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SVIM for Solving Burger's and Coupled Burger's Equations of Fractional Order

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Abstract: In this paper, we utilize Sumudu variational iteration method (SVIM) to obtain approximate solutions for fractional Burger's (FBE) and coupled fractional Burger's equations (CFBEs). The results are compared with FHPM. The results, show that the suggested algorithm is appropriate for handling linear and as well as nonlinear problems in engineering and sciences.

Keywords: Burger's equation, Caputo fractional derivative, fractional variational iteration method, Sumudu transform.

1 Introduction

Fractional calculus (FC) has played an important role in areas ranging from fundamental science to engineering in the past ten years [1–3] and has been applied to a wide class of complex problems encompassing physics, biology, mechanics, and interdisciplinary areas [4–6].

Various methods have been utilized to obtain approximate solutions of fractional PDEs, such as the FADM [7, 8], the FRDTM [9], the FVIM [10–13], the Fq-HAM [14], the FHPM [15], the FSEM [16], the FSHPM [17], and the FLVIM [18, 19] have been utilized to solve fractional differential equations.

The present paper aims is to suggest the SVIM to the the time FBE and CFBEs, as follows:

$$D_\tau^\gamma \varphi = \varphi_{\mu\mu} - \varphi\varphi_\mu, \tag{1}$$
$$\varphi(\mu,0) = g_1(\mu), \tag{2}$$

and

$$\begin{aligned} &D_\tau^{\gamma_1}\varphi - \varphi_{\mu\mu} - 2\varphi\varphi_\mu + (\varphi\psi)_\mu = 0,\\ &D_\tau^{\gamma_2}\psi - \psi_{\mu\mu} - 2\psi\psi_\mu + (\varphi\psi)_\mu = 0, \end{aligned} \tag{3}$$

$$\begin{aligned} &\varphi(\mu,0) = g_2(\mu),\\ &\psi(\mu,0) = g_3(\mu). \end{aligned} \tag{4}$$

This paper is organized as follows. In Section 2, we review the FC theory. In Section 3, the SVIM is analyzed. In Section 4, the approximate solution for the time FBE and CFBEs is obtained. Conclusion is presented in Section 5.

2 Preliminaries

Definition 1. *The R-L fractional integral operator of order $\gamma \geq 0$, of a function $\varphi(\mu) \in C_\vartheta, \vartheta \geq -1$ is [20, 21]*

$$I^\gamma \varphi(\mu) = \begin{cases} \frac{1}{\Gamma(\gamma)} \int_0^\mu (\mu-\tau)^{\gamma-1}\varphi(\tau)d\tau, & \gamma>0, \mu>0,\\ \varphi(\mu), & \gamma = 0, \end{cases}$$

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The properties of I^{γ}: For $\varphi \in C_{\vartheta}, \vartheta \geq -1, \gamma, \sigma \geq -1$, then

1.$I^{\gamma} I^{\sigma} \varphi(\mu) = I^{\gamma+\sigma} \varphi(\mu)$.

2.$I^{\gamma} I^{\sigma} \varphi(\mu) = I^{\sigma} I^{\gamma} \varphi(\mu)$.

3.$I^{\gamma} \mu^{m} = \dfrac{\Gamma(m+1)}{\Gamma(\gamma+m+1} \mu^{\gamma+m}$.

Definition 2. *The FDO of $\varphi(\mu)$ in the Caputo sense is [20, 21]*

$$\begin{aligned} D^{\gamma} \varphi(\mu) &= I^{m-\gamma} D^{m} \varphi(\mu) \\ &= \frac{1}{\Gamma(m-\gamma)} \int_{0}^{\mu} (\mu-\tau)^{m-\gamma-1} \varphi^{(m)}(\tau) d\tau, \end{aligned} \tag{5}$$

for $m-1 < \gamma \leq m$.

Definition 3. *The Mittag-Leffler function E_{γ} with $\gamma > 0$ is [20, 21]*

$$E_{\gamma}(z) = \sum_{m=0}^{\infty} \frac{z^{\gamma}}{\Gamma(m\gamma+1)}. \tag{6}$$

Definition 4. *The ST is defined over the set of function*

$$A = \left\{ \varphi(\tau) / \exists M, \omega_1, \omega_2 > 0, |\varphi(\tau)| < M e^{\frac{|\tau|}{\omega_j}}, if \tau \in (-1)^{j} \times [0, \infty) \right\},$$

by the following formula

$$S[\varphi(\tau)] = \int_{0}^{\infty} e^{-\tau} \varphi(\omega\tau) d\tau, \omega \in (-\omega_1, \omega_2). \tag{7}$$

Definition 5. *The ST of the CFD is defined as*

$$S[D_{\tau}^{m\gamma} \varphi(\mu,\tau)] = \omega^{-m\gamma} S[\varphi(\mu,\tau)] - \sum_{k=0}^{m-1} \omega^{-m\gamma+k} \varphi^{(k)}(\mu,0), m-1 < m\gamma < m. \tag{8}$$

3 Sumudu variational iteration method (SVIM)

Let us consider a general FPDEs of the form:

$$D_{\tau}^{\gamma} \varphi(\mu,\tau) + R[\varphi(\mu,\tau)] + N[\varphi(\mu,\tau)] = g(\mu,\tau), \tag{9}$$

with the initial condition

$$\varphi(\mu,0) = h(\mu) \tag{10}$$

We will see the whole process of the Lagrange multipliers in the case of an algebraic equation. The solution of the algebraic equation $f(\mu) = 0$ is

$$\mu_{n+1} = \mu_n + \lambda f(\mu_n). \tag{11}$$

The optimality condition for the extreme $\frac{\delta \mu_{n+1}}{\delta \mu_n} = 0$ is

$$\lambda = -\frac{1}{f'(\mu_n)}. \tag{12}$$

Substituting (12) in (11):

$$\mu_{n+1} = \mu_n - \frac{f(\mu_n)}{f'(\mu_n)}. \tag{13}$$

Now, we apply the ST to (9), and get

$$\frac{G_n(\omega)}{\omega^{\gamma}} - \frac{\varphi(\mu,0)}{\omega^{\gamma}} + S[R[\varphi] + N[\varphi] - g] = 0, \tag{14}$$

where $G(\omega)$ is ST of φ.

Using (11), the iteration formula (14) can be written as

$$G_{n+1}(\omega) = G_n(\omega) + \lambda(\omega)\left(\frac{G_n(\omega)}{\omega^\gamma} - \frac{\varphi}{\omega^\gamma} + S[R[\varphi] + N[\varphi] - g]\right) \tag{15}$$

Taking the variation of (15), which is given by

$$\delta[G_{n+1}(\omega)] = \delta[G_n(\omega)] + \delta\left[\lambda(\omega)\left(\frac{G_n(\omega)}{\omega^\gamma} - \frac{\varphi_n}{\omega^\gamma} + S[R[\varphi_n] + N[\varphi_n] - g]\right)\right] \tag{16}$$

Using computation of (16), we get

$$\begin{aligned}\delta[G_{n+1}(\omega)] &= \delta[G_n(\omega)] + \delta\left[\lambda(\omega)\frac{G_n(\omega)}{\omega^\gamma}\right]\\ &= \delta[G_n(\omega)] + \frac{\lambda(\omega)}{\omega^\gamma}\delta[G_n(\omega)]\\ &= 0.\end{aligned} \tag{17}$$

Hence, from (17) we get

$$\lambda(\omega) = -\omega^\gamma. \tag{18}$$

Applying the inverse ST to (15) after putting the value of $\lambda(\omega)$, we get

$$\varphi_{n+1}(\mu,\tau) = S^{-1}(\varphi_n(\mu,0)) - S^{-1}(\omega^\gamma[S[R[\varphi_m(\mu,\tau)] + N[\varphi_m(\mu,\tau)] - g(\mu,\tau)]]). \tag{19}$$

Consequently, the approximate solution may be procured using

$$\varphi(\mu,\tau) = \lim_{n\to\infty}\varphi_n(\mu,\tau) \tag{20}$$

4 Applications

Example 1. Consider the FBE

$$D_\tau^\gamma\varphi(\mu,\tau) = \varphi_{\mu\mu}(\mu,\tau) - \varphi(\mu,\tau)\varphi_\mu(\mu,\tau), \tag{21}$$

with

$$\varphi(\mu,0) = \mu. \tag{22}$$

In view of (19) and (21), we obtain

$$\begin{aligned}\varphi_{n+1}(\mu,\tau) &= S^{-1}(\varphi_n(\mu,0))\\ &+ S^{-1}\left(\omega^\gamma\left[S\left(\frac{\partial^2}{\partial\mu^2}\varphi_n(\mu,\tau) - \varphi_n(\mu,\tau)\frac{\partial}{\partial\mu}\varphi_n(\mu,\tau)\right)\right]\right)\end{aligned} \tag{23}$$

The initial iteration $\varphi_0(\mu,\tau)$ is given as follows

$$\varphi_0(\mu,\tau) = \varphi(\mu,0) = \mu. \tag{24}$$

Now, we get the first approximation, namely

$$\begin{aligned}\varphi_1(\mu,\tau) &= S^{-1}(\varphi_0(\mu,0)) + S^{-1}\left(\omega^\gamma\left[S\left(\frac{\partial^2}{\partial\mu^2}\varphi_0(\mu,\tau) - \varphi_0(\mu,\tau)\frac{\partial}{\partial\mu}\varphi_0(\mu,\tau)\right)\right]\right)\\ &= \mu + S^{-1}(\omega^\gamma[S(-\mu)])\\ &= \mu - \mu\frac{\tau^\gamma}{\Gamma(\gamma+1)}.\end{aligned} \tag{25}$$

The second approximate reads, as follows:

$$\begin{aligned}\varphi_2(\mu,\tau) &= S^{-1}(\varphi_1(\mu,0)) + S^{-1}\left(\omega^{\gamma}\left[S\left(\frac{\partial^2}{\partial\mu^2}\varphi_1(\mu,\tau) - \varphi_1(\mu,\tau)\frac{\partial}{\partial\mu}\varphi_1(\mu,\tau)\right)\right]\right)\\ &= \mu + S^{-1}\left(\omega^{\gamma}\left[S\left[-\mu + 2\mu\frac{\tau^{\gamma}}{\Gamma(\gamma+1)} - \mu\frac{\tau^{2\gamma}}{\Gamma^2(\gamma+1)}\right]\right]\right)\\ &= \mu + S^{-1}\left(\omega^{\gamma}\left[-\mu + 2\mu\omega^{\gamma} - \mu\frac{\Gamma(2\gamma+1)}{\Gamma^2(\gamma+1)}\omega^{2\gamma}\right]\right)\\ &= \mu - \mu\frac{\tau^{\gamma}}{\Gamma(\gamma+1)} + 2\mu\frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)} - \mu\frac{\Gamma(2\gamma+1)}{\Gamma^2(\gamma+1)}\frac{\tau^{3\gamma}}{\Gamma(3\gamma+1)}.\\ &\vdots\end{aligned}$$

and so on.

Then, we have

$$\varphi(\mu,\tau) = \mu - \mu\frac{\tau^{\gamma}}{\Gamma(\gamma+1)} + 2\mu\frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)} - \mu\frac{\Gamma(2\gamma+1)}{\Gamma^2(\gamma+1)}\frac{\tau^{3\gamma}}{\Gamma(3\gamma+1)} + \ldots \tag{26}$$

Example 2. Consider the following CFBEs:

$$\begin{aligned}&D_{\tau}^{\gamma_1}\varphi(\mu,\tau) - \varphi_{\mu\mu}(\mu,\tau) - 2\varphi(\mu,\tau)\varphi_{\mu}(\mu,\tau) + (\varphi\psi)_{\mu} = 0,\\ &D_{\tau}^{\gamma_2}\psi(\mu,\tau) - \psi_{\mu\mu}(\mu,\tau) - 2\psi(\mu,\tau)\psi_{\mu}(\mu,\tau) + (\varphi\psi)_{\mu} = 0,\end{aligned} \tag{27}$$

subject to initial conditions

$$\begin{aligned}&\varphi(\mu,0) = \sin(\mu),\\ &\psi(\mu,0) = \sin(\mu).\end{aligned} \tag{28}$$

Using (19), we get

$$\begin{aligned}\varphi_{n+1}(\mu,\tau) &= S^{-1}(\varphi_0(\mu,0))\\ &\quad + S^{-1}\left(\omega^{\gamma_1}\left[S\left[\frac{\partial^2}{\partial\mu^2}\varphi_n + 2\varphi_n\frac{\partial}{\partial\mu}\varphi_n - \frac{\partial}{\partial\mu}(\varphi_n\psi_n)\right]\right]\right)\\ \psi_{n+1}(\mu,\tau) &= S^{-1}(\psi_0(\mu,0))\\ &\quad + S^{-1}\left(\omega^{\gamma_2}\left[S\left[\frac{\partial^2}{\partial\mu^2}\psi_n + 2\psi_n(\mu,\tau)\frac{\partial}{\partial\mu}\psi_n - \frac{\partial}{\partial\mu}(\varphi_n\psi_n)\right]\right]\right)\end{aligned} \tag{29}$$

The initial iterations $\varphi_0(\mu,\tau)$ and $\psi_0(\mu,\tau)$ are

$$\begin{aligned}&\varphi_0(\mu,\tau) = \sin(\mu),\\ &\psi_0(\mu,\tau) = \sin(\mu).\end{aligned} \tag{30}$$

Hence, we obtain the first approximation; namely

$$\begin{aligned}\varphi_1(\mu,\tau) &= S^{-1}(\varphi_0(\mu,0)) + S^{-1}\left(\omega^{\gamma_1}\left[S\left[\frac{\partial^2}{\partial\mu^2}\varphi_0 + 2\varphi_0\frac{\partial}{\partial\mu}\varphi_0 - \frac{\partial}{\partial\mu}(\varphi_0\psi_0)\right]\right]\right)\\ \psi_1(\mu,\tau) &= S^{-1}(\psi_0(\mu,0)) + S^{-1}\left(\omega^{\gamma_2}\left[S\left[\frac{\partial^2}{\partial\mu^2}\psi_0 + 2\psi_0\frac{\partial}{\partial\mu}\psi_0 - \frac{\partial}{\partial\mu}(\varphi_0\psi_0)\right]\right]\right)\\ &= \left(1 - \frac{\tau^{\gamma_1}}{\Gamma(\gamma_1+1)}\right)\sin(\mu)\\ &= \left(1 - \frac{\tau^{\gamma_2}}{\Gamma(\gamma_2+1)}\right)\sin(\mu)\end{aligned} \tag{31}$$

The second approximation reads as follows:

$$\varphi_2(\mu,\tau) = S^{-1}(\varphi_1(\mu,0)) + S^{-1}\left(\omega^{\gamma_1}\left[S\left[\frac{\partial^2}{\partial\mu^2}\varphi_1 + 2\varphi_1\frac{\partial}{\partial\mu}\varphi_1 - \frac{\partial}{\partial\mu}(\varphi_1\psi_1\right]\right]\right)$$

$$\psi_2(\mu,\tau) = S^{-1}(\psi_1(\mu,0)) + S^{-1}\left(\omega^{\gamma_2}\left[S\left[\frac{\partial^2}{\partial\mu^2}\psi_1 + 2\psi_1\frac{\partial}{\partial\mu}\psi_1 - \frac{\partial}{\partial\mu}(\varphi_1\psi_1\right]\right]\right)$$

$$= sin(\mu) + S^{-1}\left(\omega^{\gamma_1}\left[S\left[-sin(\mu) + \frac{\tau^{\gamma_1}}{\Gamma(\gamma_1+1)})sin(\mu)\right]\right]\right) -$$

$$S^{-1}\left(\omega^{\gamma_1}\left[S\left[\frac{\tau^{\gamma_1}}{\Gamma(\gamma_1+1)})sin(2\mu) + \frac{\tau^{\gamma_2}}{\Gamma(\gamma_2+1)})sin(2\mu)\right]\right]\right)$$

$$= sin(\mu) + S^{-1}\left(\omega^{\gamma_2}\left[S\left[-sin(\mu) + \frac{\tau^{\gamma_2}}{\Gamma(\gamma_2+1)})sin(\mu)\right]\right]\right) -$$

$$S^{-1}\left(\omega^{\gamma_2}\left[S\left[\frac{\tau^{\gamma_2}}{\Gamma(\gamma_2+1)})sin(2\mu) + \frac{\tau^{\gamma_1}}{\Gamma(\gamma_1+1)})sin(2\mu)\right]\right]\right)$$

$$= sin(\mu) - \frac{\tau^{\gamma_1}}{\Gamma(\gamma_1+1)}\sin(\mu) + \frac{\tau^{2\gamma_1}}{\Gamma(2\gamma_1+1)}\sin(\mu) - \frac{\tau^{2\gamma_1}}{\Gamma(2\gamma_1+1)}sin(2\mu) +$$

$$\frac{\tau^{\gamma_1+\gamma_2}}{\Gamma(\gamma_2+\gamma_2+1)}sin(2\mu)$$

$$= sin(\mu) - \frac{\tau^{\gamma_2}}{\Gamma(\gamma_2+1)}\sin(\mu) + \frac{\tau^{2\gamma_2}}{\Gamma(2\gamma_2+1)}\sin(\mu) - \frac{\tau^{2\gamma_2}}{\Gamma(2\gamma_2+1)}sin(2\mu) +$$

$$\frac{\tau^{\gamma_1+\gamma_2}}{\Gamma(\gamma_1+\gamma_2+1)}sin(2\mu) \quad (32)$$

$$\vdots$$

and so on.

For $\gamma_1 = \gamma_2$, we have

$$\varphi(\mu,\tau) = \sin(\mu)\left(1 - \frac{\tau_1^{\gamma}}{\Gamma(\gamma_1+1)} + \frac{\tau^{2\gamma_1}}{\Gamma(2\gamma_1+1)} - \cdots\right)$$

$$\psi(\mu,\tau) = \sin(\mu)\left(1 - \frac{\tau^{\gamma_2}}{\Gamma(\gamma_2+1)} + \frac{\tau^{2\gamma_2}}{\Gamma(2\gamma_2+1)} - \cdots\right)$$

$$= E_{\gamma_1}(-\tau^{\gamma_1})sin(\mu)$$

$$= E_{\gamma_2}(-\tau^{\gamma_2})sin(\mu). \quad (33)$$

The result is same as q-HATM [14].

5 Conclusion

In this work, the SVIM has been successfully used to obtain the solutions of the FBE and CFBEs. The obtained solutions were in the form of infinite power series which can be written in a closed form. In view of the results, we can say that this technique is powerful mathematical tool for solving FPDEs.

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