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Approximate Solutions to Nonlinear Fractional Differential Equations with the Caputo-Fabrizio Derivative

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Abstract: In this work we prove the existence of solution to a nonlinear fractional initial value problem with Caputo-Fabrizio derivative. We pose an equivalent integral formulation of the problem, develop a recursive method to approximate the solution and prove its convergence. We show the performance of the procedure in some examples, including an application to hysteresis phenomena.

Keywords: Caputo-Fabrizio fractional derivative, fractional initial value problems, approximate solutions, recursive methods.

1 Introduction

In last decades numerous works devoted to fractional calculus have been published. Although the first reference to a fractional derivative dates back to the 17^{th} century, in an epistolary exchange between Leibniz and L'Hôpital, it has been in recent times that the large number of applications in different areas of science and technology gave greater relevance to calculus with non-integer orders. Through fractional calculus, phenomena such as viscoelasticity, hysteresis, diffusion in porous media, pricing, abnormal cell proliferation, etc., achieve models that seem to fit better to reality than the traditional differential equations with integer order (see [1–13], just for citing some few examples).
Several definitions have been given for a derivative of arbitrary order, $\mathscr{D}^{\alpha}$, with $\alpha \in \mathbf{R}$. A brief mention will be made of some of them.
From the classical Cauchy's formula for the n-fold iterated integral,

$$ {}_0I_t^n[f](t) = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} f(s)ds, \;\; n \in \mathbf{N}, $$

and recalling that Gamma function verifies $n\Gamma(n) = n!$, *Riemann-Liouville's integral operator of order* α, for $\alpha \in \mathbf{R}$, is an immediate generalization:

$$ {}_0I_t^{\alpha}[f](t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds. \tag{1} $$

From (1), the *Riemann-Liouville fractional derivative of order* α, with $n-1<\alpha<n$, is almost naturally defined as

$$ {}_0^{RL}\mathscr{D}_t^{\alpha}[f](t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}(\int_0^t (t-s)^{n-\alpha} f(s)ds). \tag{2} $$

Another definition is the *Caputo fractional derivative of order* α [14]:

$$ {}_0^{C}\mathscr{D}_t^{\alpha}[f](t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-1-\alpha}\frac{d^n f(s)}{ds^n}ds. \tag{3} $$

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It can be proved that both derivatives, (2) and (3), are left inverse operators to the integral operator of Riemann-Liouville of order α and are associated to the idea that "deriving α times" might be thought as "integrating $n-\alpha$ times and deriving n times"; the difference lies in the order in which these operations are performed.
These definitions are not equivalent: the operators domains are different, since different hypothesis over f are needed to guarantee their existence and, even with the appropriate conditions for f,

$$ {}_0^C\mathscr{D}_t^\alpha[f](t) = {}^{RL}_{\ 0}\mathscr{D}_t^\alpha[f](t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0^+) $$

(see, for example, [15] and [16] for a detailed treatment).
Note that both derivatives may be thought as convolution operators with singular kernels.
Less than five years ago a new definition was proposed [17]: the *Caputo-Fabrizio fractional derivative of order* α, with $\alpha \in (0,1)$, defined as

$$ {}_0^{CF}\mathscr{D}_t^\alpha[f](t) = \frac{M(\alpha)}{1-\alpha} \int_0^t f'(s) e^{-\frac{\alpha(t-s)}{1-\alpha}} ds, \tag{4} $$

with $M(\alpha)$ a normalizing factor verifying $M(0) = M(1) = 1$.
Caputo-Fabrizio definition was then generalized by Atangana and Baleanu [18], who gave the following definition of the *Atangana-Baleanu fractional derivative in Riemann-Liouville sense*,

$$ {}_0^{ABR}\mathscr{D}_t^\alpha[f](t) = \frac{M(\alpha)}{1-\alpha} \frac{d}{dt} \int_0^t f(s) E_\alpha\left(-\frac{\alpha(t-s)^\alpha}{1-\alpha}\right) ds \tag{5} $$

and the *Atangana-Baleanu fractional derivative in Caputo sense*,

$$ {}_0^{ABC}\mathscr{D}_t^\alpha[f](t) = \frac{M(\alpha)}{1-\alpha} \int_0^t f'(s) E_\alpha\left(-\frac{\alpha(t-s)^\alpha}{1-\alpha}\right) ds, \tag{6} $$

where the exponential has been replaced by the generalized Mittag-Leffler function,

$$ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}. $$

Note that these last definitions - (4), (5) and (6) - avoid singular kernels.
There are other types of fractional derivatives, like Grünwald-Letnikov's, Hadamard's, Weyl's, etc., some of them responding mostly to mathematical interest. A general framework to gather the different alternatives is proposed in [19].
All the definitions show that fractional derivative operators are not local, since they need the information of f in a whole interval of integration. If the variable is time, this means that history of the phenomenon is taken into account when deriving with a non-integer order.
In this work we consider Caputo-Fabrizio fractional derivative (CFFD). In Section 2 we comment a few of its properties. An existence result for a class of fractional initial value problem is presented in Section 3 and a recursive method to solve it appears in Section 4. We offer numerical examples in Section 5. Finally, some conclusions are presented.

2 Some comments on the Caputo-Fabrizio fractional derivative

There is no consensus as to what properties an operator must satisfy in order to be effectively considered a fractional derivative. Many of the alternative definitions have emerged to better adapt to particular physical contexts. In [20], an interesting proposal is offered about the classification of these operators.
In the case of Caputo-Fabrizio fractional derivative, an additional advantage is the non-singularity of the kernel.
In [17], [21], [22] and [23] several properties of this derivative are analyzed.
Regarding qualitative results, the operator shows a behavior similar to that of Caputo fractional derivative.
In Figure 1, for example, we show Caputo derivatives (on the left) and Caputo-Fabrizio derivatives (on the right), of different orders, for $f(t) = sin(t)$. Input $(sin(t))$ vs. output $(\mathscr{D}^\alpha[sin](t))$ is showed in Figure 2 (Caputo derivatives on the left, Caputo-Fabrizio derivatives on the right) , exhibiting a behavior that would allow - with a suitable adaptation - to describe hysteresis phenomena.

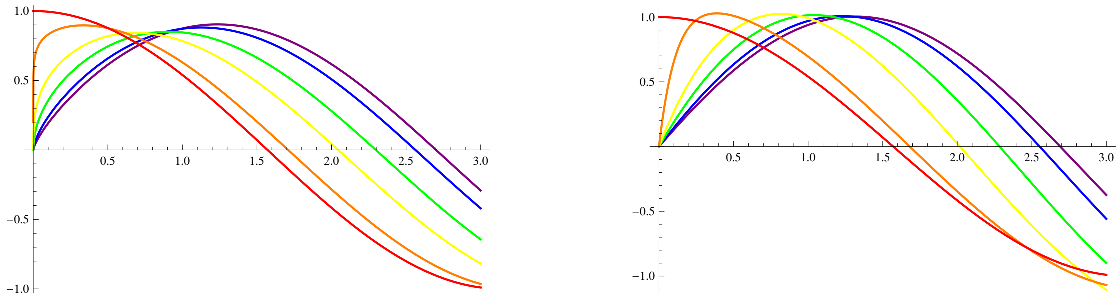

Fig. 1: Caputo and Caputo-Fabrizio derivatives of $sin(t)$ (left and right, respectively) for different fractionary orders: purple, $\alpha = \frac{1}{4}$; blue, $\alpha = \frac{1}{3}$; green, $\alpha = \frac{1}{2}$; yellow, $\alpha = \frac{2}{3}$; orange, $\alpha = \frac{9}{10}$; red, $\alpha = 1$ (ordinary derivative).

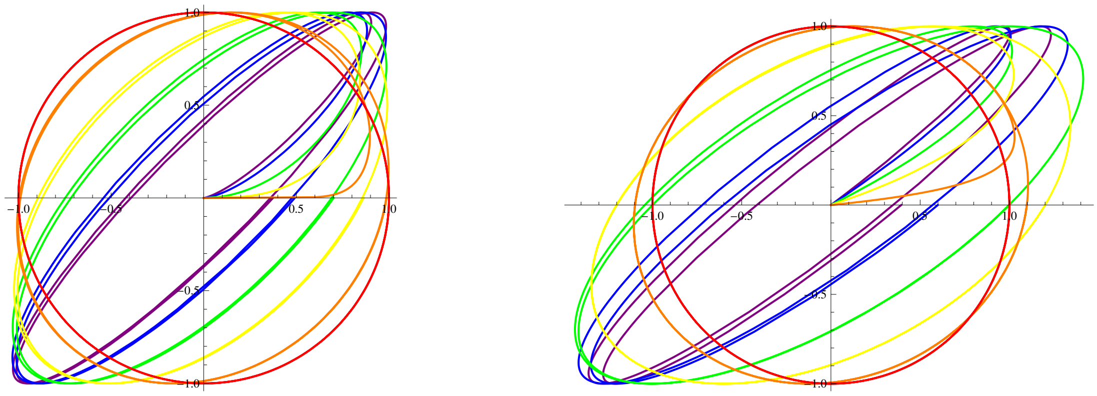

Fig. 2: $\mathscr{D}^{\alpha}[sin](t)$ vs. $sin(t)$ for different values of α (same color code). Caputo derivatives on the left and Caputo-Fabrizio derivatives on the right.

3 Existence and uniqueness of solution to a nonlinear FIVP

3.1 Introduction

Let us consider the following initial value problem:

$$\begin{cases} {}^{CF}_{0}\mathscr{D}^{\alpha}_{t}[y](t) = f(t,y(t)) \quad for \quad t \in [0,T], \\ \qquad\quad y(0) = y_0, \end{cases} \tag{7}$$

where $y(t)$ is an unknown function and ${}^{CF}_{0}\mathscr{D}^{\alpha}_{t}[y](t)$ is its Caputo-Fabrizio derivative of order $\alpha \in (0,1)$. The function $f(t,y)$ and $y_0 \in \mathbf{R}$ are given data,

In [24] an existence theorem for a continuous solution of this problem is presented.

Inspired in the traditional theorem due to Picard and Lindelöf, we will offer a slightly different result for absolutely continuous solutions: like in the ordinary differential equation case, an alternative and equivalent integral formulation of the problem is stated and the existence and uniqueness of solution follows from the Banach fixed point theorem.

For the moment we will assume that $f \in C(A)$, with $A = [0,T] \times [y_0 - b, y_0 + b]$ for some $b \in \mathbf{R}$.

As ${}^{CF}_{0}\mathscr{D}^{\alpha}_{t}[y](0) = 0$ we will also assume that $f(0,y_0) = 0$.

We will look for continuous solutions, $y(t)$, for which $y'(t)$ exists almost everywhere and is integrable on $[0,T]$ - that is to say, $y'(t) \in L^1([0,T])$. These conditions guarantee the existence of ${}^{CF}_{0}\mathscr{D}^{\alpha}_{t}[y](t)$.

We will add the requirement that $y(t) = y(0) + \int_0^t y'(\tau)d\tau$ for every $t \in [0,T]$. As we are working on the real line, all these conditions over $y(t)$ are equivalent to look for *absolutely continuous* solutions to (7), which we will denote by $y \in AC([0,T])$.

3.2 Equivalent integral problem

Under these hypotheses we can prove the following

Proposition 1. *Solving (7) in $AC([0,T])$ is equivalent to solve the following integral equation:*

$$y(t) = y_0 + \frac{\alpha}{M(\alpha)} \int_0^t f(\tau, y(\tau))d\tau + \frac{1-\alpha}{M(\alpha)} f(t, y(t)). \tag{8}$$

Proof. Suppose that $y(t) \in AC([0,T])$ verifies ${}_0^{CF}\mathscr{D}_t^{\alpha}[y](t) = u(t)$ and $y(0) = y_0$ for $u \in C([0,T])$.
Then

$$\int_0^t y'(\tau) e^{\frac{\alpha\tau}{1-\alpha}} d\tau = \frac{1-\alpha}{M(\alpha)} e^{\frac{\alpha t}{1-\alpha}} u(t).$$

It can be proved that, as $y'(\tau)e^{\frac{\alpha\tau}{1-\alpha}}$ is integrable on $[0,T]$, necessarily $\frac{1-\alpha}{M(\alpha)} e^{\frac{\alpha t}{1-\alpha}} u(t) \in AC([0,T])$ and, consequently, $u(t) \in AC([0,T])$.
Then, almost for every $t \in [0,T]$ where derivatives exist,

$$y'(t) e^{\frac{\alpha t}{1-\alpha}} = \frac{1-\alpha}{M(\alpha)} [\frac{\alpha}{1-\alpha} e^{\frac{\alpha t}{1-\alpha}} u(t) + e^{\frac{\alpha t}{1-\alpha}} u'(t)],$$

or

$$y'(t) = \frac{1-\alpha}{M(\alpha)} [\frac{\alpha}{1-\alpha} u(t) + u'(t)].$$

Then

$$y(t) - y(0) = \frac{\alpha}{M(\alpha)} \int_0^t u(\tau)d\tau + \frac{1-\alpha}{M(\alpha)} [u(t) - u(0)].$$

So, if $u(t) = f(t, y(t))$, $y(0) = y_0$ and $f(0, y_0) = 0$, we have (8).

Now suppose (8) holds, for $y = y(t) \in AC([0,T])$ and $f(t,y) \in C(A)$.
Under these conditions it can be proved that $u(t) = f(t, y(t)) \in AC([0,T])$, so its derivative, $u'(t)$, exists almost for every $t \in [0,T]$.
Then, as

$$y(t) = y_0 + \frac{\alpha}{M(\alpha)} \int_0^t u(\tau)d\tau + \frac{1-\alpha}{M(\alpha)} u(t),$$

clearly it is $y(0) = y_0$ and we obtain

$$y'(t) = \frac{\alpha}{M(\alpha)} u(t) + \frac{1-\alpha}{M(\alpha)} u'(t)$$

for every $t \in [0,T]$ where both derivatives, $y'(t)$ and $u'(t)$, exist.
As $y'(t)$ is integrable and $u(t) \in AC([0,T])$, $u'(t)$ must be integrable and so is $\frac{\alpha}{M(\alpha)} u(t) + \frac{1-\alpha}{M(\alpha)} u'(t)$.
Then,

$$\frac{M(\alpha)}{1-\alpha} \int_0^t y'(\tau) e^{\frac{-\alpha(t-\tau)}{1-\alpha}} d\tau = \int_0^t [\frac{\alpha}{1-\alpha} u(\tau) e^{\frac{-\alpha(t-\tau)}{1-\alpha}} + u'(\tau) e^{\frac{-\alpha(t-\tau)}{1-\alpha}}] d\tau.$$

Now, for every τ where u' exists, it is $\frac{\alpha}{1-\alpha} u(\tau) e^{\frac{-\alpha(t-\tau)}{1-\alpha}} + u'(\tau) e^{\frac{-\alpha(t-\tau)}{1-\alpha}} = \frac{d[u(\tau) e^{\frac{-\alpha(t-\tau)}{1-\alpha}}]}{d\tau}$ and

$${}_0^{CF}\mathscr{D}_t^{\alpha}[y](t) = \int_0^t \frac{d[u(\tau) e^{\frac{-\alpha(t-\tau)}{1-\alpha}}]}{d\tau} d\tau.$$

As $u(t) = f(t, y(t))$ and $u(0) = f(0, y_0) = 0$, we have (7).

3.3 Existence and uniqueness of the solution

Let us suppose now that f is Lipschitz continuous with respect to its second variable in $[y_0 - b, y_0 + b]$, i.e. there exists $L \in \mathbf{R}_{>0}$ so that

$$|f(t, y_1) - f(t, y_2)| \le L|y_1 - y_2|$$

for every y_1, y_2 in $[y_0 - b, y_0 + b]$ and each $t \in [0,T]$. We will indicate this by $f \in Lip_2(A)$.

Consider $h \in (0,T]$.
Given $y_0 \in \mathbf{R}$ and $f \in Lip_2(A)$, with $f(0,y_0)=0$, let us define the operator

$$\mathscr{F}(\varphi)(t) = y_0 + \frac{\alpha}{M(\alpha)} \int_0^t f(\tau,\varphi(\tau))d\tau + \frac{1-\alpha}{M(\alpha)} f(t,\varphi(t))$$

for $\varphi \in AC([0,h])$ and $t \in [0,h]$.
It can be proved that $AC([0,h])$ is a Banach space with the norm

$$||\varphi||_{AC} = ||\varphi||_\infty + V(\varphi,[0,h]),$$

for

$$V(\varphi,[0,h]) := \sup_\pi (\sum_{i=1}^n |\varphi(t_i) - \varphi(t_{i-1})|),$$

where the supreme is taken over all finite increasing subsets π of $[0,h]$: $\pi = \{t_i\}_{i=0}^n$, $n \in \mathbf{N}$, with $0 \le t_0 < t_1 < ... < t_n \le h$.

Note that, as $f(t,y(t)) \in AC([0,h])$, $\mathscr{F}(\varphi)(t)$ is absolutely continuous in $[0,h]$ and, consequently, $\mathscr{F}$ goes from $AC([0,h])$ to $AC([0,h])$.

For φ and ψ in $AC([0,h])$, $t \in [0,h]$ and $\sigma = max\{\alpha, 1-\alpha\}$, it is

$$\begin{aligned} |\mathscr{F}(\varphi)(t) - \mathscr{F}(\psi)(t)| &\le \\ &\le \tfrac{\alpha}{|M(\alpha)|} \int_0^t |f(\tau,\varphi(\tau)) - f(\tau,\psi(\tau))|d\tau + \tfrac{1-\alpha}{|M(\alpha)|}|f(t,\varphi(t)) - f(t,\psi(t))| \le \\ &\le \tfrac{\sigma}{|M(\alpha)|} \int_0^t L|\varphi(\tau) - \psi(\tau)|d\tau + \tfrac{\sigma}{|M(\alpha)|} L|\varphi(t) - \psi(t)| \le \\ &\le \tfrac{\sigma}{|M(\alpha)|} \int_0^t L||\varphi - \psi||_\infty d\tau + \tfrac{\sigma}{|M(\alpha)|} L||\varphi - \psi||_\infty \le \\ &\le \tfrac{\sigma L}{|M(\alpha)|}(h+1)||\varphi - \psi||_\infty, \end{aligned}$$

so

$$||\mathscr{F}(\varphi) - \mathscr{F}(\psi)||_\infty \le \frac{\sigma L}{|M(\alpha)|}(h+1)||\varphi - \psi||_\infty$$

if $\varphi(t), \psi(t) \in [y_0 - b, y_0 + b] \quad \forall t \in [0,h]$.
Now, we will make an additional hypothesis about f: suppose $f(t,y)$ is linear in y, that is, $f(t,y) = g_1(t)y + g_2(t)$, for g_1 and g_2 functions such that $f \in C(A) \cup Lip_2(A)$; note that it is enough that g_1 and g_2 are continuous in $[0,h]$, and then L could be chosen, simply, as an upper bound of $\{|g_1(t)|, t \in [0,h]\}$.
For $\pi = \{t_i\}_{i=0}^n$, $n \in \mathbf{N}$, with $0 \le t_0 < t_1 < ... < t_n \le h$, it is

$$\begin{aligned} \textstyle\sum_{i=1}^n |\mathscr{F}(\varphi)(t_i) - \mathscr{F}(\psi)(t_i) - (\mathscr{F}(\varphi)(t_{i-1}) - \mathscr{F}(\psi)(t_{i-1}))| &\le \\ &\le \tfrac{\sigma}{|M(\alpha)|} \textstyle\sum_{i=1}^n \int_{t_{i-1}}^{t_i} |f(\tau,\varphi(\tau)) - f(\tau,\psi(\tau))|d\tau + \\ &+ \tfrac{\sigma}{|M(\alpha)|} \textstyle\sum_{i=1}^n |f(t_i,\varphi(t_i)) - f(t_i,\psi(t_i)) - f(t_{i-1},\varphi(t_{i-1})) + f(t_{i-1},\psi(t_{i-1}))| \le \\ &\le \tfrac{\sigma}{|M(\alpha)|} Lh||\varphi - \psi||_\infty + \tfrac{\sigma}{|M(\alpha)|} \textstyle\sum_{i=1}^n |g_1(t_i)(\varphi - \psi)(t_i) - g_1(t_{i-1})(\varphi - \psi)(t_{i-1}))| \le \\ &\le \tfrac{\sigma}{|M(\alpha)|} Lh||\varphi - \psi||_\infty + \tfrac{\sigma}{|M(\alpha)|} \textstyle\sum_{i=1}^n |g_1(t_i)(\varphi - \psi)(t_i)) - g_1(t_{i-1})(\varphi - \psi)(t_i)) + \\ &+ g_1(t_{i-1})(\varphi - \psi)(t_i)) - g_1(t_{i-1})(\varphi - \psi)(t_{i-1})| \le \\ &\le \tfrac{\sigma}{|M(\alpha)|}(Lh + V(g_1,[0,h])||\varphi - \psi||_\infty + \tfrac{\sigma L}{|M(\alpha)|} V(\varphi - \psi,[0,h]), \end{aligned}$$

where we are supposing that $V(g_1,[0,h]) < \infty$ and L is an upper bound of $|g_1|$ in $[0,h]$.
Then,

$$||\mathscr{F}(\varphi) - \mathscr{F}(\psi)||_{AC} \le \frac{\sigma H}{|M(\alpha)|}||\varphi - \psi||_{AC}$$

for $H = (2h+1)L + V(g_1, [0,h])$.
So, if $\frac{\sigma H}{|M(\alpha)|} < 1$, $\mathscr{F}$ is a contractive operator from $AC([0,T])$ to $AC([0,T])$ and the Banach fixed-point theorem guarantees the existence of a unique absolutely continuous solution to (8) in $[0,h]$.
Note that the inequality $\frac{\sigma H}{|M(\alpha)|} < 1$ establishes conditions over L and determines the choice of $h \in [0,T]$.
We have proved the following:

Theorem 1. *Given the initial value problem*

$$\begin{cases} {}_0^{CF}\mathscr{D}_t^{\alpha}[y](t) = f(t,y(t)) \quad \text{for} \quad t \in [0,T], \\ y(0) = y_0, \end{cases}$$

so that

-f is linear in its second variable, i.e. $f(t,y) = g_1(t)y + g_2(t)$, with $g_1, g_2 \in C([0,T])$, and
-$f(0,y_0) = 0$,

there is $h \in (0,T]$ such that, if $\frac{\sigma H}{|M(\alpha)|} < 1$ - for $H = (2h+1)L + V(g_1,[0,h])$, $L = \sup\{|g_1(t)|, t \in [0,h]\}$, and $\sigma = \max\{\alpha, 1-\alpha\}$ - there exists a unique absolutely continuous solution $y(t)$ for $t \in [0,h] \subseteq [0,T]$.

4 Approximate solutions to the FIVP

Like in the case of ordinary differential equations, and based on the equivalent integral formulation of the FIVP, we will define a sequence of approximate solutions to (8) that converges to its unique solution.
Suppose all the hypotheses of Theorem 1 are fulfilled. We will define the recursive sequence of functions:

$$\begin{cases} \varphi_0(t) = y_0, \\ \varphi_{i+1}(t) = \mathscr{F}(\varphi_i)(t) \quad \text{for} \quad i \geq 0. \end{cases} \tag{9}$$

Now, we will conveniently choose $h_0 \in [0,h]$ so that we can prove inductively that, for each $i \in \mathbf{N}_0$, φ_i is well defined, $\varphi_i \in AC([0,h_0])$ and $\varphi_i(t) \in [y_0 - b, y_0 + b]$ for every $t \in [0,h_0]$.
After that, as $\varphi_{i+1} = \mathscr{F}(\varphi_i)$ $\forall i \geq 0$ and $\mathscr{F}$ is contractive, the sequence will converge to the solution of (8) and, equivalently, to the solution of (7) in $[0,h_0]$.

Clearly φ_0 is well defined and derivable in $[0,h_0]$ for every $h_0 \in [0,h]$, because it is constant, and $\varphi_0(t) = y_0 \in [y_0 - b, y_0 + b]$ $\forall t \in [0,h_0]$ for every $h_0 \in [0,h]$.
As $\varphi_{i+1} = \mathscr{F}(\varphi_i)$ and $\mathscr{F}$ goes from $AC([0,h_0])$ to itself, for every $h_0 \in [0,h]$, if φ_i is well defined and $\varphi_i \in AC([0,h_0])$, then φ_{i+1} is well defined and $\varphi_{i+1} \in AC([0,h_0])$.
For $t \in [0,h_0] \subseteq [0,h]$

$$|\varphi_{i+1}(t) - y_0| \leq \frac{\alpha}{|M(\alpha)|} \int_0^t |f(\tau,\varphi_i(\tau))| d\tau + \frac{1-\alpha}{|M(\alpha)|} |f(t,\varphi_i(t))| \leq \frac{\sigma}{|M(\alpha)|} B(h_0+1) < b$$

if B is a bound of f in A, $\frac{|M(\alpha)|b}{\sigma B} > 1$ and we choose $h_0 \leq \min\{h, \frac{|M(\alpha)|b}{\sigma B} - 1\}$.
We have proved the followings:

Proposition 2. *Given the fractional initial value problem (7), under the conditions of Theorem 1, the unique absolutely continuous solution $y(t)$, for $t \in [0,h_0]$, can be approximated by the sequence of functions*

$$\begin{cases} \varphi_0(t) = y_0, \\ \varphi_{i+1}(t) = y_0 + \frac{\alpha}{M(\alpha)} \int_0^t f(\tau,\varphi_i(\tau)) d\tau + \frac{1-\alpha}{M(\alpha)} f(t,\varphi_i(t)) \quad \text{for} \quad i \geq 0. \end{cases}$$

4.1 Prolongation of solutions

Note that (7) is equivalent to

$$\begin{cases} \frac{M(\alpha)}{1-\alpha} \int_{h_0}^t y'(\tau) e^{\frac{-\alpha(t-\tau)}{1-\alpha}} d\tau = f(t,y(t)) - \frac{M(\alpha)}{1-\alpha} \int_0^{h_0} y'(\tau) e^{\frac{-\alpha(t-\tau)}{1-\alpha}} d\tau \quad \text{for} \quad t \in [0,T], \\ y(h_0) = y_1, \end{cases}$$

where the datum is $y(h_0)$ instead of $y(0)$.
Then, the problem to solve can be written as

$$\begin{cases} \frac{M(\alpha)}{1-\alpha} \int_{h_0}^{t} y'(\tau) e^{\frac{-\alpha(t-\tau)}{1-\alpha}} d\tau = f(t,y(t)) - f(h_0,y(h_0)) \quad \text{for} \quad t \in [0,T], \\ y(h_0) = y_1. \end{cases}$$

Operating as in Proposition (1), but with the right hand $f_1(t,y(t)) = f(t,y(t)) - f(h_0,y(h_0))$ instead of $f(t,y(t))$, we obtain an equivalent integral problem:

$$y(t) = y_1 + \frac{\alpha}{M(\alpha)} \int_{h_0}^{t} f_1(\tau, y(\tau)) d\tau + \frac{1-\alpha}{M(\alpha)} f_1(t, y(t)).$$

Defining

$$\mathscr{F}_1(\varphi)(t) = y_1 + \frac{\alpha}{M(\alpha)} \int_{h_0}^{t} f_1(\tau, \varphi(\tau)) d\tau + \frac{1-\alpha}{M(\alpha)} f_1(t, \varphi(t))$$

and using

$$\begin{cases} \varphi_0(t) = y_1 \quad \text{for} \quad t \geq h_0, \\ \varphi_{i+1}(t) = \mathscr{F}_1(\varphi_i)(t) \quad \text{for} \quad i \geq 0, t \geq h_0, \end{cases}$$

we can obtain approximations for $t \in [h_0, h_1]$, for certain $h_1 > h_0$.
The value of y_1 is approximated by $\varphi_n(h_0)$, where n is the number of iterations chosen to approximate the solution in $[0, h_0]$ (n can be selected as the index for which $||\varphi_n - \varphi_{n-1}||_{AC} \leq \Delta$, with Δ the maximum tolerance accepted).
This process can be repeatedly applied.

5 Numerical examples

We will solve some initial value problems applying the recursive method described in the previous section, for several functions $f(t,y)$ satisfying the hypotheses of Theorem 1.

5.1 Example 1

Let us consider the following initial value problem, for $\alpha = \frac{1}{2}$ and $M(\alpha) = 1$.
The exact solution is $y_{exact}(t) = e^{\frac{t}{4}}$:

$$\begin{cases} {}_{0}^{CF}\mathscr{D}_t^{\frac{1}{2}}[y](t) = \frac{2}{5} y(1 - e^{-\frac{5}{4}t}) \quad \text{for} \quad t \in [0,5], \\ y(0) = 1. \end{cases}$$

In Figure 3 we represent the exact solution (dashed in black) and ten iterations.
For $i = 9$ it is $||\varphi_9 - y_{exact}||_{AC} \cong 0.001$ and for $i = 13$, $||\varphi_{13} - y_{exact}||_{AC} \cong 0.00001$ in $[0,5]$.
In this example $\sigma = \max\{\alpha, 1-\alpha\} = \frac{1}{2}$, so it must be $H < 2$, and $\frac{B}{b} < 2$ too. As $B = \frac{2}{5}|y|_{max}$ and $L = \frac{2}{5}$, if we choose $b = 1$ the operator is contractive for $h < \frac{2}{5}$, so we can guarantee the existence of a unique absolute continuous solution $y(t)$ for $t \in [0, \frac{2}{5})$. However, Figure 3 shows that the convergence seems to occur in a considerable wider range.

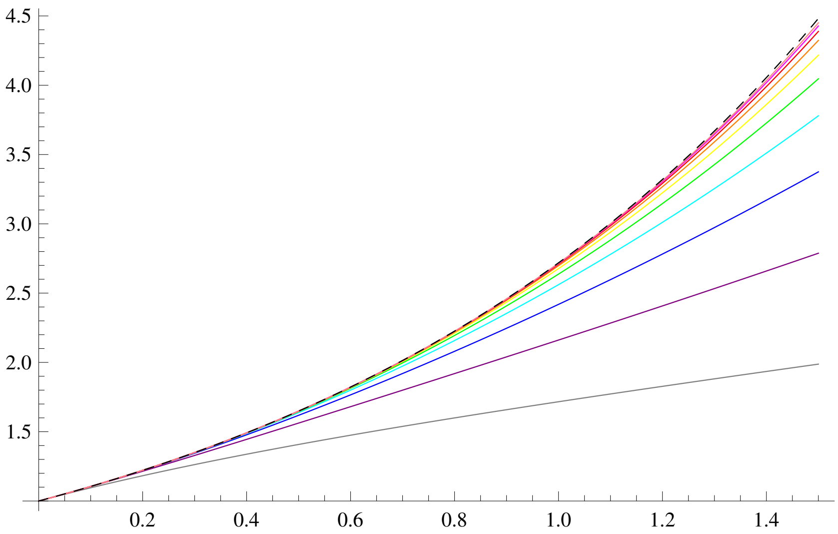

Fig. 3: The exact solution (dashed in black) and several iterations (φ_1 to φ_9, from gray to pink) for the Example 1.

5.2 Example 2

For the following initial value problem the exact solution is $y_{exact}(t) = \frac{1}{4}e^{-\frac{t}{2}}$:

$$\begin{cases} {}^{CF}_{0}\mathscr{D}_t^{\frac{2}{3}}[y](t) = -y(1-e^{-\frac{3t}{2}}) \quad for \quad t \in [0,5], \\ y(0) = \frac{1}{4}. \end{cases}$$

In Figure 4, the exact solution (dashed in black) and ten iterations are represented.
In Figure 5 we show, in red, three prolongations of the solution, using the procedure described in Section 4.1, each one with nine iterations; dashed in black, the exact solution.
It is $||\varphi_{10} - y_{exact}||_{AC} \cong 0.008$ in $[0,2]$, $||\varphi_9 - y_{exact}||_{AC} \cong 0.001$ in $[2,3]$, $||\varphi_9 - y_{exact}||_{AC} \cong 0.0002$ in $[3,4]$ and $||\varphi_9 - y_{exact}||_{AC} \cong 0.0003$ in $[4,5.5]$.

5.3 Example 3

In [21] the author gives an expression for the exact solution to the problem

$$\begin{cases} {}^{CF}_{0}\mathscr{D}_t^{\alpha}[y](t) - \lambda y(t) = g(t) \quad for \quad t \in [0,T], \\ y(0) = 0. \end{cases}$$

If $\lambda(1-\alpha) \neq 1$ it is

$$y(t) = \frac{1-\alpha}{1-\lambda(1-\alpha)}g(t) + \frac{\alpha}{(1-\lambda(1-\alpha))^2}\int_0^t g(\tau)e^{\frac{\lambda\alpha(t-\tau)}{1-\lambda(1-\alpha)}}d\tau.$$

For $\alpha = \frac{1}{2}$, $\lambda = \frac{1}{3}$ and $g(t) = sin(t)$ we show in Figure 6 the exact solution dashed in blue and, in red, the approximation after twelve iterations. It is $||\varphi_{12} - y_{exact}||_{AC} \cong 0.004$ in $[0,10]$.

5.4 Example 4

The solution of the following initial value problem, for $\alpha = \frac{1}{4}$ and $M(\alpha) = 1$, is unknown:

$$\begin{cases} {}^{CF}_{0}\mathscr{D}_t^{\frac{1}{4}}[y](t) = ye^{-t}\sin(t) \quad for \quad t \in [0,5], \\ y(0) = 1. \end{cases}$$

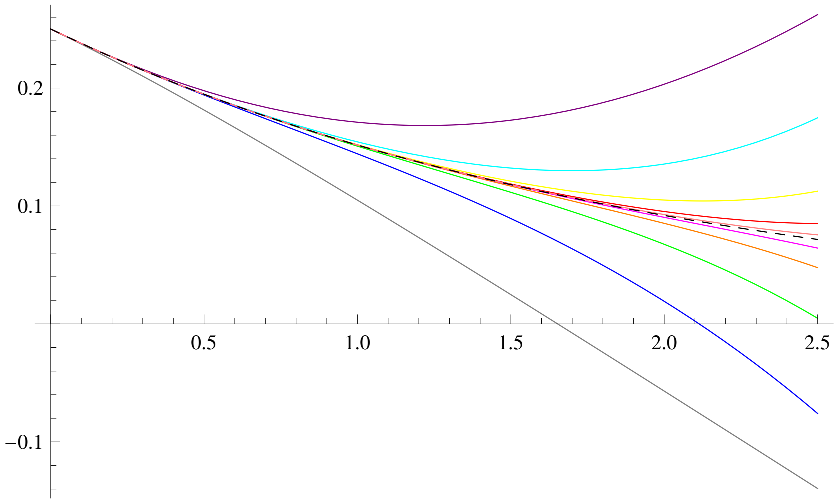

Fig. 4: The exact solution (dashed in black) and several iterations (φ_1 to φ_{10}, alternating at both sides) for the Example 2.

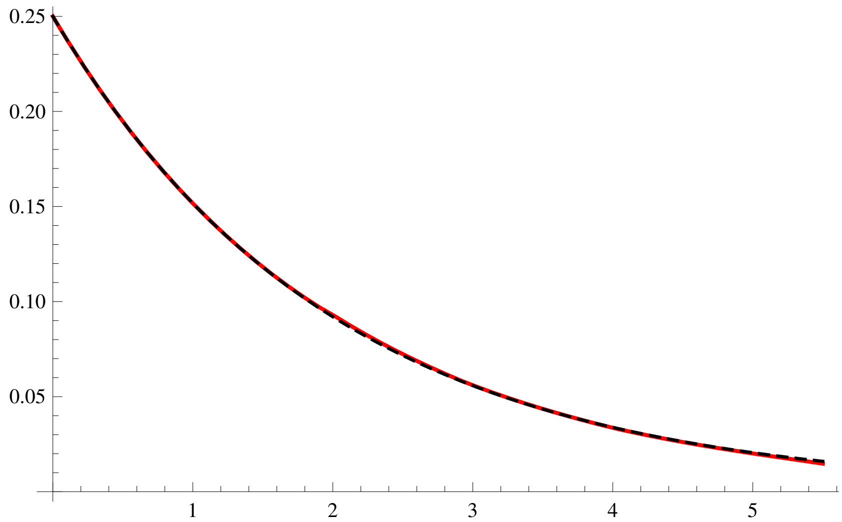

Fig. 5: The exact solution (dashed in black) and three prolongations (intervals $[0,2],[2,3],[3,4]$ and $[4,5]$) for the Example 2.

In Figure 7 we show eight iterations. From φ_6 to φ_8 the curves are practically indistinguishable. It is $||\varphi_8-\varphi_7||_{AC} \cong 0.00003$ in $[0,5]$. The solution seems to have a horizontal asymptote (near 1.167), which is reasonable: as $y(t)$ remains bounded in $[0,T]$, $||f(t,y(t))|| = ||ye^{-t}\sin(t)|| \underset{t\to\infty}{\longrightarrow} 0$ and the Caputo-Fabrizio fractional derivative is null if and only if the function is a constant.

5.5 *Example 5*

In this example we will illustrate an application to hysteresis phenomena.
In [25], a fractional differential equation is proposed to model magnetization processes in ferromagnetic materials. A thermodynamic base is provided to ensure the viability of the model, and the authors also simulate hysteresis phenomena that are in agreement with experimental data.
If **M** is the magnetization vector and **H** is the magnetic field that produces it, in diamagnetic and paramagnetic materials

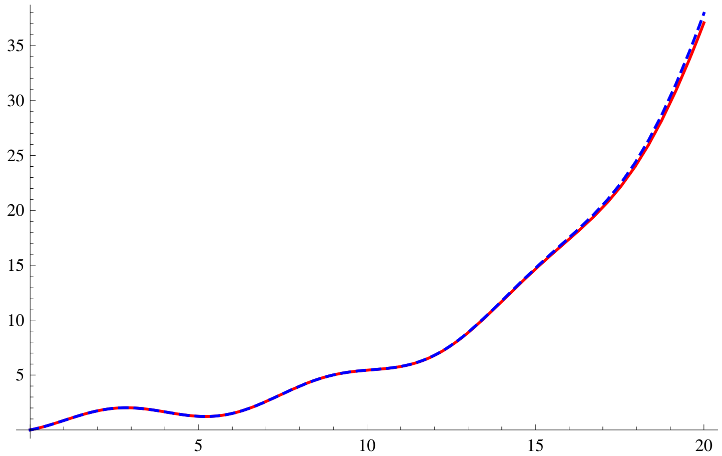

Fig. 6: the 12^{th} iteration and the exact solution (dashed in blue) for the problem of Example 2.

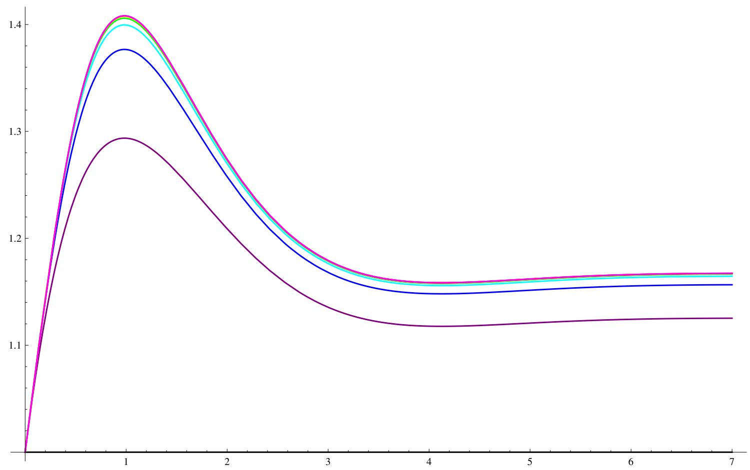

Fig. 7: Several iterations for the Example 4. The approximations accumulate on the pink curve.

the relationship between them is linear,

$$\mathbf{M}(t) = \chi_m \mathbf{H}(t),$$

where χ_m is the so-called magnetic susceptibility of the medium.
In ferromagnetic materials this relationship is no linear and the hysteresis phenomena, related to the *material memory*, are observed (magnetization remains after magnetic field is reduced to 0). In one dimension, the proposed model for this case is

$$c_1 \mathscr{D}_t^{\alpha}[M](t) + c_2 M^3(t) + c_3 M(t) = H(t),$$

where c_1, c_2 and c_3 are constants that have to do with constitutive properties of the material, temperature, and the so-called Curie temperature over which there is no "memory". $\mathscr{D}_t^{\alpha}$ might be the Caputo fractional derivative or the Caputo-Fabrizio fractional derivative.
With the recursive method we have presented in Section 4, and the prolongation procedure we have shown in Section 4.1, we solved the equation

$${}^{CF}_{0}\mathscr{D}_t^{\frac{1}{2}}[M] = f(t,M)$$

with $f(t,M) = \frac{1}{13}[5cos(\frac{t}{5}) - 5e^{-t} + M]$, which corresponds to $c_1 = c_2 = 1$, $c_3 = \frac{2}{5}$ and $H(t) = \frac{1}{13}[5cos(\frac{t}{5}) - 5e^{-t}] + \frac{31}{65}sin(\frac{t}{5}) + sin^3(\frac{t}{5})$; the exact solution is $M(t) = sin(\frac{t}{5})$ for $M(0) = 0$.

In Figure 8 we represent $M_{approx.}(t)$ vs. $H_{approx.}(t)$, with $H_{approx.}(t) = f(t, M_{approx.}(t)) + \frac{2}{5}M_{approx.}(t) + M^3_{approx.}(t)$, giving rise to a typical hysteresis cycle. $M(t)$ was obtained by the iterative method after ten iterations and two prolongations.

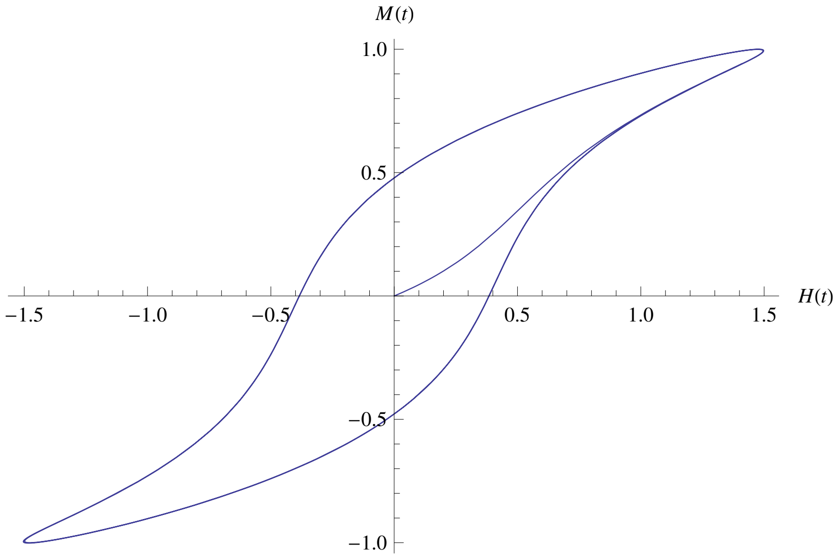

Fig. 8: $M(t)$ **vs.** $H(t)$**.**

6 Conclusion

In this work we have proved a theorem about the existence of an absolutely continuous solution to the initial value problem ${}^{CF}_{0}\mathscr{D}^{\alpha}_{t}[y](t) = f(t, y(t))$, with $y(0) = y_0$, when $f(t, y)$ is linear in its second variable. A recursive method to build analytical (not numerical) approximate solutions to this problem have also been developed, and a general demonstration of its convergence was offered. Both, existence theorem and recursive method, are inspired in the traditional theorem for ordinary differential equations due to Picard and Lindelöf, conveniently adapted to FIVP. Several examples with very good performance of the method have been shown.
Regarding the future work, we are analyzing the possibility of combining the proposed method with existing techniques, to obtain approximate solutions to boundary value problems involving fractional partial differential equations.

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