Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/080305

Inference for a Constant-Stress Accelerated Life Testing for Power Generalized Weibull Distribution under Progressive Type-II Censoring

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Received: 14 Mar. 2019, Revised: 26 May. 2019, Accepted: 15 Jun. 2019
Published online: 1 Nov 2019

Abstract: In this paper, constant-stress Accelerated Life Testing (ALT) is studied when the lifetime of test units follows Power Generalized Weibull (PGW) distribution. The Maximum Likelihood Estimates (MLEs) and Bayes Estimates (BEs) of the model parameters are obtained under type-II progressive censoring. Moreover, the approximate and credible Confidence Intervals (CIs) of the parameters are derived. The optimal stress level is discussed under D-optimality criterion. Furthermore, a real dataset is analyzed to show the suggested methods. Moreover, this real dataset is used to show the role of PGW distribution as an alternative to the other well-known distributions. Finally, simulation studies are conducted to demonstrate the precision of the MLEs and BEs for the parameters of PGW distribution.

Keywords: accelerated life testing, progressive type-II censoring, power generalized Weibull distribution, Bayes estimation, maximum likelihood estimation, credible confidence intervals, simulation study.

1 Introduction

Experiments of reliability and life testing are done to investigate data of failure time which occurs under the normal operating conditions. Due to the hardness of collecting such data which needs long time, we have tended to use ALT in order to obtain adequate failure data in a compact time. In ALT, experiments are done at greater than normal levels of stress to expedite failure occurring. Then, the collected life data is investigated and used to estimate the life characteristics under normal operating conditions. The stress in ALT can be applied in different ways, the most commonly-used methods are constant-stress, step-stress and progressive-stress. Nelson [32] explained the advantages and disadvantages of each of such classifications.

The constant-stress ALT is practiced by operating every unit at a constant high stress till either failure occurs or the test is stopped. Constant-stress models were reviewed by various authors; see Mohie El-Din et al. [27], Kim and Bai [20], Watkins and John [39] and Mohie El-Din et al. [28]. Abdel-Hamid [1] studied the constant partially-accelerated life tests for Burr type-XII distribution with type-II progressive censoring. Jaheen et al. [19] examined the constant partially ALT under progressive type-II censoring for generalized exponential distribution. Guan et al. [16] obtained the optimal constant-stress accelerated life tests with uncensored sampling for the generalized exponential distribution. Mohie El-Din et al. [30] introduced the geometric process as a constant-stress accelerated model.

The step-stress ALT is practiced by increasing the stress on each unit gradually by pre-specified times or according to the occurrence of a fixed number of failures. The step-stress models were analyzed widely in the literature; see Miller and Nelson [23], Bai et al. [9], Gouno et al. [15] and Mohie El-Din et al. [26]. Balakrishnan et al. [13] considered the

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simple step-stress ALT under type-II censoring, considering a cumulative exposure model for exponential distribution. Mohie El-Din et al. [24] utilized the simple step-stress ALT under progressive first-failure censoring, considering a tampered random variable model for Weibull distribution. Mohie El-Din et al. [25] discussed BE for step-stress ALT of PGW distribution under progressive censoring, using a tampered random variable model.

The progressive-stress ALT is employed by increasing the stress on each test unit continuously in time. If an ALT involves linearly increasing stress, this test is referred to as a ramp-stress test. Yin and Sheng [40] obtained the MLEs of parameters of the exponential progressive-stress model. Abdel-Hamid and AL-Hussaini [2] applied the progressive-stress ALT under progressive censoring for Weibull distribution. Abdel-Hamid and Abushul [3] obtained the BE of exponentiated exponential distribution under type-II progressive hybrid censoring, considering the inverse power law and the cumulative exposure model. Mohie El-Din et al. [29] considered progressive-stress ALT for the extension of the exponential distribution. Abd El-Raheem [5] discussed the optimal design of multiple progressive-stress ALT for generalized half normal distribution.

In life testing and reliability inspections, tests are often ended before all units fail. As a result, the censored data is used to reduce test time and cost. The most two traditional Censoring Schemes (CSs) in life testing and reliability experiments are type-I and type-II censoring. Lately, progressive type-II CS has become quite familiar with analyzing highly reliable data. This type of CS can be represented as follows: assume n identical items are set on a life test, the integer $m < n$ is a pre-specified number of failures, and $R_1, R_2, ..., R_m$ are m pre-fixed integers satisfying $R_1 + R_2 + ... + R_m + m = n$. At the time of the primary failure $t_{1:m:n}, R_1$ of the surviving units is randomly removed. Furthermore, at the time of the second failure $t_{2:m:n}, R_2$ of the surviving units is randomly withdrawn and so on. At the time of the mth failure $t_{m:m:n}$, the test is stopped and whole surviving $R_m = n - m - (R_1 + ... + R_{m-1})$ units are withdrawn. For more features about progressive type-II censoring, see Balakrishnan and Aggarwala [12].

The purpose of this study is to apply the constant-stress ALT to units whose lifetime follows PGW distribution under type-II progressive censoring. MLEs, BEs and some inferences for the parameters of the supposed model are studied. The article is prepared as follows: In Section 2, a representation of the lifetime model and test assumptions are displayed. In Section 3, the MLEs of the model parameters are derived. In Section 4, the BEs of model parameters using MCMC method are obtained. In Section 5, the approximate and credible confidence bounds for the model parameters are established. The optimal stress level is discussed in Section 6. In Section 7, a real dataset is analyzed to demonstrate the suggested methods in Sections 3, 4 and 5. Section 8 includes the simulation outcomes. The conclusion is given in Section 9.

2 Model description and test assumptions

2.1 Power generalized Weibull distribution

The PGW distribution is an extension of Weibull distribution. It was founded by Bagdonavicius and Nikulin [10] as a baseline distribution for the accelerated failure time model. It includes distributions with unimodal and bathtub hazard shapes. Also, it allows for a broader class of monotone hazard rate. Besides, it is a right skewed heavy tailed distribution which is not very common in lifetime model. The PGW distribution can be a possible alternative to the exponentiated Weibull distribution for modeling lifetime data, see Nikulin and Haghighi[34]. In Section 7, we present a real example in constant-stress ALT, wherein the PGW distribution is a possible alternative to Weibull, extension of the exponential, generalized exponential, and exponentiated Weibull distributions. This example illustrates the applicability of PGW distribution in lifetime studies and its role as an alternative to the other well-known distributions. These reasons have motivated us to study the constant-stress model with PGW distribution under type-II progressive censoring. As a consequence of the importance of PGW distribution, many authors considered the PGW distribution as a lifetime model. Nikulin and Haghighi [33] proposed a chi-squared type statistic to test the validity of the generalized power Weibull distribution based on the head-and-neck cancer censored data. Alloyarova et al.[8] constructed the Hsuan-Robson-Mirvaliev (HRM) statistic for testing the hypothesis based on moment-type estimators and investigated its properties. Nikulin and Haghighi [35] obtained MLEs of the parameters and illustrated the flexibility of the model by using Efron's [14] head-and-neck cancer clinical trial data. Bagdonavicius and Nikulin [11] proposed chi-squared goodness of fit test for right censored data and applied the proposed test to PGW distribution. Voinov et al. [38] constructed modified chi-squared goodness of fit tests for PGW probability distribution. Mohie El-Din et al. [25] obtained MLEs and BEs based on progressive censoring using step-stress partially-accelerated life tests. Further, they obtained the approximate and the bootstrap confidence intervals of the estimators. Recently, Kumar and Dey [21] studied PGW distribution based on order statistics.

The PGW distribution is determined by the probability density function (pdf):

$$f(t) = \gamma\nu\sigma^{\nu}t^{\nu-1}(1+(\sigma t)^{\nu})^{\gamma-1}\exp\{1-(1+(\sigma t)^{\nu})^{\gamma}\}, \qquad t,\gamma,\nu,\sigma > 0, \tag{1}$$

the corresponding cumulative distribution function (cdf) is

$$F(t) = 1-\exp\{1-(1+(\sigma t)^{\nu})^{\gamma}\}, \qquad t,\gamma,\nu,\sigma > 0, \tag{2}$$

and the corresponding hazard rate function (hrf) is given by

$$h(t) = \gamma\nu\sigma^{\nu}t^{\nu-1}(1+(\sigma t)^{\nu})^{\gamma-1}. \tag{3}$$

There are three special cases of the PGW distribution which are

1-Weibull distribution when $\gamma = 1$.
2-Extension of the exponential distribution [31] when $\nu = 1$.
3-Exponential distribution when $\gamma = 1$ and $\nu = 1$.

2.2 Assumptions and test procedures

The constant-stress ALT under a progressively CS is set as follows: Let S_0 be the use-stress level , and let $S_1 < S_2 < ... < S_k$ be the k accelerated stress levels. Under each constant-stress level S_i, $i = 1,2,...,k$, n_i identical units are tested. Prior to the experiment, the number $m_i(\leq n_i)$, $i = 1,2,...,k$ is fixed, and the progressive censoring scheme $(R_{i1},R_{i2},...,R_{im_i})$ with $R_{ij} \geq 0$, $i = 1,2,...,k$, $j = 1,2,...,m_i$ and $\sum_{j=1}^{m_i} R_{ij} + m_i = n_i$ is specified. Under each stress level S_i, $i = 1,2,...,k$, at the time of the first failure $t_{i1:m_i:n_i}$, R_{i1} units are randomly withdrawn from the remaining $n_i - 1$ surviving units. At the time of the second failure $t_{i2:m_i:n_i}$, R_{i2} units from the remaining $n_i - 2 - R_{i1}$ units are randomly withdrawn. The test continues until the m_ith failure time $t_{im_i:m_i:n_i}$. At failure time $t_{im_i:m_i:n_i}$, all remaining units $R_{im_i} = n_i - m_i - \sum_{j=1}^{m_i-1} R_{ij}$ are removed. When $R_{ij} = 0$, $i = 1,2,...,k$, $j = 1,2,...,m_i - 1$, then $R_{im_i} = n_i - m_i$, which corresponds to the classical constant-stress ALT with type-II CS. When $R_{ij} = 0$, $i = 1,2,...,k$, $j = 1,2,...,m_i$, then $n_i = m_i$, which corresponds to the classical constant-stress ALT with a complete sampling. With these notations the observed progressive censored data under the stress level S_i are $t_{i1:m_i:n_i} < t_{i2:m_i:n_i} < ... < t_{im_i:m_i:n_i}$, $i = 1,2,...,k$.

The following assumptions are used throughout the paper in the framework of constant-stress ALT:

1. Under each constant-stress level S_i, $i = 0,1,...,k$, the failure time T_i follows PGW distribution.
2. The linked function between the life characteristic σ and the stress S takes one of the following shapes:
   - Arrhenius model: $\ln(\sigma) = a + \frac{b}{-S}$, $b > 0$, where S is the temperature.
   - Inverse power model: $\ln(\sigma) = a + b[\ln(S)]$, $b > 0$, where S is the voltage.
   - Exponential model: $\ln(\sigma) = a + bS$, $b > 0$, where S is a weathering variable.

   For further information on these accelerated models, see Nelson [32]. Thus, $\ln(\sigma)$ is a linear function of the stress function $\phi(S) = \frac{1}{-S}$, $\ln(S)$ or S for the above three models. Furthermore, we assume that the linked function between the parameter σ_i and the stress level S_i is

$$\ln(\sigma_i) = a + b\phi_i,\ i = 0,1,...,k, \tag{4}$$

where a and $b(>0)$ are unknown parameters, and $\phi_i = \phi(S_i)$ is an increasing function of S. From the life-stress-linked function in (4), the parameter σ_i can be expressed as

$$\sigma_i = \sigma_0\exp\{b(\phi_i - \phi_0)\} = \sigma_0\theta^{h_i}, \qquad i = 0,1,...,k, \tag{5}$$

where σ_0 is the parameter of the PGW distribution under use-stress level S_0, $\theta = \exp\{b(\phi_1 - \phi_0)\} = \frac{\sigma_1}{\sigma_0} > 1$ is the acceleration factor from S_0 to S_1 and the transformed stress level

$$h_i = \frac{\phi_i - \phi_0}{\phi_1 - \phi_0}, \tag{6}$$

so that $1 \leq h_i < \infty$, $i = 1,2,...,k$.

3 Estimation via maximum likelihood method

This section contains the MLEs of the model parameters γ, σ_0, ν and θ which are obtained under progressive type-II censoring. Assuming $t_{ij} = t_{ij:m_i:n_i}$, $i = 1,2,\cdots,k$ and $j = 1,2,\cdots,m_i$ be the observed data under the stress level S_i. Then the likelihood function of γ, σ_0, ν and θ is given by

$$L(\gamma,\sigma_0,\nu,\theta) = \prod_{i=1}^{k} C_i \prod_{j=1}^{m_i} f_{T_i}(t_{ij})\,[1-F_{T_i}(t_{ij})]^{R_{ij}}\,, \tag{7}$$

where $C_i = n_i\,(n_i-1-R_{i1})\,(n_i-2-R_{i1}-R_{i2})\cdots\left(n_i-m_i+1-\sum_{j=1}^{m_i-1}R_{ij}\right)$.

From (1) and (2) in (7), we get

$$L(\gamma,\sigma_0,\nu,\theta) = \prod_{i=1}^{k} C_i \prod_{j=1}^{m_i} \gamma\nu(\sigma_0\theta^{h_i})^{\nu} t_{ij}^{\nu-1}(1+(\sigma_0\theta^{h_i}t_{ij})^{\nu})^{\gamma-1}Exp[(R_{ij}+1)(1-(1+(\sigma_0\theta^{h_i}t_{ij})^{\nu})^{\gamma})], \tag{8}$$

So, the log-likelihood function is written as

$$\begin{aligned}\ell(\gamma,\sigma_0,\nu,\theta) = &\sum_{i=1}^{k}\log C_i + (\log\gamma+\log\nu+\nu\log\sigma_0)\sum_{i=1}^{k} m_i + \nu\log\theta\sum_{i=1}^{k} m_i h_i \\ &+(\nu-1)\sum_{i=1}^{k}\sum_{j=1}^{m_i}\log t_{ij} + (\gamma-1)\sum_{i=1}^{k}\sum_{j=1}^{m_i}\left[\log(1+(\sigma_0\theta^{h_i}t_{ij})^{\nu})\right] \\ &+\sum_{i=1}^{k}\sum_{j=1}^{m_i}\left[(R_{ij}+1)(1-(1+(\sigma_0\theta^{h_i}t_{ij})^{\nu})^{\gamma})\right],\end{aligned} \tag{9}$$

the likelihood equations of γ, σ_0, θ and ν are respectively

$$\left.\begin{aligned}
\frac{\partial\ell}{\partial\gamma} =& \frac{\sum_{i=1}^{k} m_i}{\gamma} + \sum_{i=1}^{k}\sum_{j=1}^{m_i}\left[\log(1+(\sigma_0\theta^{h_i}t_{ij})^{\nu})\right] - \sum_{i=1}^{k}\sum_{j=1}^{m_i}\left[(R_{ij}+1)(1+(\sigma_0\theta^{h_i}t_{ij})^{\nu})^{\gamma}\log(1+(\sigma_0\theta^{h_i}t_{ij})^{\nu})\right] = 0,\\
\frac{\partial\ell}{\partial\sigma_0} =& \frac{\nu\sum_{i=1}^{k} m_i}{\sigma_0} + (\gamma-1)\sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{\left[\nu\theta^{h_i}t_{ij}(\sigma_0\theta^{h_i}t_{ij})^{\nu-1}\right]}{(1+(\sigma_0\theta^{h_i}t_{ij})^{\nu})}\\
&-\nu\gamma\sum_{i=1}^{k}\sum_{j=1}^{m_i}\left[\theta^{h_i}t_{ij}(R_{ij}+1)(1+(\sigma_0\theta^{h_i}t_{ij})^{\nu})^{\gamma-1}(\sigma_0\theta^{h_i}t_{ij})^{\nu-1}\right] = 0,\\
\frac{\partial\ell}{\partial\nu} =& \frac{\nu\sum_{i=1}^{k} h_i m_i}{\theta} + (\gamma-1)\sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{\left[\nu\sigma_0\theta^{h_i-1}h_i t_{ij}(\sigma_0\theta^{h_i}t_{ij})^{\nu-1}\right]}{(1+(\sigma_0\theta^{h_i}t_{ij})^{\nu})}\\
&-\gamma\sum_{i=1}^{k}\sum_{j=1}^{m_i}\left[t_{ij}\nu\sigma_0\theta^{h_i-1}h_i(R_{ij}+1)(1+(\sigma_0\theta^{h_i}t_{ij})^{\nu})^{\gamma-1}(\sigma_0\theta^{h_i}t_{ij})^{\nu-1}\right] = 0,\\
&\text{and}\\
\frac{\partial\ell}{\partial\theta} =& \frac{\sum_{i=1}^{k} m_i}{\nu} + \log\sigma_0\sum_{i=1}^{k} m_i + \sum_{i=1}^{k}\sum_{j=1}^{m_i}\log t_{ij} + \log\theta\sum_{i=1}^{k} m_i h_i + (\gamma-1)\sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{\left[\log(\nu\theta^{h_i}\sigma_0 t_{ij})(\sigma_0\theta^{h_i}t_{ij})^{\nu}\right]}{(1+(\sigma_0\theta^{h_i}t_{ij})^{\nu})}\\
&-\gamma\sum_{i=1}^{k}\sum_{j=1}^{m_i}\left[\log(\sigma_0\theta^{h_i}t_{ij})(R_{ij}+1)(1+(\sigma_0\theta^{h_i}t_{ij})^{\nu})^{\gamma-1}(\sigma_0\theta^{h_i}t_{ij})^{\nu}\right] = 0.
\end{aligned}\right\} \tag{10}$$

Now, we have a system of four nonlinear equations in four unknown parameters γ, σ_0, ν and θ. It is evident that a closed-form solution is too difficult to be obtained. So, an iterative procedure such as Newton-Raphson can be considered to get numerical solutions of the nonlinear system in (10).

4 Bayes inference

BEs of the model parameters γ, σ_0, ν and θ under progressive type-II censoring are obtained by considering the Square Error (SE) loss function and Linear Exponential loss function (LINEX). Informative priors are used to obtain the BEs. Since for PGW distribution not a single conjugate prior is known till date. Therefore, we assume that γ, σ_0, ν and θ are independent with informative priors as follows:

$$\pi_1(\gamma) \propto \gamma^{\mu_1-1}e^{-\lambda_1\gamma}, \qquad \gamma > 0,\ \mu_1,\ \lambda_1 > 0, \tag{11}$$

$$\pi_2(\sigma_0) \propto \sigma_0^{\mu_2-1}e^{-\lambda_2\sigma_0}, \qquad \sigma_0 > 0,\ \mu_2,\ \lambda_2 > 0, \tag{12}$$

$$\pi_3(\nu) \propto \nu^{\mu_3-1}e^{-\lambda_3\nu}, \qquad \nu > 0,\ \mu_3,\ \lambda_3 > 0, \tag{13}$$

$$\pi_4(\theta) \propto e^{-\beta(\theta-1)}, \qquad \theta > 1,\ \beta > 0. \tag{14}$$

The gamma prior is used for its flexibility and it accommodates different shapes reflected in prior beliefs. The hyper-parameters μ_i and λ_i, $i = 1,2,3$, can be easily evaluated if we consider any two independent information for γ, σ_0 and ν respectively. The informative priors in (11)-(14) can be converted as non-informative priors when $\mu_i = \lambda_i = 0$, $i = 1,2,3$ and $\beta = \frac{\ln\theta}{(\theta+1)}$.

From (11)-(14), the joint prior of the parameters γ, σ_0, ν and θ is given by:

$$\pi(\gamma,\sigma_0,\nu,\theta) \propto \gamma^{\mu_1-1}\sigma_0^{\mu_2-1}\nu^{\mu_3-1}e^{-(\gamma\lambda_1+\sigma_0\lambda_2+\nu\lambda_3+\beta(\theta-1))}, \qquad \gamma,\sigma_0,\nu > 0,\ \theta > 1. \tag{15}$$

The joint posterior density function of the parameters γ, σ_0, ν and θ can be written from (8) and (15) as follows:

$$\begin{aligned}\pi^*(\gamma,\sigma_0,\nu,\theta) &\propto L(\gamma,\sigma_0,\nu,\theta)\ \pi(\gamma,\sigma_0,\nu,\theta)\\ &\propto \gamma^{(\mu_1-1)+\sum_{i=1}^k m_i}\sigma_0^{(\mu_2-1)+\nu\sum_{i=1}^k m_i}\nu^{(\mu_3-1)+\sum_{i=1}^k m_i}\theta^{\nu\sum_{i=1}^k h_i m_i}e^{-(\gamma\lambda_1+\sigma_0\lambda_2+\nu\lambda_3+\beta(\theta-1))}\times\\ &\prod_{i=1}^k\prod_{j=1}^{m_i}(1+(\sigma_0\theta^{h_i}t_{ij})^\nu)^{\gamma-1}\exp\left\{(R_{ij}+1)\left(1-(1+(\sigma_0\theta^{h_i}t_{ij})^\nu)^\gamma\right)\right\}.\end{aligned} \tag{16}$$

The BEs of the function $U(\Theta) = U(\gamma,\sigma_0,\nu,\theta)$ under SE and LINEX loss functions are given respectively by

$$\widetilde{U}_{SE}(\Theta) = E(U(\Theta)), \tag{17}$$

and

$$\widetilde{U}_{LINEX}(\Theta) = -\frac{1}{c}\log[E(e^{-c\,U(\Theta)})], \tag{18}$$

where $E(.)$ is the expected value and $c \neq 0$ is the shape parameter of LINEX loss function.
Regrettably, we cannot compute the expectations in (17) and (18) explicitly. Therefore, Markov Chain Monte Carlo (MCMC) method is used to approximate these expectations.

4.1 MCMC approach

In this subsection, MCMC technique is applied to generate samples from the posterior distribution and then compute the BEs of γ, σ_0, ν and θ.

From the joint posterior density function in (16), the conditional posterior distributions of γ, σ_0, ν and θ are given respectively by:

$$P_1(\gamma|\sigma_0,\nu,\theta) \propto \gamma^{(\mu_1-1)+\sum_{i=1}^{k} m_i} e^{-\gamma\lambda_1} \prod_{i=1}^{k}\prod_{j=1}^{m_i}(1+(\sigma_0\theta^{h_i}t_{ij})^{\nu})^{\gamma} \times \exp\left\{(R_{ij}+1)\left(1-(1+(\sigma_0\theta^{h_i}t_{ij})^{\nu})^{\gamma}\right)\right\}, \tag{19}$$

$$P_2(\sigma_0|\gamma,\nu,\theta) \propto \sigma_0^{(\mu_2-1)+\nu\sum_{i=1}^{k} m_i} e^{-\sigma_0\lambda_2} \prod_{i=1}^{k}\prod_{j=1}^{m_i}(1+(\sigma_0\theta^{h_i}t_{ij})^{\nu})^{\gamma-1} \times \exp\left\{(R_{ij}+1)\left(1-(1+(\sigma_0\theta^{h_i}t_{ij})^{\nu})^{\gamma}\right)\right\}, \tag{20}$$

$$P_3(\nu|\gamma,\sigma_0,\theta) \propto \nu^{(\mu_3-1)+\sum_{i=1}^{k} m_i} e^{-\nu\lambda_3} \prod_{i=1}^{k}\prod_{j=1}^{m_i}(1+(\sigma_0\theta^{h_i}t_{ij})^{\nu})^{\gamma-1} \times \exp\left\{(R_{ij}+1)\left(1-(1+(\sigma_0\theta^{h_i}t_{ij})^{\nu})^{\gamma}\right)\right\}, \tag{21}$$

$$P_4(\theta|\gamma,\sigma_0,\nu) \propto \theta^{\nu\sum_{i=1}^{k} h_i m_i} e^{-\beta(\theta-1)} \prod_{i=1}^{k}\prod_{j=1}^{m_i}(1+(\sigma_0\theta^{h_i}t_{ij})^{\nu})^{\gamma-1} \times \exp\left\{(R_{ij}+1)\left(1-(1+(\sigma_0\theta^{h_i}t_{ij})^{\nu})^{\gamma}\right)\right\}. \tag{22}$$

The conditional posterior distributions of γ, σ_0, ν and θ cannot be reduced analytically to well-known distributions. Therefore, Metropolis-Hasting algorithm is used to generate random samples from these distributions; see Upadhyay and Gupta [37].

The following algorithm can be used to compute BEs of $U = U(\gamma,\sigma_0,\nu,\theta)$ under SE and LINEX loss functions.

Algorithm(1)

1.Begin with an initial guess point of $(\gamma,\sigma_0,\nu,\theta)$ say $(\gamma^{(0)},\sigma_0^{(0)},\nu^{(0)},\theta^{(0)})$.
2.Set $i=1$.
3.Generate $\gamma^{(i)}$, $\sigma_0^{(i)}$, $\nu^{(i)}$ and $\theta^{(i)}$ from equations (19), (20), (21) and (22) respectively.
4.Set $i=i+1$.
5.Repeat steps ((2)-(4)) N times.
6.The approximate means of U and e^{-cU} are given respectively by

$$E(U) = \frac{1}{N-M}\sum_{i=M+1}^{N} U(\gamma^{(i)},\sigma_0^{(i)},\nu^{(i)},\theta^{(i)}), \tag{23}$$

$$E(e^{-cU}) = \frac{1}{N-M}\sum_{i=M+1}^{N} \exp\{-cU(\gamma^{(i)},\sigma_0^{(i)},\nu^{(i)},\theta^{(i)})\}, \tag{24}$$

where M is the burn-in period.

5 Confidence intervals

The approximate and credible CIs of the parameters γ, σ_0, ν and θ are derived in this section.

5.1 Normal approximation confidence interval

The approximate CIs of the four parameters are deduced by the asymptotic distributions of the MLEs of the unknown parameters $\Theta = (\gamma, \sigma_0, \nu, \theta)$. This asymptotic distribution of the MLEs of Θ was introduced by Miller [22].

$$\left((\hat{\gamma}-\gamma),(\hat{\sigma_0}-\sigma_0),(\hat{\nu}-\nu),(\hat{\theta}-\theta)\right) \sim \mathbf{N}(0,\sigma_{ij}),$$

where σ_{ij}, $i, j = 1,2,3,4$, is the variance-covariance matrix of the unknown parameters $\Theta = (\gamma, \sigma_0, \nu, \theta)$. The approximate $100\,(1-\alpha)\%$ two sided CI of ϑ is given by:

$$\left(\hat{\vartheta}_l,\ \hat{\vartheta}_u\right) = \hat{\vartheta} \pm Z_{1-\alpha/2}\sqrt{\hat{\sigma}_{ii}}, \qquad i = 1,2,3,4, \tag{25}$$

where ϑ is γ, σ_0, ν or θ and Z_q is the $100q-th$ percentile of a standard normal distribution.

5.2 Credible confidence intervals

A $100(1-\alpha)\%$ Bayesian credible or posterior interval of a random quantity ϑ is the interval that has the posterior probability $(1-\alpha)$, such ϑ lies in the interval where

$$p(l \le \vartheta \le u) = \int_l^u \pi^*(\vartheta|\mathbf{t})d\vartheta = 1-\alpha.$$

The following algorithm is used to obtain credible CIs of γ, σ_0, ν and θ.

Algorithm (2)

1. Perform steps $((1)-(6))$ in algorithm (1).
2. Repeat the first step K times and arrange the results in ascending order as $\{\tilde{\gamma}^{[1]}, \tilde{\gamma}^{[2]}, \cdots, \tilde{\gamma}^{[K]}\}$, $\{\tilde{\sigma_0}^{[1]}, \tilde{\sigma_0}^{[2]}, \cdots, \tilde{\sigma_0}^{[K]}\}$, $\{\tilde{\nu}^{[1]}, \tilde{\nu}^{[2]}, \cdots, \tilde{\nu}^{[K]}\}$ and $\{\tilde{\theta}^{[1]}, \tilde{\theta}^{[2]}, \cdots, \tilde{\theta}^{[K]}\}$.

Then, the $100\,(1-\alpha)\%$ credible CI of ϑ is expressed by

$$\left(\tilde{\vartheta}_l,\ \tilde{\vartheta}_u\right) = \left(\tilde{\vartheta}^{[\alpha K/2]},\ \tilde{\vartheta}^{[(1-\alpha/2)K]}\right), \qquad \text{where } \vartheta \text{ is } \gamma,\ \sigma_0,\ \nu \text{ or } \theta. \tag{26}$$

6 Optimal stress level

Through the past three decades, the problem of optimal design ALT has received a great consideration in the reliability literature, see, for example, Miller and Nelson [23], Bai et al. [9] and Gouno et al. [15]. Han and Ng [17] introduced a comparative study between the optimal design of constant and step-stress ALT for exponential distribution under type-I censoring. Guan et al. [16] derived the optimal plans of constant-stress ALTs for the generalized exponential distribution. Han [18] considered time- and cost-constrained optimal designs of constant and step-stress ALTs for the exponential distribution. Abdel-Hamid and AL-Hussaini [4] considered the problem of optimally designing a step-stress partially ALT for progressively type-I censored data from generalized pareto distribution. Mohie El-Din et al. [28] obtained the optimal designs of constant-stress ALTs for Lindley distribution. Abd El-Raheem [6] derived the optimal designs of constant-stress ALT for the extension of the exponential distribution. Abd El-Raheem [7] expanded his results in Abd El-Raheem [6] to the censored data.

In this section, we investigate the problem of choosing the optimal transformed stress level h_i, $i = 1,2,...,k$, of constant-stress ALT for progressively type-II censored data from PGW distribution. For simplicity of discussion, we only consider the case $k = 2$ stress levels (h_1, h_2) in the life test. Because the smallest transformed stress level is fixed at $h_1 = 1$, the problem becomes to solve the optimal stress level of h_2 which fulfills the criterion.

6.1 D-optimality

The D-optimality criterion is frequently used in designing ALT by maximizing the determinant of the Fisher information matrix. The local Fisher information matrix, **F**, for MLEs $(\hat{\gamma}, \hat{\sigma_0}, \hat{\nu}, \hat{\theta})$ is the 4×4 symmetric matrix of negative second partial derivatives of $\ell(\gamma, \sigma_0, \nu, \theta)$ with respect to γ, σ_0, ν, and θ, see Nelson [32]. If $\vartheta_1 = \gamma$, $\vartheta_2 = \sigma_0$, $\vartheta_3 = \nu$, and $\vartheta_4 = \theta$, then

$$\mathbf{F} = \left(-\frac{\partial^2 \hat{\ell}(\gamma, \sigma_0, \nu, \theta)}{\partial \vartheta_i \partial \vartheta_j} \right)_{4\times4}, \tag{27}$$

where the hate $\hat{}$ indicates that the derivative is calculated at $(\hat{\gamma}, \hat{\sigma_0}, \hat{\nu}, \hat{\theta})$. The optimal transformed stress level h_2^* can be obtained by

$$\text{Maximize}\{det(\mathbf{F}(\hat{\gamma}, \hat{\sigma_0}, \hat{\nu}, \hat{\theta}))\}. \tag{28}$$

7 Application

In this section, we demonstrate the proposed procedures in this article with a real-data example. Moreover, the real dataset is used to show that PGW can be a better model than the Extension of the Exponential (EE) distribution, Weibull (W) distribution, Generalized Exponential (GE) distribution and the Exponentiated Weibull (EW) distribution.

7.1 Example

The progressively-censored data in Table 7.1 represents the failure times in hours of transformers at high voltage, see Nelson [32] (page 161). In this life test the design voltage is $14.4KV$. In Table 7.1, + denotes censored data.

Table 7.1: The failure times in hours of transformer life testing at high voltage

$35.4KV$	$42.4KV$	$46.7KV$
40.1	0.6	3.1
59.4	13.4	8.3
71.2	15.2	8.9
166.5	19.9	9.0
204.7	25.0	13.6
229.7	30.2	14.9
308.3	32.8	16.1
537.9	44.4	16.9
1002.3+	50.2+	21.3
1002.3+	56.2	48.1+

By engineering experience, inverse power model is sufficient to describe the acceleration voltage relationship. So, the acceleration model can be expressed as

$$\ln(\sigma_i) = a + b\ln(S_i),\ b > 0, \qquad i = 0, 1, 2, 3. \tag{29}$$

In this example, $S_0 = 14.4KV$, $S_1 = 35.4KV$, $S_2 = 42.4KV$, $S_3 = 46.7KV$ and $\phi_i = \ln(S_i)$, $i = 0, 1, 2, 3$.
For the data in Table 7.1 the progressive censoring schemes R_{ij}, $i = 1, 2, 3$, $j = 1, ..., m_i$ of each stress level are as follows:

- Under $S_1 = 35.4KV$: $n_1 = 10$, $m_1 = 8$ and $R_{1j} = 0$, $j = 1, ..., 7$, $R_{18} = 2$.
- Under $S_2 = 42.4KV$: $n_2 = 10$, $m_2 = 9$ and $R_{2j} = 0$, $j = 1, ..., 7$, $R_{28} = 1$, $R_{29} = 0$.
- Under $S_3 = 46.7KV$: $n_3 = 10$, $m_3 = 9$ and $R_{3j} = 0$, $j = 1, ..., 8$, $R_{39} = 1$.

Modified Kolmogorov-Smirnov goodness of fit test for progressively type-II censored data is used to check the validity of the five distributions PGW, EE, W, GE and EW with the data in Table 7.1. The modified Kolmogorov-Smirnov statistic for progressive type-II censored data was introduced by Pakyari and Balakrishnan [36]. Let $T_{1:m:n} < T_{2:m:n} < ... < T_{m:m:n}$ be a progressively type-II censored sample with progressive censoring scheme $(R_1, R_2, ..., R_m)$ from a continuous distribution function $F(t, \vartheta)$. Then the modified Kolmogorov-Smirnov statistic for progressive type-II censored data is given by

$$D_{m:n} = \max\{D^+_{m:n},\ D^-_{m:n}\}, \tag{30}$$

where

$$D^+_{m:n} = \max_i \{v_{i:m:n} - u_{i:m:n}\},$$

$$D^-_{m:n} = \max_i \{u_{i:m:n} - v_{i-1:m:n}\},$$

where $v_{i:m:n} = E(U_{i:m:n})$ is the expected value of the i-th type-II progressively-censored order statistic from the uniform(0, 1) distribution, given by

$$v_{i:m:n} = 1 - \prod_{j=m-i+1}^{m} \left(\frac{j + R_{m-j+1} + \ldots + R_m}{j + 1 + R_{m-j+1} + \ldots + R_m} \right),$$

and $u_{i:m:n} = F(t_{i:m:n}, \hat{\vartheta})$ for $i = 1, 2, \ldots, m$.

The values of test statistic $(D_{m:n})$ and the corresponding P-values for each stress level S_i, $i = 1, 2, 3$ are presented in Table 7.2. Because of all P-values are greater than 0.05, the five models provide good fit to the given data. To compare between the five models, we use Akaike Information Criterion (AIC) as a tool to compare different models. So, the method of maximum likelihood is used to obtain the estimates of the parameters of the five distributions PGW, EE, W, GE and EW. The AIC and the estimated parameters for the five distributions are summarized in Table 7.3. It is clear that the PGW distribution provides a better fit compared to EE, W, GE and EW distributions regarding AIC.

Table 7.2: Test statistic and the corresponding P-value of each stress level for the five models

Stress (voltage)	Distribution	35.4KV	42.4KV	46.7KV
Test statistic $(D_{m:n})$	PGW	0.1722	0.2316	0.1997
P-value		0.757	0.36	0.524
Test statistic $(D_{m:n})$	EE	0.1715	0.2361	0.2014
P-value		0.822	0.366	0.535
Test statistic $(D_{m:n})$	W	0.1814	0.2250	0.1969
P-value		0.654	0.496	0.552
Test statistic $(D_{m:n})$	GE	0.1871	0.2224	0.1957
P-value		0.57	0.525	0.571
Test statistic $(D_{m:n})$	EW	0.1871	0.2246	0.1967
P-value		0.587	0.522	0.558

Table 7.3: The AIC and estimated parameters for the five models

Distribution	$F(t)$	AIC	Estimated parameters
PGW	$1-\exp\{1-(1+(\sigma t)^{\nu})^{\gamma}\}$	279.476	$\hat{\gamma}=0.5047$, $\hat{\sigma_0}=0.0060$, $\hat{\nu}=0.99$ and $\hat{\theta}=38.435$
EE	$1-\exp\{1-(1+\sigma t)^{\gamma}\}$	279.926	$\hat{\gamma}=0.4841$, $\hat{\sigma_0}=0.0012$ and $\hat{\theta}=22.7436$
W	$1-\exp\{-(\sigma t)^{\gamma}\}$	284.246	$\hat{\gamma}=0.7512$, $\hat{\sigma_0}=0.0005$ and $\hat{\theta}=15.063$
GE	$(1-\exp\{-(\sigma t)\})^{\gamma}$	294.138	$\hat{\gamma}=0.34446$, $\hat{\sigma_0}=0.000171$ and $\hat{\theta}=17.0303$
EW	$(1-\exp\{-(\sigma t)^{\nu}\})^{\gamma}$	303.545	$\hat{\gamma}=0.3712$, $\hat{\sigma_0}=0.0005$, $\hat{\nu}=0.9843$ and $\hat{\theta}=5.9936$

The MLEs and BEs of the parameters γ, σ_0, ν and θ of the PGW distribution are introduced in Table 7.4. For this dataset, non-informative priors are considered for Bayesian analysis. From the results in Table 7.4, we noticed that BEs of σ_0 and θ give more accurate results than the MLEs under SE loss function through the lengths of the CIs.

Table 7.4: MLEs and BEs along with their lengths of 95% CIs inside the parentheses of γ, σ_0, ν and θ for the real dataset

$\hat{\vartheta}$	MLE	BE
$\hat{\gamma}$	0.5047 (0.5225)	1.5840(2.8341)
$\hat{\sigma_0}$	0.0006(0.0023)	0.00142(0.0017)
$\hat{\nu}$	0.9900(1.0326)	2.00261 (2.5401)
$\hat{\theta}$	38.435(88.9851)	17.564(52.6284)

8 Simulation studies

A comparison between the performances of the MLEs and BEs under SE and LINEX loss functions in terms of their Mean Square Errors (MSEs) and Relative Absolute Biases (RABs) for various choices of n_i, m_i and R_{ij}, $i = 1,2,\cdots,k$, $j = 1,2,\cdots,m_i$ is performed through the simulation study. Moreover, the 95% asymptotic and credible CIs are calculated. The progressive censoring schemes which are used in the simulation studies are shown in Table 8.1. Table 8.2 introducing MSEs of the MLEs and BEs in the case of informative priors of the model parameters. Moreover, Table 8.3 introduces RABs of the MLEs and BEs in the case of informative priors of the model parameters. Table 8.4 includes lengths and coverage probabilities of 95% approximate and credible CIs. Finally, Table 8.5 displays the average optimal stress level h_k^* over 1000 repetition according to D-optimality criterion.
The following algorithm is performed to obtain the MLEs, BEs, approximate and credible confidence intervals of the four parameters.

Algorithm (3)

1.Specify the values of n_i, m_i, k, $S_0, S_1,\cdots,S_k$ and c.
2.For given values of the prior parameters (μ_1,λ_1), (μ_2,λ_2), (μ_3,λ_3) and β generate γ, σ_0, ν and θ from (11), (12), (13) and (14) respectively.
3.Generate k simple random samples of size m_i from Uniform$(0,1)$ distribution, $(U_{i1},U_{i2},\cdots,U_{im_i})$, $i = 1,2,\cdots,k$.
4.Determine the values of the censored schemes, R_{ij}, $i = 1,2,\cdots,k$, and $j = 1,2,\cdots,m_i$ such that $\sum_{j=1}^{m_i} R_{ij} = n_i - m_i$.
5.Set $E_{ij} = U_{ij}^{1/(j+\sum_{d=m_i-j+1}^{m_i} R_{id})}$, $j = 1,2,\cdots,m_i$, and $i = 1,2,\cdots,k$.
6.Obtain the progressive type-II censored samples $(U^*_{i1},U^*_{i2},\cdots,U^*_{im_i})$, where $U^*_{ij} = 1-\prod_{d=m_i-j+1}^{m_i} E_{id}$, $j = 1,2,\cdots,m_i$, $i = 1,2,\cdots,k$.
7.Use step 6, to generate random samples $(t_{i1},\cdots,t_{im_i})$, $i = 1,2,\cdots,k$, from equations (2) and (5) as follows:

$$t_{ij} = \frac{1}{\sigma_0 \theta^{h_i}} \left[\left(1-\log(1-U^*_{ij})\right)^{\frac{1}{\gamma}} - 1 \right]^{\frac{1}{\nu}}, \quad j = 1,2,\cdots,m_i,\ i = 1,2,\cdots,k.$$

8.Use the progressive censored data to obtain the MLEs of the model parameters by solving the nonlinear system (10).
9.Compute the BEs of the model parameters relative to SE and LINEX loss functions, using algorithm (1), with $N = 11000$ and $M = 1000$.
10.Compute the approximate confidence bounds with confidence level 95% for the four parameters γ, σ_0, ν and θ.
11.Compute 95% credible confidence intervals using algorithm (2) .
12.Replicate the steps $((3)-(11))$, 1000 times.
13.Compute the average values of the MSEs and RABs associated with the MLEs and BEs of the four parameters.
14.Do steps ((1)-(13)) with different values of prior parameters, n_i, m_i and R_{ij}, $j = 1,2,\cdots,m_i$, $i = 1,2,\cdots,k$.

Table 8.1: The progressive censoring schemes used in the simulation studies

n_i	m_i	C.S	$(R_{i1},\cdots,R_{im_i})$	C.S	$(R_{i1},\cdots,R_{im_i})$	C.S	$(R_{i1},\cdots,R_{im_i})$
$n_i=\begin{cases}29 & i=1\\16 & i=2\\13 & i=3\\7 & i=4\end{cases}$	$m_i=\begin{cases}25 & i=1\\13 & i=2\\11 & i=3\\6 & i=4\end{cases}$	[1]	$R_{ij}=\begin{cases}4 & i=1,j=1\\3 & i=2,j=1\\2 & i=3,j=1\\1 & i=4,j=1\\0 & other\ wise\end{cases}$	[2]	$R_{ij}=\begin{cases}4 & i=1,j=m_1\\3 & i=2,j=m_2\\2 & i=3,j=m_3\\1 & i=4,j=m_4\\0 & other\ wise\end{cases}$	[3]	$R_{ij}=\begin{cases}1 & i=1,j=17,\cdots,20\\1 & i=2,j=9,10,11\\1 & i=3,j=8,9\\1 & i=4,j=5\\0 & other\ wise\end{cases}$
$n_i=\begin{cases}35 & i=1\\20 & i=2\\15 & i=3\\10 & i=4\end{cases}$	$m_i=\begin{cases}30 & i=1\\15 & i=2\\12 & i=3\\8 & i=4\end{cases}$	[4]	$R_{ij}=\begin{cases}5 & i=1,j=1\\5 & i=2,j=1\\3 & i=3,j=1\\2 & i=4,j=1\\0 & other\ wise\end{cases}$	[5]	$R_{ij}=\begin{cases}5 & i=1,j=m_1\\5 & i=2,j=m_2\\3 & i=3,j=m_3\\2 & i=4,j=m_4\\0 & other\ wise\end{cases}$	[6]	$R_{ij}=\begin{cases}1 & i=1,j=16,\cdots,20\\1 & i=2,j=8,\cdots,12\\1 & i=3,j=7,8,9\\1 & i=4,j=5,6\\0 & other\ wise\end{cases}$
$n_i=\begin{cases}45 & i=1\\25 & i=2\\20 & i=3\\10 & i=4\end{cases}$	$m_i=\begin{cases}37 & i=1\\20 & i=2\\16 & i=3\\8 & i=4\end{cases}$	[7]	$R_{ij}=\begin{cases}8 & i=1,j=1\\5 & i=2,j=1\\4 & i=3,j=1\\2 & i=4,j=1\\0 & other\ wise\end{cases}$	[8]	$R_{ij}=\begin{cases}8 & i=1,j=m_1\\5 & i=2,j=m_2\\4 & i=3,j=m_3\\2 & i=4,j=m_4\\0 & other\ wise\end{cases}$	[9]	$R_{ij}=\begin{cases}1 & i=1,j=28,\cdots,35\\1 & i=2,j=14,\cdots,18\\1 & i=3,j=11,\cdots,14\\1 & i=4,j=6,7\\0 & other\ wise\end{cases}$
$n_i=\begin{cases}50 & i=1\\30 & i=2\\25 & i=3\\15 & i=4\end{cases}$	$m_i=\begin{cases}44 & i=1\\25 & i=2\\21 & i=3\\12 & i=4\end{cases}$	[10]	$R_{ij}=\begin{cases}6 & i=1,j=1\\5 & i=2,j=1\\4 & i=3,j=1\\3 & i=4,j=1\\0 & other\ wise\end{cases}$	[11]	$R_{ij}=\begin{cases}6 & i=1,j=m_1\\5 & i=2,j=m_2\\4 & i=3,j=m_3\\3 & i=4,j=m_4\\0 & other\ wise\end{cases}$	[12]	$R_{ij}=\begin{cases}1 & i=1,j=37,\cdots,42\\1 & i=2,j=16,\cdots,20\\1 & i=3,j=16,\cdots,19\\1 & i=4,j=10,11,12\\0 & other\ wise\end{cases}$

Table 8.2: MSEs for MLEs and BEs under SE (BSE) and LINEX (BLINEX) loss functions of γ, σ_0, ν and θ with true values of parameters ($\gamma = 0.9181$, $\sigma_0 = 1.0108$, $\nu = 1.102$ and $\theta = 3.2102$), values of the prior parameters ($\mu_1 = 84.29$, $\mu_2 = 102.17$, $\mu_3 = 1214.40$, $\lambda_1 = 91081$, $\lambda_2 = 101.08$, $\lambda_3 = 1102$ and $\beta = 0.452$), the number of stress levels ($k = 4$), and $S_0 = 50$, $S_1 = 70$, $S_2 = 100$, $S_3 = 120$ and $S_4 = 160$

$\sum_{i=1}^{k} n_i$	$\sum_{i=1}^{k} m_i$	$C.S$	ϑ	MLE	BSE	BLINEX		
						$c=-2$	$c=.001$	$c=2$
65	55	[1]	γ	1.4661	0.0135	0.0125	0.0135	0.0145
			σ_0	0.8104	0.0098	0.0118	0.0098	0.0080
			ν	0.1052	0.0086	0.0088	0.0085	0.0084
			θ	0.0637	0.1871	0.2269	0.1871	0.1541
		[2]	γ	1.2999	0.0138	0.0127	0.0138	0.0151
			σ_0	1.0308	0.0113	0.0135	0.0113	0.0094
			ν	0.0991	0.0086	0.0088	0.0086	0.0084
			θ	0.0820	0.1438	0.1759	0.1438	0.1171
		[3]	γ	1.6429	0.0144	0.0130	0.0145	0.0153
			σ_0	0.9136	0.0063	0.0118	0.0062	0.0080
			ν	0.0761	0.0085	0.0087	0.0085	0.0083
			θ	0.0686	0.1401	0.1767	0.1436	0.1163
	65		γ	1.0798	0.0120	0.0212	0.0120	0.0118
			σ_0	1.3024	0.0072	0.0147	0.0072	0.0069
			ν	0.0877	0.0079	0.0012	0.0079	0.0021
			θ	0.0840	0.1304	0.2025	0.1304	0.0861
80	65	[4]	γ	1.3576	0.0068	0.0061	0.0068	0.0076
			σ_0	0.6766	0.0053	0.0067	0.0052	0.0040
			ν	0.0817	0.0056	0.0055	0.0055	0.0052
			θ	0.0605	0.1163	0.1523	0.1163	0.0882
		[5]	γ	1.3187	0.0091	0.0081	0.0091	0.0101
			σ_0	1.1521	0.0057	0.0071	0.0057	0.0044
			ν	0.0564	0.0054	0.0056	0.0054	0.0053
			θ	0.0677	0.1392	0.1771	0.1392	0.1086
		[6]	γ	1.3078	0.0064	0.0082	0.0063	0.0101
			σ_0	0.6471	0.0055	0.0069	0.0055	0.0043
			ν	0.0597	0.0053	0.0056	0.0054	0.0052
			θ	0.0652	0.1418	0.1809	0.1418	0.1102
	80		γ	0.9873	0.0062	0.0129	0.0062	0.0049
			σ_0	0.5521	0.0052	0.0085	0.0052	0.0029
			ν	0.0547	0.0052	0.0080	0.0052	0.0064
			θ	0.0540	0.1029	0.1383	0.1029	0.0850

9 Conclusion

Constant-stress ALT model for PGW distribution is introduced under progressive type-II censored data. MLEs and BEs in the case of informative priors of the model parameters γ, σ_0, ν and θ are determined through a real dataset. Point estimation of the model parameters γ, σ_0, ν and θ has been investigated through maximum likelihood and Bayes methods in terms of their MSEs and RABs. Also, approximate and credible CIs are established for the model parameters γ, σ_0, ν and θ. The calculations are done depending on different sample sizes and three different progressive censoring schemes, one of them represents the traditional type-II censoring. From the results in Tables (8.2)-(8.4), we have noticed the following:

1. The MSEs and RABs of MLEs and BEs of the parameters decrease as the sample size increases, except for few cases. This may be due to variation in data.
2. The BEs of γ, σ_0 and ν give more accurate results than MLEs through the MSEs and RABs.
3. The MLEs of θ give more accurate results through the MSEs than BEs.
4. The BEs of γ, σ_0, ν and θ under LINEX loss function ($c = .001$) have the same MSEs and RABs as compared with estimates under SE loss function, except for few cases.
5. The lengths of approximate and credible CIs decrease as the sample size increases.
6. The credible CIs of γ, σ_0, ν and θ give more accurate results than approximate CIs through lengths.

(continued)

$\sum_{i=1}^{k} n_i$	$\sum_{i=1}^{k} m_i$	$C.S$	ϑ	MLE	BSE	BLINEX		
						$c=-2$	$c=.001$	$c=2$
100	81	[7]	γ	1.2380	0.0050	0.0044	0.0050	0.0057
			σ_0	0.5391	0.0032	0.0043	0.0032	0.0023
			ν	0.0546	0.0035	0.0036	0.0035	0.0034
			θ	0.0445	0.1035	0.1386	0.1035	0.0765
		[8]	γ	0.9093	0.0055	0.0048	0.0055	0.0063
			σ_0	0.9930	0.0033	0.0044	0.0033	0.0024
			ν	0.0523	0.0035	0.0036	0.0035	0.0034
			θ	0.0596	0.0969	0.1277	0.0969	0.0730
		[9]	γ	1.016	0.0044	0.0038	0.0044	0.0051
			σ_0	0.6497	0.0032	0.0043	0.0032	0.0023
			ν	0.0597	0.0035	0.0036	0.0035	0.0033
			θ	0.0587	0.1076	0.1406	0.1076	0.0816
	100		γ	0.6537	0.0076	0.0031	0.0076	0.0048
			σ_0	0.3938	0.0031	0.0054	0.0031	0.0024
			ν	0.0371	0.0033	0.0051	0.0033	0.0029
			θ	0.0441	0.0984	0.1872	0.0985	0.1171
120	102	[10]	γ	0.8749	0.0034	0.0030	0.0034	0.0039
			σ_0	0.4082	0.0022	0.0031	0.0022	0.0015
			ν	0.0352	0.0025	0.0026	0.0025	0.0024
			θ	0.0564	0.0834	0.1188	0.0834	0.0577
		[11]	γ	0.8965	0.0043	0.0040	0.0044	0.0047
			σ_0	0.6416	0.0020	0.0028	0.0020	0.0014
			ν	0.0368	0.0025	0.0026	0.0025	0.0024
			θ	0.0452	0.1017	0.1017	0.1017	0.0742
		[12]	γ	0.7483	0.0038	0.0032	0.0038	0.0045
			σ_0	0.5140	0.0020	0.0029	0.0020	0.0014
			ν	0.0391	0.0025	0.0026	0.0025	0.0024
			θ	0.0304	0.0939	0.1310	0.0939	0.0663
	120		γ	0.8311	0.0030	0.0026	0.0030	0.0036
			σ_0	0.3378	0.0024	0.0034	0.0024	0.0017
			ν	0.0286	0.0025	0.0026	0.0025	0.0024
			θ	0.0416	0.1179	0.1602	0.1179	0.0864

Table 8.3: RABs for MLEs and BEs of γ, σ_0, ν and θ with true values of parameters ($\gamma = 0.9181$, $\sigma_0 = 1.0108$, $\nu = 1.102$ and $\theta = 3.2102$), values of the prior parameters ($\mu_1 = 84.29$, $\mu_2 = 102.17$, $\mu_3 = 1214.40$, $\lambda_1 = 91081$, $\lambda_2 = 101.08$, $\lambda_3 = 1102$ and $\beta = 0.452$), the number of stress levels ($k = 4$), and $S_0 = 50$, $S_1 = 70$, $S_2 = 100$, $S_3 = 120$ and $S_4 = 160$

$\sum_{i=1}^{k} n_i$	$\sum_{i=1}^{k} m_i$	$C.S$	ϑ	MLE	BSE	BLINEX		
						$c=-2$	$c=.001$	$c=2$
65	55	[1]	γ	0.6926	0.0129	0.0176	0.0129	0.0082
			σ_0	0.8838	0.7449	0.7447	0.7449	0.745
			ν	0.2856	0.1334	0.1381	0.1334	0.1292
			θ	0.4617	0.4311	0.4221	0.4311	0.4451
		[2]	γ	0.7263	0.0197	0.0246	0.0197	0.0149
			σ_0	0.8653	0.7294	0.7297	0.7294	0.7105
			ν	0.2334	0.1285	0.1458	0.1286	0.1016
			θ	0.4291	0.3945	0.3815	0.3945	0.4125
		[3]	γ	0.7045	0.0161	0.0210	0.0162	0.0114
			σ_0	0.8650	0.7299	0.7355	0.7297	0.7194
			ν	0.2454	0.1241	0.1203	0.1240	0.1279
			θ	0.4423	0.3219	0.3201	0.3218	0.4018

(continued)

$\sum_{i=1}^{k} n_i$	$\sum_{i=1}^{k} m_i$	$C.S$	ϑ	MLE	BSE	BLINEX		
						$c=-2$	$c=.001$	$c=2$
	65		γ	0.6843	0.0203	0.0250	0.0203	0.0156
			σ_0	0.8698	0.7389	0.7388	0.7389	0.7290
			ν	0.2828	0.1705	0.1743	0.1705	0.1669
			θ	0.4465	0.2596	0.2460	0.2596	0.2642
80	65	[4]	γ	0.6726	0.0106	0.0127	0.0106	0.0089
			σ_0	0.8713	0.7159	0.7249	0.7159	0.7021
			ν	0.1483	0.1141	0.1215	0.1141	0.1109
			θ	0.4554	0.3856	0.3761	0.3856	0.3916
		[5]	γ	0.7032	0.0142	0.0174	0.0142	0.0116
			σ_0	0.8576	0.7283	0.7418	0.7283	0.7130
			ν	0.2244	0.1264	0.1354	0.1264	0.1009
			θ	0.4167	0.3514	0.3340	0.3514	0.3604
		[6]	γ	0.6974	0.0107	0.0091	0.0107	0.0184
			σ_0	0.8583	0.7107	0.6840	0.7107	0.7401
			ν	0.2359	0.1154	0.1008	0.1153	0.1249
			θ	0.4336	0.3009	0.3105	0.3008	0.2954
	80		γ	0.6609	0.0170	0.0220	0.0170	0.0121
			σ_0	0.8561	0.7260	0.7306	0.7260	0.7150
			ν	0.2252	0.1390	0.1399	0.1390	0.1385
			θ	0.4295	0.2153	0.2022	0.2153	0.2225
100	81	[7]	γ	0.6001	0.0087	0.0109	0.0087	0.0079
			σ_0	0.8674	0.6429	0.6655	0.6429	0.5912
			ν	0.1246	0.0674	0.0677	0.0674	0.0666
			θ	0.4401	0.3551	0.3540	0.3551	0.3564
		[8]	γ	0.7001	0.0101	0.0108	0.0100	0.0079
			σ_0	0.8266	0.5673	0.5845	0.5673	0.5312
			ν	0.1046	0.0684	0.0696	0.0683	0.0642
			θ	0.6541	0.0086	0.0074	0.0086	0.0105
		[9]	γ	0.6254	0.0056	0.0091	0.0056	0.0039
			σ_0	0.5812	0.3210	0.3432	0.3209	0.3050
			ν	0.1023	0.0185	0.0214	0.0184	0.0158
			θ	0.3006	0.1954	0.1749	0.1954	0.2012
	100		γ	0.5891	0.0097	0.0102	0.0096	0.0074
			σ_0	0.7641	0.6008	0.6051	0.6008	0.6001
			ν	0.2021	0.1092	0.1103	0.1092	0.1021
			θ	0.2985	0.1950	0.1708	0.1950	0.1992
120	102	[10]	γ	0.5708	0.0062	0.0081	0.0062	0.0054
			σ_0	0.6753	0.4691	0.4715	0.4690	0.4502
			ν	0.0944	0.0155	0.0198	0.0155	0.0116
			θ	0.3581	0.2157	0.2003	0.2156	0.2258
		[11]	γ	0.6181	0.0087	0.0092	0.0086	0.0064
			σ_0	0.7125	0.4381	0.4594	0.4381	0.4175
			ν	0.0927	0.0482	0.0580	0.0482	0.0326
			θ	0.5090	0.0058	0.0044	0.0057	0.0061
		[12]	γ	0.5524	0.0042	0.0049	0.0042	0.0037
			σ_0	0.5004	0.2501	0.2611	0.2500	0.2368
			ν	0.0873	0.0097	0.0109	0.0097	0.0068
			θ	0.2814	0.1758	0.1704	0.1758	0.1823
	120		γ	0.5001	0.0052	0.0067	0.0051	0.0048
			σ_0	0.6213	0.5801	0.5417	0.5801	0.4197
			ν	0.1892	0.0972	0.0996	0.0971	0.0854
			θ	0.2712	0.1758	0.1675	0.1758	0.1802

Table 8.4: Lengths and coverage probabilities of 95% approximate (Appro), credible (Cred) CIs for γ, σ_0, v and θ with true values ($\gamma = 0.9181$, $\sigma_0 = 1.0108$, $v = 1.102$ and $\theta = 3.2102$), the number of stress levels ($k = 4$), and $S_0 = 50$, $S_1 = 70$, $S_2 = 100$, $S_3 = 120$ and $S_4 = 160$

$\sum_{i=1}^{k} n_i$	$\sum_{i=1}^{k} m_i$	$C.S$	ϑ	Length of CI		Coverage probability of CI	
				Appro	Cred	Appro	Cred
65	55	[1]	γ	5.0732	0.3189	0.9250	0.992
			σ_0	0.0654	0.0121	0.808	1
			v	0.4980	0.1498	0.933	1
			θ	1.2085	0.0028	0.95	0.933
		[2]	γ	4.2532	0.38	0.875	0.992
			σ_0	0.0575	0.0123	0.792	1
			v	0.3951	0.1238	0.983	1
			θ	0.9661	0.0022	0.95	0.875
		[3]	γ	4.7135	0.3795	0.925	1
			σ_0	0.0620	0.0128	0.767	1
			v	0.4504	0.1387	0.942	1
			θ	1.0812	0.0025	0.975	0.908
	65		γ	3.4704	0.3755	0.917	1
			σ_0	0.0519	0.0117	0.808	1
			v	0.3713	0.1447	0.958	1
			θ	0.8485	0.0018	0.933	0.867
80	65	[4]	γ	3.9314	0.3049	0.917	0.975
			σ_0	0.0548	0.0118	0.775	1
			v	0.4072	0.1413	0.925	1
			θ	0.9648	0.0022	0.958	0.867
		[5]	γ	3.3843	0.3772	0.933	0.975
			σ_0	0.0493	0.0120	0.758	1
			v	0.3288	0.1291	0.925	1
			θ	0.7841	0.0017	0.933	0.817
		[6]	γ	4.0317	0.3793	0.892	0.992
			σ_0	0.0537	0.0116	0.817	1
			v	0.3684	0.1187	0.967	1
			θ	0.8813	0.0019	0.925	0.8
	80		γ	4.0179	0.3780	0.9	0.95
			σ_0	0.0554	0.0125	0.875	1
			v	0.4002	0.1517	0.967	1
			θ	0.8911	0.0020	0.925	0.683
100	81	[7]	γ	3.6879	0.2922	0.925	0.95
			σ_0	0.0491	0.0109	0.808	1
			v	0.3748	0.1407	0.908	1
			θ	0.8877	0.0019	0.975	0.775
		[8]	γ	3.8079	0.3794	0.9	0.933
			σ_0	0.0492	0.0123	0.858	1
			v	0.3505	0.1589	0.967	1
			θ	0.8327	0.0018	0.925	0.733
		[9]	γ	3.4477	0.3758	0.9	0.9
			σ_0	0.0464	0.0115	0.9	1
			v	0.3331	0.1379	0.942	1
			θ	0.7894	0.0018	0.908	0.717
	100		γ	3.7465	0.3801	0.925	0.833
			σ_0	0.0509	0.0129	0.925	0.992
			v	0.3762	0.1538	0.967	1
			θ	0.8514	0.0021	0.908	0.558

(continued)

$\sum_{i=1}^{k} n_i$	$\sum_{i=1}^{k} m_i$	$C.S$	ϑ	Length of CI		Coverage probability of CI	
				Appro	Cred	Appro	Cred
120	102	[10]	γ	3.1611	0.2736	0.942	0.767
			σ_0	0.0465	0.0116	0.825	1
			ν	0.3350	0.1486	0.908	1
			θ	0.7572	0.0016	0.892	0.542
		[11]	γ	3.0229	0.3785	0.95	0.742
			σ_0	0.0424	0.0115	0.808	1
			ν	0.2929	0.1331	0.933	1
			θ	0.6853	0.0015	0.892	0.633
		[12]	γ	4.3055	0.3790	0.908	0.75
			σ_0	0.0526	0.0113	0.842	1
			ν	0.3885	0.1199	0.95	1
			θ	0.9620	0.0043	0.958	0.55
	120		γ	3.3556	0.3748	0.958	0.45
			σ_0	0.0481	0.0121	0.85	0.967
			ν	0.3477	0.1406	0.892	1
			θ	0.7696	0.0018	0.95	0.317

Table 8.5: Average optimal stress level h_k^* with true values of parameters ($\gamma = 0.9181,\ \sigma_0 = 1.0108,\ \nu = 1.102$ and $\theta = 3.2102$), and the number of stress levels $(k = 2)$

$\sum_{i=1}^{k} n_i$	$\sum_{i=1}^{k} m_i$	$C.S$	h_k^*	$C.S$	h_k^*	$C.S$	h_k^*
65	55	[1]	2.5767	[2]	2.8531	[3]	3.7889
80	65	[4]	2.8323	[5]	3.0221	[6]	2.8506
100	81	[7]	2.7871	[8]	1.6616	[9]	2.9682
120	102	[10]	1.8697	[11]	1.6979	[12]	2.1555

7. The coverage probabilities of credible CIs of γ, σ_0 and ν are greater than the corresponding coverage probabilities of approximate CIs.
8. The coverage probabilities of approximate CIs of the parameter θ are greater than the corresponding coverage probabilities of credible CIs.

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