} D_i\right)^{\frac{1}{n+1}},$$

where F is defined as the geometric mean of the product spacing function and

$$D_i = \begin{cases} D_1 = G(x_1) \\ D_i = G(x_i) - G(x_{i-1}), \quad i = 2,...,n \\ D_{n+1} = 1 - G(x_n). \end{cases} \tag{33}$$

such that $\sum D_i = 1$, then the product spacing function is

$$F = \left(G(x_1,\Theta)(1 - G(x_n,\Theta)) \prod_{i=2}^{n} [G(x_{i-1},\Theta) - G(x_i,\Theta)]\right)^{\frac{1}{n+1}},$$

the natural logarithm of the product spacing function is

$$\log(F) = \frac{1}{n+1}\left[\log(G(x_1,\Theta)) + \log(1 - G(x_n,\Theta)) + \sum_{i=2}^{n} \log(G(x_{i-1},\Theta) - G(x_i,\Theta))\right]. \tag{34}$$

To obtain the normal equations for the unknown parameters, we differentiate equation (34) partially with respect to the parameters Θ and equate to zero

$$\frac{\partial \log(F)}{\partial \theta} = \frac{1}{n+1}\left[\varsigma(x_1,\Theta) - \frac{\varsigma(x_n,\Theta)G(x_n,\Theta)}{1 - G(x_n,\Theta)} + \sum_{i=2}^{n} \frac{\varsigma(x_{i-1},\Theta)G(x_{i-1},\Theta) - \varsigma(x_i,\Theta)G(x_i,\Theta)}{G(x_{i-1},\Theta) - G(x_i,\Theta)}\right], \tag{35}$$

$$\begin{aligned} \frac{\partial \log(F)}{\partial \alpha} = \frac{1}{n+1}\Bigg[\rho(x_1,\Theta) - \frac{\Im(x_n,\Theta) - \aleph(x_n,\Theta)G(x_n,\Theta)}{1 - G(x_n,\Theta)} \\ + \sum_{i=2}^{n} \frac{\Im(x_{i-1},\Theta) - \Im(x_i,\Theta) - \aleph(x_{i-1},\Theta)G(x_{i-1},\Theta) + \aleph(x_i,\Theta)G(x_i,\Theta)}{G(x_{i-1},\Theta) - G(x_i,\Theta)}\Bigg], \end{aligned} \tag{36}$$

$$\begin{aligned} \frac{\partial \log(F)}{\partial \beta} = \frac{1}{n+1}\Bigg[\wp(x_1,\Theta) - \frac{\pounds(x_n,\Theta) - \pounds(x_n,\Theta)G(x_n,\Theta)}{1 - G(x_n,\Theta)} \\ + \sum_{i=2}^{n} \frac{\pounds(x_{i-1},\Theta) - \pounds(x_i,\Theta) + \pounds(x_{i-1},\Theta)G(x_{i-1},\Theta) - \pounds(x_i,\Theta)G(x_i,\Theta)}{G(x_{i-1},\Theta) - G(x_i,\Theta)}\Bigg], \end{aligned} \tag{37}$$

and

$$\frac{\partial \log(F)}{\partial \lambda} = \frac{1}{n+1}\left[\Re(x_1,\Theta) - \frac{\frac{\pounds(x_n,\Theta)}{\lambda \log(x_n)} + \frac{\pounds(x_n,\Theta)}{\lambda \log(x_n)}G(x_n,\Theta)}{1-G(x_n,\Theta)}\right.$$

$$\left. + \sum_{i=2}^{n} \frac{\frac{\pounds(x_i,\Theta)}{\lambda \log(x_i)} - \frac{\pounds(x_{i-1},\Theta)}{\lambda \log(x_{i-1})} - \frac{\pounds(x_{i-1},\Theta)}{\lambda \log(x_{i-1})}G(x_{i-1},\Theta) + \frac{\pounds(x_i,\Theta)}{\lambda \log(x_i)}G(x_i,\Theta)}{G(x_{i-1},\Theta)-G(x_i,\Theta)}\right]. \tag{38}$$

The above-mentioned nonlinear equations cannot be solved analytically,so the $\hat{\Theta}_{MPS}$ of Θ can use any iterative procedure techniques, such as conjugate-gradient algorithms, to obtain the numerical solution.

$$\varsigma(x_i,\Theta) = \frac{\alpha^{e^{-\lambda x_i^{-\beta}}} - \alpha}{(\alpha-1)\theta-(\theta-1)(\alpha^{e^{-\lambda x_i^{-\beta}}}-1)},$$

$$\rho(x_i,\Theta) = \frac{\alpha^{e^{-\lambda x_i^{-\beta}}-1}e^{-\lambda x_i^{-\beta}}}{\alpha^{e^{-\lambda x_i^{-\beta}}}-1} - \frac{\theta-\alpha^{e^{-\lambda x_i^{-\beta}}-1}e^{-\lambda x_i^{-\beta}}(\theta-1)}{(\alpha-1)\theta-(\theta-1)(\alpha^{e^{-\lambda x_i^{-\beta}}}-1)},$$

$$\Im(x_i,\Theta) = \frac{\alpha^{e^{-\lambda x_i^{-\beta}}-1}e^{-\lambda x_i^{-\beta}}}{(\alpha-1)\theta-(\theta-1)(\alpha^{e^{-\lambda x_i^{-\beta}}}-1)},$$

$$\aleph(x_i,\Theta) = \frac{\theta-\alpha^{e^{-\lambda x_i^{-\beta}}-1}e^{-\lambda x_i^{-\beta}}(\theta-1)}{(\alpha-1)\theta-(\theta-1)(\alpha^{e^{-\lambda x_i^{-\beta}}}-1)},$$

$$\wp(x_i,\Theta) = \frac{\lambda \alpha^{e^{-\lambda x_i^{-\beta}}} e^{-\lambda x_i^{-\beta}} x_i^{-\beta}\log(\alpha)\log(x_i)}{\alpha^{e^{-\lambda x_i^{-\beta}}}-1} + \pounds(x_i,\Theta),$$

$$\pounds(x_i,\Theta) = \frac{\lambda \alpha^{e^{-\lambda x_i^{-\beta}}} e^{-\lambda x_i^{-\beta}} x_i^{-\beta}\log(\alpha)\log(x_i)}{(\alpha-1)\theta-(\theta-1)(\alpha^{e^{-\lambda x_i^{-\beta}}}-1)},$$

and

$$\wp(x_i,\Theta) = \frac{\alpha^{e^{-\lambda x_i^{-\beta}}} e^{-\lambda x_i^{-\beta}} x_i^{-\beta}\log(\alpha)}{\alpha^{e^{-\lambda x_i^{-\beta}}}-1} + \frac{\pounds(x_i,\Theta)}{\lambda \log(x_i)}.$$

5.5 Bayesian estimation

Bayesian methods in statistical inference depend on choice of the prior distribution and the loss function. However, prior distribution parameters may depend on the hyper parameters. On the other hand, the loss function is important in Bayesian methods. Most of the Bayesian inference procedures have been developed under the symmetric and asymmetric loss functions. One of the most widely used symmetric loss function is the squared error loss (SEL) function and the most widely used asymmetric loss function is the LINEX loss function. The joint prior density function of Θ can be written, as follows:

$$\pi(\Theta) = \frac{b_1^{a_1}}{\Gamma(a_1)}\frac{b_2^{a_2}}{\Gamma(a_2)}\frac{b_3^{a_3}}{\Gamma(a_3)}\frac{b_4^{a_4}}{\Gamma(a_4)}\alpha^{a_1-1}\lambda^{a_2-1}\beta^{a_3-1}\theta^{a_4-1}e^{-(b_1\alpha+b_2\lambda+b_3\beta+b_4\theta)}. \tag{39}$$

The joint posterior density function of Θ can be obtained from (19) and (39)

$$\pi(\Theta|\underline{x}) = \frac{\ell(\underline{x}|\Theta).\pi(\Theta)}{\int_\Theta \ell(\underline{x}|\Theta).\pi(\Theta)}. \tag{40}$$

The Bayes estimate of Θ, say $p(\alpha,\lambda,\beta,\theta)$ using SEL function is given by

$$\hat{p}_{B-SEL}(\alpha,\lambda,\beta,\theta) = E_{(\alpha,\lambda,\beta,\theta|\underline{x})}[p(\alpha,\lambda,\beta,\theta)] = \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty p(\alpha,\lambda,\beta,\theta)\times\pi(\Theta|\underline{x})d\alpha d\lambda d\beta d\theta. \quad (41)$$

The Bayes estimate of Θ, say $p(\alpha,\lambda,\beta,\theta)$ using LINEX loss function is given by

$$\hat{p}_{B-LINEX}(\alpha,\lambda,\beta,\theta) = \frac{-1}{v}\log[E_{(\alpha,\lambda,\beta,\theta|\underline{x})}[e^{-vp(\alpha,\lambda,\beta,\theta)}]] = \frac{-1}{v}\log[\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-vp(\alpha,\lambda,\beta,\theta)}\times\pi(\Theta|data)d\alpha d\lambda d\beta d\theta], \quad v\neq 0. \quad (42)$$

For more details about the Bayesian estimation, see for example, Riaz et al. [16,17,18] and Hassan and Zaky [19].

It is noticeable that the ratio of two integrals given by (41) and (42) cannot be obtained in an explicit form. In this case, we use the Markov Chain Monte Carlo (MCMC) technique to compute an approximate value of integrals in (41) and (42). A lot of studies, such as Okasha et al. [20] and Almetwally et al. [21,22,23,24] addressed MCMC technique. The Metropolis algorithm was first proposed in Metropolis et al. [25] and was then generalized by Hastings [26].

5.6 Simulation

To compare between the non-Bayesian and Bayesian estimations methods, the parameters of MOAPIW distribution are estimated using Monte Carlo simulation which is implemented by *R* language. The data were generated from the MOAPIW Distribution for life time of different values of parameters α, β, θ, and λ such as: case 1: α=1.5, β=1.5, θ=1.6 and λ=1.5; case 2: α=0.5, β=1.5, θ=1.6 and λ=1.5;

Table 6: Bias and MSE of parameters α=1.5, β=1.5, θ=1.6 and λ=1.5.

		MLE	LSE	WLSE	MPS	SEL	LINEX v=1.5	LINEX v=0.5
n	Parameters	Bias MSE	Bias MSE	Bias MSE	Bias MSE	Bias MSE	Bias MSE	Bias MSE
50	α	0.1640	0.0125	0.0603	0.0693	-0.2325	-0.3531	-0.2750
		0.7752	0.0850	0.1960	0.2274	0.3260	0.3755	0.3401
	β	0.0110	-0.0256	-0.0361	-0.1261	-0.0778	-0.1140	-0.0900
		0.1265	0.0581	0.0724	0.0931	0.0480	0.0526	0.0492
	θ	0.4703	-0.0141	0.0435	-0.0535	-0.0703	-0.1794	-0.1083
		2.6654	0.3377	0.7387	0.8423	0.2429	0.2281	0.2327
	λ	0.2415	0.1011	0.1696	0.2419	0.2557	0.1582	0.2219
		0.9412	0.2195	0.4656	0.7326	0.2261	0.1512	0.1968
100	α	0.1048	0.0240	0.0582	0.0533	-0.0141	-0.0272	-0.0185
		0.4560	0.0733	0.1427	0.1410	0.0351	0.0359	0.0353
	β	-0.0140	-0.0209	-0.0221	-0.0818	-0.0146	-0.0222	-0.0172
		0.0983	0.0358	0.0418	0.0493	0.0171	0.0173	0.0171
	θ	0.2941	0.0210	0.0722	-0.0281	-0.0219	-0.0313	-0.0251
		1.3502	0.2813	0.5288	0.5035	0.0288	0.0302	0.0293
	λ	0.1062	0.0533	0.0773	0.1143	0.0179	0.0080	0.0146
		0.4969	0.1301	0.2404	0.3444	0.0214	0.0208	0.0212
200	α	0.0926	0.0353	0.0400	0.0385	-0.0069	-0.0106	-0.0082
		0.2964	0.0530	0.0881	0.0766	0.0069	0.0098	0.0096
	β	-0.0103	-0.0079	-0.0103	-0.0446	-0.0060	0.0032	0.0051
		0.0650	0.0186	0.0219	0.0221	0.0060	0.0065	0.0065
	θ	0.1934	0.0473	0.0623	-0.0107	0.0170	0.0131	0.0157
		0.7611	0.1967	0.3292	0.2692	0.0103	0.0101	0.0102
	λ	0.0454	0.0212	0.0498	0.0531	-0.0005	-0.0041	-0.0017
		0.2773	0.0856	0.1477	0.1467	0.0091	0.0090	0.0090

Table 7: Bias and MSE of parameters α=0.5, β=1.5, θ=1.6 and λ=1.5.

		MLE	LSE	WLSE	MPS	SEL	LINEX v=1.5	LINEX v=0.5
n	Parameters	Bias MSE	Bias MSE	Bias MSE	Bias MSE	Bias MSE	Bias MSE	Bias MSE
50	α	0.5305	0.0924	0.1797	0.0786	0.1777	0.1054	0.1525
		0.8610	0.1523	0.3056	0.3064	0.1513	0.1014	0.1321
	β	0.0597	-0.0095	-0.0033	-0.1701	-0.0471	-0.0897	-0.0614
		0.1433	0.0678	0.0818	0.1382	0.0488	0.0485	0.0482
	θ	0.2194	0.0634	0.1166	-0.0657	-0.1319	-0.2321	-0.1664
		1.5229	0.1573	0.3742	0.4907	0.2591	0.2772	0.2627
	λ	0.1006	0.0117	0.0247	0.2485	0.1370	0.0614	0.1109
		0.5935	0.1615	0.2683	0.4942	0.1298	0.1018	0.1186
100	α	0.3601	0.1088	0.1711	0.0311	0.0039	-0.0049	0.0010
		0.4441	0.1375	0.2471	0.2100	0.0210	0.0199	0.0206
	β	0.0076	0.0028	-0.0001	-0.1380	-0.0214	-0.0282	-0.0237
		0.0965	0.0457	0.0579	0.0955	0.0157	0.0161	0.0158
	θ	0.1001	0.0871	0.1131	0.0707	-0.0031	-0.0150	-0.0071
		0.7519	0.1271	0.2793	0.2800	0.0239	0.0251	0.0242
	λ	0.0703	-0.0318	-0.0101	0.1850	0.0082	-0.0016	0.0049
		0.3801	0.1071	0.1859	0.3206	0.0220	0.0215	0.0218
200	α	0.2578	0.0678	0.1336	-0.0213	-0.0115	-0.0150	-0.0127
		0.2859	0.0730	0.1588	0.1396	0.0078	0.0079	0.0079
	β	-0.0256	0.0023	0.0087	-0.1082	-0.0080	-0.0112	-0.0091
		0.0906	0.0245	0.0351	0.0626	0.0060	0.0061	0.0060
	θ	0.0467	0.0509	0.0806	0.0724	0.0064	-0.0105	-0.0077
		0.4709	0.0609	0.1593	0.1406	0.0113	0.0115	0.0114
	λ	0.0501	-0.0154	-0.0147	0.1591	0.0055	0.0023	0.0041
		0.2886	0.0635	0.1334	0.2176	0.0091	0.0091	0.0091

case 3: α=1.5, β=1.5, θ=0.6 and λ=1.5; case 4: α=0.5, β=1.5, θ=0.6 and λ=1.5; case 5: α=1.5, β=2.85, θ=1.6 and λ=3.5. The bias and MSE are computed by generating 1000 replications samples size n =50, 100 and 200 from the MOAPIW distribution. Tables 6 - 10 reveal that for each method, the biases and the MSE's decrease as sample size n increases. Also, the Bayesian estimation is the best estimation.

Table 8: Bias and MSE of parameters α=1.5, β=1.5, θ=0.6 and λ=1.5.

		MLE	LSE	WLSE	MPS	SEL	LINEX v=1.5	LINEX v=0.5
n	Parameters	Bias MSE	Bias MSE	Bias MSE	Bias MSE	Bias MSE	Bias MSE	Bias MSE
50	α	0.0509	0.0504	0.0769	0.0912	-0.1351	-0.2721	-0.1839
		0.3250	0.0266	0.0627	0.0887	0.4118	0.4249	0.4115
	β	0.1031	0.0243	0.0255	-0.1233	0.0177	-0.0226	0.0041
		0.2089	0.0865	0.0970	0.1484	0.0559	0.0515	0.0540
	θ	0.4948	0.1088	0.1907	0.1083	0.2168	0.1437	0.1914
		1.2462	0.3476	0.4328	0.4785	0.1476	0.0927	0.1265
	λ	0.0141	-0.0216	0.0116	0.2950	0.0452	-0.0311	0.0191
		0.6491	0.1974	0.2842	0.7717	0.1406	0.1196	0.1320
100	α	0.0560	0.0360	0.0629	0.0656	0.0174	0.0057	0.0135
		0.2867	0.0151	0.0357	0.0453	0.0276	0.0271	0.0273
	β	0.0471	0.0093	0.0283	-0.1017	0.0101	0.0021	0.0074
		0.1329	0.0483	0.0646	0.0956	0.0145	0.0142	0.0144
	θ	0.3047	0.0915	0.1736	0.0469	0.0053	-0.0020	0.0029
		0.5961	0.1282	0.2741	0.2642	0.0141	0.0135	0.0139
	λ	-0.0083	-0.0180	-0.0320	0.2189	0.0136	0.0050	0.0107
		0.4423	0.1148	0.2009	0.4885	0.0203	0.0199	0.0201
200	α	0.0261	0.0199	0.0422	0.0442	0.0228	0.0184	0.0214
		0.1374	0.0059	0.0194	0.0206	0.0103	0.0100	0.0102
	β	-0.0113	0.0121	0.0293	-0.0723	0.0003	-0.0033	-0.0009
		0.1463	0.0255	0.0394	0.0559	0.0077	0.0078	0.0078
	θ	0.1807	0.0655	0.1314	0.0127	0.0098	0.0065	0.0087
		0.3143	0.0579	0.1625	0.1461	0.0078	0.0076	0.0078
	λ	-0.0135	-0.0209	-0.0356	0.1590	0.0023	-0.0013	0.0011
		0.3005	0.0718	0.1393	0.1856	0.0081	0.0081	0.0081

Table 9: Bias and MSE of parameters α=0.5, β=1.5, θ=0.6 and λ=1.5.

		MLE	LSE	WLSE	MPS	SEL	LINEX v=1.5	LINEX v=0.5
n	Parameters	Bias MSE	Bias MSE	Bias MSE	Bias MSE	Bias MSE	Bias MSE	Bias MSE
50	α	0.2761	0.0216	0.0411	0.0137	0.1385	0.0574	0.1100
		0.2919	0.0376	0.0748	0.1482	0.1375	0.0863	0.1171
	β	0.1029	-0.0094	-0.0093	-0.1961	-0.0085	-0.0529	-0.0236
		0.2412	0.0425	0.0665	0.1592	0.0673	0.0639	0.0655
	θ	0.3413	0.0669	0.1050	0.1069	0.1812	0.1140	0.1575
		0.7061	0.1085	0.1882	0.3136	0.1634	0.1131	0.1438
	λ	-0.0130	0.0152	0.0488	0.3882	0.0566	-0.0183	0.0312
		0.5155	0.0905	0.1654	0.6147	0.1522	0.1310	0.1432
100	α	0.2091	0.0039	0.0347	-0.0297	0.0315	0.0215	0.0281
		0.2172	0.0224	0.0553	0.1194	0.0231	0.0216	0.0226
	β	0.0421	-0.0082	0.0037	-0.1827	-0.0245	-0.0318	-0.0269
		0.1914	0.0294	0.0505	0.1255	0.0158	0.0163	0.0160
	θ	0.2527	0.0580	0.1056	0.0844	0.0114	0.0028	0.0085
		0.4930	0.0703	0.1464	0.2145	0.0206	0.0194	0.0202
	λ	0.0083	0.0065	0.0087	0.3483	0.0174	0.0078	0.0142
		0.4097	0.0579	0.1252	0.4696	0.0220	0.0217	0.0219
200	α	0.1683	0.0104	0.0369	-0.0467	-0.0043	-0.0083	-0.0057
		0.1829	0.0185	0.0440	0.1060	0.0120	0.0121	0.0120
	β	0.0187	0.0019	0.0185	-0.1541	0.0071	0.0040	0.0061
		0.2069	0.0242	0.0435	0.1053	0.0053	0.0052	0.0053
	θ	0.2186	0.0528	0.0994	0.0820	0.0107	0.0072	0.0096
		0.4423	0.0589	0.1192	0.1738	0.0070	0.0067	0.0069
	λ	-0.0263	-0.0068	-0.0137	0.2973	0.0032	-0.0002	0.0021
		0.3594	0.0527	0.1102	0.3893	0.0090	0.0090	0.0090

Table 10: Bias and MSE of parameters α=1.5, β=2.85, θ=1.6 and λ=3.5.

		MLE	LSE	WLSE	MPS	SEL	LINEX v=1.5	LINEX v=0.5
n	Parameters	Bias MSE	Bias MSE	Bias MSE	Bias MSE	Bias MSE	Bias MSE	Bias MSE
50	α	0.2357	0.0515	0.1519	0.0490	-0.2129	-0.3268	-0.2525
		1.1678	0.1255	0.3012	0.1669	0.2790	0.3253	0.2918
	β	0.0455	-0.0323	-0.0313	-0.2168	-0.2250	-0.3105	-0.2544
		0.3928	0.2558	0.2728	0.2990	0.1771	0.2298	0.1932
	θ	0.6028	0.0389	0.1943	-0.0791	-0.0445	-0.1540	-0.0836
		3.6870	0.4271	1.1577	0.6377	0.2266	0.2292	0.2333
	λ	0.3686	0.0826	0.0843	0.0488	0.0448	-0.0951	-0.0044
		2.6932	0.6063	1.2919	1.0508	0.3287	0.3116	0.1319
100	α	0.1956	0.0653	0.0937	0.0342	-0.0292	-0.0422	-0.0335
		0.8693	0.0997	0.1819	0.0994	0.0411	0.0431	0.0417
	β	0.0117	-0.0194	-0.0268	-0.1410	-0.0296	-0.0387	-0.0326
		0.2400	0.1447	0.1621	0.1513	0.0256	0.0266	0.0259
	θ	0.4079	0.0942	0.1354	0.0488	-0.0059	-0.0179	-0.0099
		1.9676	0.3799	0.6972	0.3658	0.0318	0.0320	0.0318
	λ	0.1265	-0.0235	0.0146	-0.0269	0.0216	0.0102	0.0178
		1.4216	0.4211	0.6747	0.5245	0.0362	0.0349	0.0357
200	α	0.1196	0.0388	0.1268	0.0283	-0.0201	-0.0250	-0.0218
		0.4335	0.0437	0.1831	0.0506	0.0116	0.0118	0.0116
	β	0.0123	-0.0023	0.0135	-0.0755	-0.0054	-0.0090	-0.0066
		0.1239	0.0719	0.0928	0.0660	0.0094	0.0093	0.0093
	θ	0.2463	0.0628	0.2238	-0.0203	0.0097	0.0059	0.0084
		0.9625	0.1568	0.6917	0.1824	0.0092	0.0091	0.0092
	λ	0.0624	-0.0195	-0.0721	-0.0293	-0.0125	-0.0169	-0.0140
		0.7850	0.1711	0.6017	0.2588	0.0123	0.0125	0.0123

6 Fitting reliability data

We present the numerical results of the parameter estimation of MOAPIW distribution of two real data.

Example 1.

Lawless [27] discussed the data set of failure which consists of failure time or censoring time for 17 appliances. Data set: 1167, 1925, 1990, 2223, 2400, 2471, 2551, 2568, 2694, 3034, 3112, 3214, 3478, 3504, 4329, 6976, 7846. We have estimated parameters of MOAPIW distribution and computed the Kolmogorov-Smirnov (K-S) distance, Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC) statistics for this distribution and different distributions in Table 11.

Table 11: MLEs, KS, p-values and different model criterion for failure times data.

Distributions	MOAPIW	APIW	MOIW	IW
α	276.212	115.01	-	-
λ	625.01	195.35	4.8723	105.349
β	1.83	0.879	0.4199	0.6043
θ	612.66	-	3.94	-
$K-S$	0.262	0.3109	0.857	0.370
$p-value$	0.1611	0.059	8.33×10^{-15}	0.0135
AIC	328.749	332.936	360.610	337.616
$CAIC$	331.606	334.782	362.456	338.474
BIC	332.082	335.435	363.110	339.283
$HQIC$	329.080	333.184	360.899	337.782

We have estimated parameters of MOAPIW distribution and computed the Kolmogorov-Smirnov (K-S) distance under different estimation methods in Table 12.

Table 12: Estimate and standard error of parameters for different methods of estimation.

Parameters	MLE estimate std	LSE estimate std	WLSE estimate std	MPS estimate std	Bayesian estimate std
α	276.21 3.67×10^{-6}	274.53 7.43×10^{-6}	693.4 2.54×10^{-7}	270.1 1.6×10^{-5}	276.4 1.54×10^{-8}
β	1.830 4.66×10^{-2}	1.83 9.26×10^{-2}	2.1 9.19×10^{-3}	1.82 1.98×10^{-1}	1.81 5.19×10^{-4}
θ	612.626 9.49×10^{-6}	623.25 1.87×10^{-5}	1700.8 6.84×10^{-7}	612.1 4.04×10^{-5}	612.69 1.56×10^{-7}
λ	625.011 9.3×10^{-6}	636.17 1.84×10^{-5}	1712.5 6.8×10^{-7}	625.1 3.95×10^{-5}	624.94 2.36×10^{-7}
K-S	0.26211	0.2603	0.2603	0.25534	0.2743
P-Value	0.1611	0.1666	0.1666	0.1827	0.1271

Example 2.

Gacula and Kubala [28] represented the data set of failure times. This consists of failure times of a certain product and contains 26 observations. Data Set: 24, 24, 26, 26, 32, 32, 33, 33, 33, 35, 41, 42, 43, 47, 48, 48, 48, 50, 52, 54, 55, 57, 57, 57, 57, 61. The MLEs estimates of the parameters, different model criterion and the values of the K-S statistic with p-values are reported in Table 13.

Table 13: MLEs, K-S, p-values and different model criterion for failure times data.

Distributions	MOAPIW	APIW	MOIW	IW
α	270.623	99.355	-	-
λ	610.688	172.620	340.220	159.480
β	3.9	1.86	3.162	1.427
θ	601.468	-	333.241	-
$K-S$	0.188	0.308	0.999	0.363
$p-value$	0.318	0.014	0.000	0.002
AIC	218.054	231.408	222.156	236.084
$CAIC$	219.959	232.499	223.247	236.606
BIC	223.087	235.182	225.931	238.601
$HQIC$	219.503	232.495	223.243	236.809

We have estimated parameters of MOAPIW distribution and computed the Kolmogorov-Smirnov (K-S) distance under different estimation methods in Table 14.

Table 14: Estimate and standard error of parameters for different methods of estimation.

	MLE	LSE	WLSE	Bayesian
Parameters	estimate std	estimate std	estimate std	estimate std
α	270.623 1.42×10^{-5}	275.1 3.02×10^{-5}	567.46 1.05×10^{-6}	270.74 4.99×10^{-5}
β	3.9 8.37×10^{-2}	3.92 1.18×10^{-1}	4.39 1.46×10^{-2}	3.89 5.19×10^{-4}
θ	601.468 3.65×10^{-5}	624.7 7.63×10^{-5}	1371.4 2.27×10^{-6}	601.42 3.97×10^{-6}
λ	610.688 3.59×10^{-5}	637.58 7.48×10^{-5}	1382.2 2.77×10^{-6}	610.63 1.4×10^{-5}
K-S	0.187	0.1904	0.1904	0.1933
P-Value	0.3177	0.3025	0.3025	0.2857

7 Conclusion

In this paper, we achieved two goals: The first one is to introduce a new distribution of Marshall Olkin alpha power inverse Weibull (MOAPIW) distribution and study some of its properties, such as hazard rate function, reversed hazard rate function, mean residual life, mean inactivity time, quantiles, moments, R$\acute{e}$nyi entropy and order statistics . The second goal is to estimate the unknown parameters of the new distribution and provide some applications in the context of statistics. The results ware elucidated by a real data set and showed that the new distribution provides a better fit than some other known distributions.

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Conflict of Interest

The author(s) professed that no conflicts of interest for this publication ,research and authorship of this article.

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