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Estimation of Parameters of Inverse Lomax Distribution under Type-II Censoring Scheme

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Abstract: In this paper, Bayesian and E-Bayesian estimation of unknown parameters of Inverse Lomax probability distribution based on Type-II censored samples has been obtained under De Groot Loss Function (DLF), Al-Bayyati Loss function (ABLF), Entropy Loss Function (ELF), Linex Loss Function (LLF) and Minimum Expected Loss Function (MELF). To derive these estimators, we have considered a conjugate gamma prior and uniform hyperprior distributions. The Monte Carlo simulation method has been used to generate a Type-II censored data from Inverse Lomax probability distribution. The performance of the estimators has been tested and compared by computing mean square error (MSE). Further, the relationship between the E-Bayesian estimators has been established in the form of theorems.

Keywords: Inverse Lomax Distribution, Bayesian, E-Bayesian, Type-II censoring, De Groot Loss Function(DLF), Al-Bayyati Loss Function (ABLF), Entropy Loss Function (ELF), Linex Loss Function (LLF), Minimum Expected Loss Function (MELF), Markov Chain Monte Carlo

1 Introduction

Bayesian approach has opened new vistas to the tools and techniques of the estimation of parameter(s). The effectiveness of this approach is more or less linked with choice of prior. A suitable prior distribution plays an effective role in reducing error in the estimation. A number of authors had considered the problem of estimation of the unknown parameter by using Bayesian approach which is otherwise a quite difficult task, by using methods of classical statistics. Zellner [1] had studied Bayesian estimation and prediction using asymmetric loss function. Basu and Ebrahimi [2] provided Bayesian approach for life testing and reliability estimation by using asymmetric loss functions. Bayesian estimation for three parameter Weibull distribution was studied by Green et al. [3]. Ahmed et al. [4] developed an approximate method of Bayesian estimation for the shape parameters of a mixture of two Weibull distributions under Type-II censoring. Balakrishnan and Aggarwala [5] studied Bayesian inference for Type-II doubly censored Rayleigh data. Bayesian estimation of progressively censored data from Burr model has been discussed by Mousa and Jaheen [6]. Parameter estimation of modified Weibull distribution under Progressively Type-II censored samples was provided by Ng [7]. Wu et al. [8] developed Bayesian inference for Rayleigh distribution under progressively Type-II censored scheme. Singh and Kumar [9] provided Bayesian estimate of the unknown parameter of exponential distribution under multiple Type-II censoring scheme. Kundu and Howlader [10] developed Bayesian inference and prediction of inverse Weibull distribution under Type-II censored data.Yadav et al. [11] developed estimates of unknown parameters, reliability and hazard functions by using Bayesian approach for hybrid censoring scheme. Thus, in light of the above quoted research, one can comment that Bayesian estimation has found great importance in literature more than in classical estimation. Another major approach used for estimation is Expected-Bayesian (E-Bayesian), which is a new approach of Bayesian estimation introduced by Han [12]. Han [13] developed the E-Bayesian and hierarchical Bayesian estimates of reliability and unknown parameters of exponential distribution under Type-I censoring, by using quadratic loss functions. Jaheen and Okasha [14] obtained E-Bayesian estimation of Bur type-3 model based on Type-II censoring. Okasha [15] developed Bayesian and E-Bayesian method for estimating scale parameter, reliability and hazard function of Weibull distribution under Type-II censored samples by considering the squared error loss functions. Reyad and

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Ahmed [16] derived E-Bayes estimators of Gumbel distribution under censoring. Wang and Chen [17] studied the properties of the Bayes and E-Bayes estimates of the system reliability parameter with zero-failure data. Singh et al. [18] derived prediction of inverse Weibull distribution based on Type-II censored sample. Devi et al. [19] studied E-Bayesian estimators of truncated geometric distribution and shown that E-Bayes estimators are better than Bayes estimators. Lee et al. [20] provided estimates for the generalized exponential distribution based on multiple progressive Type-II censored samples. Reyad and Ahmed [21] obtained E-Bayesian and Bayesian estimates of unknown parameters for the Kumarswamy distribution based on Type-II censoring. E-Bayesian and hierarchical Bayesian estimates of the failure probability and the parameter of Pareto distribution were investigated by Han [22]. Rabie and Li [23] studied Bayesian and E-Bayesian estimators of Burr-X distribution under squared error and Linex loss functions.
We have considered Inverse Lomax Distribution (ILD) which has wide applications in Stochastic modeling of decreasing failure rate of life components and one of the most frequently used distribution in economics, geography, actuarial and medical fields, see Kleiber and Kotz [24]. Kleiber [25] studied ILD to get Lorenz ordering relationship among ordered statistics. Rahman and Aslam [26] analyzed order statistics of two component mixture of ILD of Type-II censoring through Bayesian approach. Singh et al. [27] developed reliability estimates of ILD under Type-II censoring scheme. Rahman and Aslam [28] studied the estimation of two-components mixture inverse Lomax model via Bayesian approach. Recently, Reyad and Othman [29] constructed E-Bayesian estimates of two component mixture of ILD under Type-I censoring. Sing et al. [30] studied Bayesian estimation of the stress-strength reliability parameter of ILD.
In this article, Bayesian and E-Bayesian approaches have been used to obtain the estimate of unknown parameter α of the ILD. The article organised: in section 2, we have discussed the model of ILD under censoring scheme. In sections 3 and 4, we have developed the Bayesian and E-Bayesian estimates of the distribution using different error loss functions. In section 5, we established the relation among the obtained E-Bayesian estimators, and performance of the different estimators have been compared by using simulation techniques. Section 6 has comprised of numerical findings of results obtained in previous sections.

2 The Inverse Lomax Model

Kleiber [25] introduced the Inverse Lomax Distribution (ILD) and defined it. If the random variable Z follows the Lomax distribution, then the distribution of $X = \frac{1}{Z}$ will be Inverse Lomax distribution. The cumulative distribution function and probability density function of ILD are defined as

$$F(x) = \left(1 + \frac{\beta}{x}\right)^{-\alpha}; \quad x \geq 0, \alpha, \beta \geq 0 \tag{1}$$

where α is the shape parameter, and β is the scale parameter of the distribution
and

$$f(x) = \frac{\alpha\beta}{x^2}\left(1 + \frac{\beta}{x}\right)^{-(1+\alpha)}; \quad x \geq 0, \alpha, \beta \geq 0 \tag{2}$$

Reliability function R(t) of ILD is

$$R(t) = 1 - \left(1 + \frac{\beta}{t}\right)^{-\alpha} \tag{3}$$

Using (1) and (2), hazard rate of the distribution will be

$$H(t) = \frac{f(x)}{R(t)}$$

$$H(t) = \frac{\alpha\beta\left(1 + \frac{\beta}{x}\right)^{-(1+\alpha)}}{x^2\left(1 + \frac{\beta}{t}\right)^{-\alpha}} \tag{4}$$

Balakrishnan and Aggarwala [5], in their book, have extensively discussed different censoring schemes and their applications. Suppose that n independent items are put on a test at time zero, and failure times of the first r units are observed, the ordered r-failure are denoted by $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(r)}$, then joint likelihood function of (2) is obtained by integrating out $x_{(r+1)} \leq x_{(r+2)} \leq \cdots \leq x_{(n)}$, which is given as

$$L(\alpha,\beta/\underline{x}) = \frac{n!}{(n-r)!}\prod_{i=1}^{r} f(x_i)\Big(1-F(x_i)\Big)^i, \quad x_{(1)} < x_{(2)} < \cdots < x_{(r)} \tag{5}$$

$$L(\alpha,\beta/\underline{x}) = \frac{n!}{(n-r)!}(\alpha\beta)^r \prod_{i=1}^{r}\left(\frac{x_i^{-2}}{1+\frac{\beta}{x_i}}\right)\exp\left\{-\alpha\sum_{i=1}^{r} log\left(1+\left(\frac{\beta}{x_i}\right)\right)+ (n-r)log\left(1+\left(\frac{\beta}{x_r}\right)\right)\right\}$$

Let $S_r = \sum_{i=1}^{r} log\left(1+\left(\frac{\beta}{x_i}\right)\right)+(n-r)log\left(1+\left(\frac{\beta}{x_r}\right)\right)$, then

$$L(\alpha,\beta/\underline{x}) = \frac{n!}{(n-r)!}(\alpha\beta)^r \prod_{i=1}^{r}\left(\frac{x_i^{-2}}{1+\frac{\beta}{x_i}}\right)\exp\Big(-\alpha S_r\Big) \quad ; \tag{6}$$

which is the joint likelihood function of (2).

3 Bayesian Estimation of Parameters of ILD for Different Loss Functions

In this section, we have obtained Bayes estimators of unknown parameter α of the ILD under Type-II censoring scheme. The estimators of the parameter have been obtained by using gamma informative prior under a different loss function. The gamma prior distribution with shape parameter c and scale parameter p is defined as

$$g(\alpha|c,p) = \frac{p^c}{\Gamma(c)}\alpha^{(c-1)}e^{-p\alpha} \quad ; \quad c \geq 0, p \geq 0 \tag{7}$$

Taking the product of the joint likelihood (6) and (7), we get the posterior distribution as

$$\pi(\alpha|x) = \frac{kp^c\beta^r}{\Gamma(c)}\prod_{i=1}^{r}\frac{x_i^{-2}}{\left(1+\frac{\beta}{x_i}\right)}\alpha^{c+r-1}exp\Big\{-\alpha(S_r+p)\Big\} \tag{8}$$

On simplification, the posterior density function of α will be

$$\pi(\alpha|x) = \frac{(S_r+p)^{c+r}}{\Gamma(c+r)}\alpha^{c+r-1}exp\Big\{-\alpha(S_r+p)\Big\} \tag{9}$$

Next, we will derive the Bayes estimators of α for different loss functions.

3.1 Bayesian Estimation Under De Groot Loss Function

If $\hat{\alpha}$ is an estimator of α, the De Groot [31] loss function is defined as

$$L(\hat{\alpha}) = \frac{\alpha-\hat{\alpha}}{\hat{\alpha}} \tag{10}$$

Bayes estimate, using De Groot loss function, will be

$$\hat{\alpha}_{DG} = \frac{E(\alpha^2|x)}{E(\alpha|x)} \tag{11}$$

We have

$$E(\alpha^2|x) = \frac{(c+r+1)(c+r)}{(S_r+p)^2} \tag{12}$$

and

$$E(\alpha|x) = \frac{(c+r)}{(S_r+p)} \tag{13}$$

So, by using (12) and(13), we get the Bayes estimator of α under De Groot loss function as

$$\hat{\alpha}_{DG} = \frac{(c+r+1)}{(S_r+p)} \tag{14}$$

3.2 Bayesian Estimation Under Al-Bayyati Loss Function

Al-Bayyati [32] introduced additional parameter c_1 which determines a flatter loss function and is a generalization of the squared error loss function. Al-Bayyati loss function is defined as

$$L(\alpha,\hat{\alpha}) = \alpha^{c_1}(\hat{\alpha}-\alpha)^2 \quad ; \quad c_1 \varepsilon R \tag{15}$$

Bayes estimator under Al-Bayyati loss function will be

$$\begin{aligned} E(L(\alpha,\hat{\alpha})) &= \int_0^\infty \alpha^{c_1}(\hat{\alpha}-\alpha)^2 \pi(\alpha|x)\,d\alpha \\ &= \frac{(S_r+p)^{c+r}}{\Gamma(c+r)} \frac{\Gamma(c+r+c_1)}{(S_r+p)^{c+r+c_1}} \left\{ \hat{\alpha}^2 + \frac{(c+r+c_1)(c+r+c_{1+1})}{(S_r+p)^2} \right. \\ &\quad \left. - \frac{2\hat{\alpha}(c+r+c_1)}{S_r+p} \right\} \end{aligned}$$

After simplification, we get Bayes estimator of α under Al-Bayyati loss function as

$$\hat{\alpha}_{AB} = \frac{(c+r+c_1)}{S_r+p} \tag{16}$$

3.3 Bayesian Estimation of α Under Minimum Expected Loss Function

Tummala and Sathe [33] suggested Minimum Expected Loss Function (MELF) which is defined as

$$L(\alpha,\hat{\alpha}) = \frac{(\hat{\alpha}-\alpha)^2}{\alpha^2} \tag{17}$$

Bayes estimator of α under MELF will be

$$\begin{aligned} E(L(\alpha,\hat{\alpha})) &= \int_0^\infty \frac{(\hat{\alpha}-\alpha)^2}{\alpha^2} \pi(\alpha|x)\,d\alpha \\ &= \frac{\hat{\alpha}^2\Gamma(c+r+2)}{(c+r-2)(c+r-2)\Gamma(c+r-2)(S_r+p)} + 1 \\ &\quad - \frac{2\hat{\alpha}\Gamma(c+r-1)(S_r+p)^{c+r}}{(c+r)\Gamma(c+r-1)(S_r+p)^{c+r-1}} \end{aligned}$$

On solving the integral, the Bayes estimator will be

$$\hat{\alpha}_{ME} = \frac{(c+r-2)}{(S_r+p)} \tag{18}$$

3.4 Bayesian Estimation Under Linex Loss Function

Zellner [1] defined Linex Loss Function (LLF) with two parameter m $>0, w \neq 0$ as

$$L(\alpha,\hat{\alpha}) = m\Big[exp\big(w(\alpha-\hat{\alpha})\big) - w(\hat{\alpha}-\alpha) - 1\Big] \quad ; \tag{19}$$

where m is the scale parameter and w is the shape parameter of loss function. Without loss of generality, we assume $m = 1$, the Bayes estimators of α will be given by

$$\hat{\alpha}_{LL} = -\frac{1}{w} log\big(E(e^{-w\alpha}|x)\big) \tag{20}$$

Bayes estimator of α under LLF will be

$$\hat{\alpha}_{LL} = -\frac{1}{w} log\left\{\left(\frac{S_r+p}{S_r+p+w}\right)^{c+r}\right\} \tag{21}$$

3.5 Bayesian Estimation Under Entropy Loss Function

Dey et al. [34] defined Entropy Loss Function (ELF) as

$$L(\alpha,\hat{\alpha}) \propto \left(\frac{\hat{\alpha}}{\alpha}\right) - log\left(\frac{\hat{\alpha}}{\alpha}\right) - 1 \tag{22}$$

The Bayes estimator of α for this loss function will be

$$\hat{\alpha}_{EL} = \Big[E(\alpha^{-1}/x)\Big]^{-1}$$

$$\Big[E(\alpha^{-1}/x)\Big] = \int_0^\infty \alpha^{-1}\pi(\alpha|x)\,d\alpha$$

So, Bayes estimates of α under ELF will be

$$\hat{\alpha}_{EL} = \frac{c+r-1}{S_r+p} \tag{23}$$

In this section, the expression have been derived for the Bayes estimators under different loss functions. In the next section, we have derived expression for E-Bayes estimators using different prior.

4 E-Bayesian Estimation

Based on Han [35], prior parameters c and p must be selected to guarantee that $g(\alpha|c,p)$ in (7) is a decreasing function of α. So to ensure these conditions $\frac{\partial g(\alpha|c,p)}{\partial\alpha}$ must be negative.

$$\frac{\partial g(\alpha|c,p)}{\partial\alpha} = \frac{p^c}{\Gamma c}\alpha^{c-2}e^{-bc}\Big[(c-1) - p\alpha\Big] \tag{24}$$

Note that if c>0 , p> 0 and $\alpha > 0$ and then $0 < c < 1 and p > 1$ results in $\frac{\partial g(\alpha|c,p)}{\partial\alpha} < 0$. Thus, the prior distribution $g(\alpha|c,p)$ is a decreasing function of α. Given 0<c<1 , as p grows larger, the tail of the gamma density function will become thinner. However, as far as the robustness of Bayesian estimation concerns us, Berger [36] provided that the thinner tailed prior distribution often reduces the robustness of Bayesian estimate. For this reason, upper bound λ is set for parameter p, where λ is a constant to be determined. Therefore, hyperparameters c and p should be selected with the

restriction of $0<c<1$ and $0<p<\lambda$.
The E-Bayesian estimate of α (expectation of Bayesian estimate of α) can be written as

$$\begin{aligned}\hat{\alpha}_{EB} &= E(\alpha|\underline{x}) \\ &= \int_0^1 \int_\Omega \hat{\alpha}_B \pi(\alpha,c,p)\,dp\,dc; \end{aligned} \tag{25}$$

where Ω is the domain of p, and $\hat{\alpha}_B$ is the Bayes estimate of α under the different loss function. E-Bayesian estimates of α depends on hyperparameters c and p given below in, (26) to (28). These distributions are used to study the impact of the different prior distributions on the E-Bayesian estimate of α. We have used the uniform hyperprior distribution as the distribution of hyperparameters c and p.

$$\pi_1(c,p) = \frac{2(\lambda-p)}{\lambda^2}; \qquad 0<c<1, 0<p<\lambda \tag{26}$$

$$\pi_2(c,p) = \frac{1}{\lambda}; \qquad 0<c<1, 0<p<\lambda \tag{27}$$

$$\pi_3(c,p) = \frac{2p}{\lambda^2}; \qquad 0<c<1, 0<p<\lambda \tag{28}$$

Using above hyperprior distribution, next, we have derived the E-Bayesian of the parameters.

4.1 E-Bayesian Estimation Under De Groot Loss Function

E-Bayes estimate of α, under De Groot loss function based on $\pi_1(\alpha,c,p)$, denoted as $\hat{\alpha}_{EBDG_1}$, is obtained by using (14) and (26) as

$$\begin{aligned}\hat{\alpha}_{EBDG_1} &= \int_0^1 \int_0^\lambda \frac{(c+r+1)}{(S_r+p)}.2\frac{(\lambda-p)}{\lambda^2}dp\,dc \\ &= \frac{2}{\lambda^2}\int_0^1 \left[(c+r+1)(\lambda+S_r)\left\{log(S_r+\lambda)-logS_r\right\}-\lambda\right]dc \\ &= \frac{2r+3}{\lambda^2}\left[(\lambda+S_r)\left\{log(S_r+\lambda)-logS_r\right\}-\lambda\right] \end{aligned} \tag{29}$$

Similarly, $\hat{\alpha}_{EBDG_2}$ is the E-Bayes estimate of α under De Groot loss function based on $\pi_2(\alpha,c,p)$ and obtained by using (14) and (27) as

$$\hat{\alpha}_{EBDG_2} = \int_0^1 \int_0^\lambda \frac{(c+r+1)}{(S_r+p)}.\frac{1}{\lambda}dp\,dc \tag{30}$$

$$\begin{aligned} &= \frac{1}{\lambda}\int_0^1 (c+r+1)\left[log\left(\frac{S_r+\lambda}{S_r}\right)\right]dc \\ &= \frac{2r+3}{2\lambda}\left[log\left(\frac{S_r+\lambda}{S_r}\right)\right] \end{aligned} \tag{31}$$

Similarly, $\hat{\alpha}_{EBDG_3}$ is the E-Bayes estimate of α under De Groot loss function based on $\pi_3(\alpha,c,p)$ and is obtained by using (14) and (28) as

$$\begin{aligned}\hat{\alpha}_{EBDG_3} &= \int_0^1 \int_0^\lambda \frac{(c+r+1)}{(S_r+p)}.\frac{2p}{\lambda^2}dp\,dc \\ &= \frac{2}{\lambda}\int_0^1 (c+r+1)\left[\lambda - S_r log\left(\frac{S_r+\lambda}{S_r}\right)\right]dc \\ &= \frac{2r+3}{\lambda^2}\left[\lambda - S_r log\left(\frac{S_r+\lambda}{S_r}\right)\right] \end{aligned} \tag{32}$$

4.2 E-Bayesian Estimation Under Al-Bayyati Loss function

We obtained E-Bayes estimates of α under Al-Bayyati loss function based on $\pi_1(\alpha,c,p)$, denoted by $\hat{\alpha}_{EBAB_1}$, by using (16) and (26) as

$$\begin{aligned}\hat{\alpha}_{EBAB_1} &= \int_0^1\int_0^\lambda \frac{(c+r+c_{-1})}{(S_r+p)}.\frac{2(\lambda-p)}{\lambda^2}dp\,dc\\ &= \frac{2}{\lambda^2}\int_0^1\left[(c+r+c_1)(\lambda+S_r)\left\{log(S_r+\lambda)-logS_r\right\}-\lambda\right]dc\\ &= \frac{2r+2c_1+1}{\lambda^2}\left[(\lambda+S_r)\left\{log(S_r+\lambda)-logS_r\right\}-\lambda\right] \end{aligned} \tag{33}$$

Similarly, $\hat{\alpha}_{EBAB_2}$ is the E-Bayes estimates of α under Al-Bayyati loss function based on $\pi_2(\alpha,c,p)$ and obtained by using (14) and (27) as

$$\begin{aligned}\hat{\alpha}_{EBAB_2} &= \int_0^1\int_0^\lambda \frac{(c+r+c_1)}{(S_r+p)}.\frac{1}{\lambda}dp\,dc\\ &= \frac{1}{\lambda}\int_0^1(c+r+c_1)\left[log\left(\frac{S_r+\lambda}{S_r}\right)\right]dc\\ &= \frac{2r+2c_1+1}{2\lambda}\left[log\left(\frac{S_r+\lambda}{S_r}\right)\right] \end{aligned} \tag{34}$$

Similarly, $\hat{\alpha}_{EBAB_3}$ is the E-Bayes estimates of α under Al-Bayyati loss function based on $\pi_3(\alpha,c,p)$ is obtained by using (14) and (28) as

$$\begin{aligned}\hat{\alpha}_{EBAB_3} &= \int_0^1\int_0^\lambda \frac{(c+r+c_1)}{(S_r+p)}.\frac{2p}{\lambda^2}dp\,dc\\ &= \frac{2}{\lambda}\int_0^1(c+r+c_1)\left[\lambda-S_r log\left(\frac{S_r+\lambda}{S_r}\right)\right]dc\\ &= \frac{2r+2c_1+1}{\lambda^2}\left[\lambda-S_r log\left(\frac{S_r+\lambda}{S_r}\right)\right] \end{aligned} \tag{35}$$

4.3 E-Bayesian Estimation of α Under Minimum Expected Loss Function

E-Bayes estimation of α, using $\pi_1(\alpha,c,p)$ under Minimum Expected Loss Function denoted by $\hat{\alpha}_{EBME_1}$, is obtained by using (18) and (26) as

$$\begin{aligned}\hat{\alpha}_{EBME_1} &= \int_0^1\int_0^\lambda \frac{(c+r-2)}{(S_r+p)}.\frac{2(\lambda-p)}{\lambda^2}dp\,dc\\ &= \frac{2}{\lambda^2}\int_0^1\left[(c+r-2)(\lambda-S_r)\left\{log(S_r+\lambda)-logS_r\right\}-\lambda\right]dc \end{aligned}$$

On simplification , we get

$$\hat{\alpha}_{EBME_1} = \frac{2r-3}{\lambda^2}\left[(\lambda-S_r)\left\{log(S_r+\lambda)-logS_r\right\}-\lambda\right] \tag{36}$$

Similarly, $\hat{\alpha}_{EBME_2}$ is the E-Bayes estimation of α using $\pi_2(\alpha,c,p)$ under Minimum Expected Loss Function is obtained by using (18) and (27) as

$$\hat{\alpha}_{EBME_2} = \int_0^1 \int_0^{\lambda} \frac{(c+r-2)}{(S_r+p)}.\frac{1}{\lambda} dp\, dc$$
$$= \frac{1}{\lambda} \int_0^1 (c+r-2)\left[log\left(\frac{S_r+\lambda}{S_r}\right)\right] dc$$

We get

$$= \frac{2r-3}{2\lambda}\left[log\left(\frac{S_r+\lambda}{S_r}\right)\right] \tag{37}$$

Similarly, $\hat{\alpha}_{EBME_3}$ is the E-Bayes estimation of α using $\pi_3(\alpha,c,p)$ under Minimum Expected Loss Functions obtained by using (18) and (28) as

$$\hat{\alpha}_{EBME_3} = \int_0^1 \int_0^{\lambda} \frac{(c+r-2)}{(S_r+p)}.\frac{2p}{\lambda^2} dp\, dc$$
$$= \frac{2}{\lambda} \int_0^1 (c+r-2)\left[\lambda - S_r log\left(\frac{S_r+\lambda}{S_r}\right)\right] dc$$
$$= \frac{2r-3}{\lambda^2}\left[\lambda - S_r log\left(\frac{S_r+\lambda}{S_r}\right)\right] \tag{38}$$

4.4 E-Bayesian Estimation af α Under Entropy Loss Function

E-Bayes estimates of α under Entropy Loss Function based on $\pi_1(\alpha,c,p)$, denoted by $\hat{\alpha}_{EBEL_1}$, is obtained by using (23) and (26) and is given as

$$\hat{\alpha}_{EBEL_1} = \int_0^1 \int_0^{\lambda} \frac{(c+r-1)}{(S_r+p)}.\frac{2(\lambda-p)}{\lambda^2} dp\, dc$$
$$= \frac{2}{\lambda^2} \int_0^1 \left[(c+r-1)(\lambda-S_r)\left\{log(S_r+\lambda) - logS_r\right\} - \lambda\right] dc$$

On solving the above integral, we get

$$\hat{\alpha}_{EBEL_1} = \frac{2r-1}{\lambda^2}\left[(\lambda-S_r)\left\{log(S_r+\lambda) - logS_r\right\} - \lambda\right] \tag{39}$$

Similarly, $\hat{\alpha}_{EBEL_2}$ is the E-Bayes estimates of α under Entropy Loss Function based on $\pi_2(\alpha,c,p)$ and is obtained by using (23) and (27) as

$$\hat{\alpha}_{EBEL_2} = \int_0^1 \int_0^{\lambda} \frac{(c+r-1)}{(S_r+p)}.\frac{1}{\lambda} dp\, dc$$
$$= \frac{1}{\lambda} \int_0^1 (c+r-1)\left[log\left(\frac{S_r+\lambda}{S_r}\right)\right] dc$$

On simplification, we get

$$\hat{\alpha}_{EBEL_2} = \frac{2r-1}{2\lambda}\left[log\left(\frac{S_r+\lambda}{S_r}\right)\right] \tag{40}$$

Similarly, $\hat{\alpha}_{EBEL_3}$ is the E-Bayes estimates of α under Entropy Loss Function based on $\pi_3(\alpha,c,p)$ and is obtained by using (23) and (28) as

$$\hat{\alpha}_{EBEL_3} = \int_0^1 \int_0^{\lambda} \frac{(c+r-1)}{(S_r+p)} \cdot \frac{2p}{\lambda^2} dp\, dc$$
$$= \frac{2}{\lambda} \int_0^1 (c+r-1) \left[\lambda - S_r log\left(\frac{S_r+\lambda}{S_r}\right)\right] dc$$
$$= \frac{2r-1}{\lambda^2} \left[\lambda - S_r log\left(\frac{S_r+\lambda}{S_r}\right)\right] \tag{41}$$

5 Relation Among E-Bayesian Estimators of α Under Different Loss Function

In this section, we have established the relation among the different E-Bayesian estimators obtained in an earlier section. $\hat{\alpha}_{EBDG_i}, \hat{\alpha}_{EBAB_i}, \hat{\alpha}_{EBME_i}$ and $\hat{\alpha}_{EBEL_i}$, i=1,2,3, are the different E-Bayes estimators based on different loss function as has been defined in the previous section. We have established the relation between these estimators in the form of theorems as follows.

Theorem 1. *E-Bayes estimators of parameter $\alpha(\hat{\alpha}_{EBDG_i}, i = 1,2,3)$ under De Groot Loss Function obeys the following relationships.*

$$i)\quad \hat{\alpha}_{EBDG_3} < \hat{\alpha}_{EBDG_2} < \hat{\alpha}_{EBDG_1}$$
$$ii)\ \lim_{S_r\to\infty} \hat{\alpha}_{EBDG_1} = \lim_{S_r\to\infty} \hat{\alpha}_{EBDG_2} = \lim_{S_r\to\infty} \hat{\alpha}_{EBDG_3}$$

Proof (i): From (29) and (30) , we have

$$\hat{\alpha}_{EBDG_1} - \hat{\alpha}_{EBDG_2} = \frac{2r+3}{\lambda^2}\left[log\left(\frac{S_r+\lambda}{S_r}\right)\left(\frac{S_r}{\lambda}+\frac{1}{2}\right) - 1\right] \tag{42}$$

From (30) and (32),

$$\hat{\alpha}_{EBDG_2} - \hat{\alpha}_{EBDG_3} = \frac{2r+3}{\lambda^2}\left[log\left(\frac{S_r+\lambda}{S_r}\right)\left(\frac{S_r}{\lambda}+\frac{1}{2}\right) - 1\right] \tag{43}$$

From (42) and (43), we get

$$\hat{\alpha}_{EBDG_1} - \hat{\alpha}_{EBDG_2} = \hat{\alpha}_{EBDG_2} - \hat{\alpha}_{EBDG_3}$$
$$= \frac{2r+3}{\lambda^2}\left[log\left(\frac{S_r+\lambda}{S_r}\right)\left(\frac{S_r}{\lambda}+\frac{1}{2}\right) - 1\right] \tag{44}$$

$$Using \quad log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \ldots\ for\, z\varepsilon\,(-1,1) \tag{45}$$

Let $z = \frac{\lambda}{S_r}$; when $0 < \frac{\lambda}{S_r} < 1$, we get

$$\left[log\left(\frac{S_r+\lambda}{S_r}\right)\left(\frac{S_r}{\lambda}+\frac{1}{2}\right) - 1\right] = \left[\frac{\lambda^2}{12{S_r}^2} + \frac{\lambda^3}{6{S_r}^3} + \ldots\right] > 0 \tag{46}$$

So, from (44) and (46), we get

$$\hat{\alpha}_{EBDG_1} - \hat{\alpha}_{EBDG_2} = \hat{\alpha}_{EBDG_2} - \hat{\alpha}_{EBDG_3} > 0$$

$$\implies \quad \hat{\alpha}_{EBDG_3} < \hat{\alpha}_{EBDG_2} < \hat{\alpha}_{EBDG_1} \qquad \square$$

Proof (ii): From (44),

$$\begin{aligned}\lim_{S_r \to \infty}\left(\hat{\alpha}_{EBDG_1} - \hat{\alpha}_{EBDG_2}\right) &= \lim_{S_r \to \infty}\left(\hat{\alpha}_{EBDG_2} - \hat{\alpha}_{EBDG_3}\right) \\ &= \lim_{S_r \to \infty} \frac{2r+3}{\lambda^2}\left[log\left(\frac{S_r+\lambda}{S_r}\right)\left(\frac{S_r}{\lambda}+\frac{1}{2}\right)-1\right] \\ &= 0 \end{aligned} \tag{47}$$

Thus,

$$\lim_{S_r \to \infty} \hat{\alpha}_{EBDG_1} = \lim_{S_r \to \infty} \hat{\alpha}_{EBDG_2} = \lim_{S_r \to \infty} \hat{\alpha}_{EBDG_3} \qquad \square$$

Theorem 2. *E-Bayes estimator of parameter* $\alpha(\hat{\alpha}_{EBAB_i}, i = 1,2,3)$ *under Al-Bayyati loss function obeys the following relationships.*

$$\begin{aligned} &i) \quad \hat{\alpha}_{EBAB_3} < \hat{\alpha}_{EBAB_2} < \hat{\alpha}_{EBAB_1} \\ &ii) \lim_{S_r \to \infty} \hat{\alpha}_{EBAB_1} = \lim_{S_r \to \infty} \hat{\alpha}_{EBAB_2} = \lim_{S_r \to \infty} \hat{\alpha}_{EBAB_3} \end{aligned} \tag{48}$$

Proof (i): From (33) and (4.2), we have

$$\hat{\alpha}_{EBAB_1} - \hat{\alpha}_{EBAB_2} = \frac{2r+2c_1+1}{\lambda^2}\left[log\left(\frac{S_r+\lambda}{S_r}\right)\left(\frac{S_r}{\lambda}+\frac{1}{2}\right)-1\right] \tag{49}$$

From (4.2) and (35),

$$\hat{\alpha}_{EBAB_2} - \hat{\alpha}_{EBAB_3} = \frac{2r+2c_1+1}{\lambda^2}\left[log\left(\frac{S_r+\lambda}{S_r}\right)\left(\frac{S_r}{\lambda}+\frac{1}{2}\right)-1\right] \tag{50}$$

From (49) and (50), we get

$$\begin{aligned}\hat{\alpha}_{EBAB_1} - \hat{\alpha}_{EBAB_2} &= \hat{\alpha}_{EBAB_2} - \hat{\alpha}_{EBAB_3} \\ &= \frac{2r+2c_1+1}{\lambda^2}\left[log\left(\frac{S_r+\lambda}{S_r}\right)\left(\frac{S_r}{\lambda}+\frac{1}{2}\right)-1\right]\end{aligned} \tag{51}$$

By using (45), let $z = \frac{\lambda}{S_r}$; when $0 < \frac{\lambda}{S_r} < 1$, we get

$$\left[log\left(\frac{S_r+\lambda}{S_r}\right)\left(\frac{S_r}{\lambda}+\frac{1}{2}\right)-1\right] = \left[\frac{\lambda^2}{12{S_r}^2}+\frac{\lambda^3}{6{S_r}^3}+\ldots\right] > 0 \tag{52}$$

So, from (51) and (52), we get

$$\hat{\alpha}_{EBAB_1} - \hat{\alpha}_{EBAB_2} = \hat{\alpha}_{EBAB_2} - \hat{\alpha}_{EBAB_3} > 0$$

$$\implies \quad \hat{\alpha}_{EBAB_3} < \hat{\alpha}_{EBAB_2} < \hat{\alpha}_{EBAB_1} \qquad \square$$

Proof (ii): From (51),

$$\begin{aligned}\lim_{S_r \to \infty}\left(\hat{\alpha}_{EBAB_1} - \hat{\alpha}_{EBAB_2}\right) &= \lim_{S_r \to \infty}\left(\hat{\alpha}_{EBAB_2} - \hat{\alpha}_{EBAB_3}\right) \\ &= \lim_{S_r \to \infty} \frac{2r+2c_1+1}{\lambda^2}\left[log\left(\frac{S_r+\lambda}{S_r}\right)\left(\frac{S_r}{\lambda}+\frac{1}{2}\right)-1\right] \\ &= 0 \end{aligned} \tag{53}$$

$$\Longrightarrow \quad \lim_{S_r\to\infty} \hat{\alpha}_{EBAB_1} = \lim_{S_r\to\infty} \hat{\alpha}_{EBAB_2} = \lim_{S_r\to\infty} \hat{\alpha}_{EBAB_3} \qquad \square \tag{54}$$

Theorem 3. *E-Bayes estimator of parameter $\alpha(\hat{\alpha}_{EBME_i}, i = 1,2,3)$ under Minimum Expected Loss Function obeys the following relationships.*

$$i) \quad \hat{\alpha}_{EBME_3} < \hat{\alpha}_{EBME_2} < \hat{\alpha}_{EBME_1}$$
$$ii) \lim_{S_r\to\infty} \hat{\alpha}_{EBME_1} = \lim_{S_r\to\infty} \hat{\alpha}_{EBME_2} = \lim_{S_r\to\infty} \hat{\alpha}_{EBME_3}$$

Proof (i): From (36) and (37) , we have

$$\hat{\alpha}_{EBME_1} - \hat{\alpha}_{EBME_2} = \frac{2r-3}{\lambda^2}\left[log\left(\frac{S_r+\lambda}{S_r}\right)\left(\frac{S_r}{\lambda}+\frac{1}{2}\right)-1\right] \tag{55}$$

From (37) and (38), we have

$$\hat{\alpha}_{EBME_2} - \hat{\alpha}_{EBME_3} = \frac{2r-3}{\lambda^2}\left[log\left(\frac{S_r+\lambda}{S_r}\right)\left(\frac{S_r}{\lambda}+\frac{1}{2}\right)-1\right] \tag{56}$$

From (55) and (56), we get

$$\begin{aligned}\hat{\alpha}_{EBME_1} - \hat{\alpha}_{EBME_2} &= \hat{\alpha}_{EBME_2} - \hat{\alpha}_{EBME_3} \\ &= \frac{2r-3}{\lambda^2}\left[log\left(\frac{S_r+\lambda}{S_r}\right)\left(\frac{S_r}{\lambda}+\frac{1}{2}\right)-1\right]\end{aligned} \tag{57}$$

From equation(45), we get

$$\left[log\left(\frac{S_r+\lambda}{S_r}\right)\left(\frac{S_r}{\lambda}+\frac{1}{2}\right)-1\right] = \left[\frac{\lambda^2}{12\,{S_r}^2}+\frac{\lambda^3}{6\,{S_r}^3}+\dots\right] > 0 \tag{58}$$

So, from (57) and (58), we get

$$\hat{\alpha}_{EBME_1} - \hat{\alpha}_{EBME_2} = \hat{\alpha}_{EBME_2} - \hat{\alpha}_{EBME_3} > 0$$

$$\Longrightarrow \quad \hat{\alpha}_{EBME_3} < \hat{\alpha}_{EBME_2} < \hat{\alpha}_{EBME_1} \qquad \square$$

Proof (ii): From (57),

$$\begin{aligned}\lim_{S_r\to\infty}\left(\hat{\alpha}_{EBME_1} - \hat{\alpha}_{EBME_2}\right) &= \lim_{S_r\to\infty}\left(\hat{\alpha}_{EBME_2} - \hat{\alpha}_{EBME_3}\right) \\ &= \lim_{S_r\to\infty}\frac{2r-3}{\lambda^2}\left[log\left(\frac{S_r+\lambda}{S_r}\right)\left(\frac{S_r}{\lambda}+\frac{1}{2}\right)-1\right] \\ &= 0\end{aligned} \tag{59}$$

Thus,

$$\lim_{S_r\to\infty} \hat{\alpha}_{EBME_1} = \lim_{S_r\to\infty} \hat{\alpha}_{EBME_2} = \lim_{S_r\to\infty} \hat{\alpha}_{EBME_3} \qquad \square$$

Theorem 4. *E-Bayes estimator of parameter $\alpha(\hat{\alpha}_{EBEL_i}, i = 1,2,3)$ under Entropy Loss Function obeys the following relationships.*

$$i) \quad \hat{\alpha}_{EBEL_3} < \hat{\alpha}_{EBEL_2} < \hat{\alpha}_{EBEL_1}$$
$$ii) \lim_{S_st\to\infty} \hat{\alpha}_{EBEL_1} = \lim_{S_r\to\infty} \hat{\alpha}_{EBEL_2} = \lim_{S_r\to\infty} \hat{\alpha}_{EBEL_3}$$

Proof (i): From (39) and (40) , we have

$$\hat{\alpha}_{EBEL_1} - \hat{\alpha}_{EBEL_2} = \frac{2r-1}{\lambda^2}\left[log\left(\frac{S_r+\lambda}{S_r}\right)\left(\frac{S_r}{\lambda}+\frac{1}{2}\right)-1\right] \tag{60}$$

From (40) and (41), we have

$$\hat{\alpha}_{EBEL_2} - \hat{\alpha}_{EBEL_3} = \frac{2r-1}{\lambda^2}\left[log\left(\frac{S_r+\lambda}{S_r}\right)\left(\frac{S_r}{\lambda}+\frac{1}{2}\right)-1\right] \tag{61}$$

From (60) and (61), we get

$$\begin{aligned}\hat{\alpha}_{EBEL_1} - \hat{\alpha}_{EBEL_2} &= \hat{\alpha}_{EBEL_2} - \hat{\alpha}_{EBEL_3} \\ &= \frac{2r-1}{\lambda^2}\left[log\left(\frac{S_r+\lambda}{S_r}\right)\left(\frac{S_r}{\lambda}+\frac{1}{2}\right)-1\right]\end{aligned} \tag{62}$$

Using (45), let $z = \frac{\lambda}{S_r}$; when $0<\frac{\lambda}{S_r}<1$, we get

$$\left[log\left(\frac{S_r+\lambda}{S_r}\right)\left(\frac{S_r}{\lambda}+\frac{1}{2}\right)-1\right] = \left[\frac{\lambda^2}{12{S_r}^2}+\frac{\lambda^3}{6{S_r}^3}+\ldots\right] > 0 \tag{63}$$

So, from (62) and (63), we get

$$\hat{\alpha}_{EBEL_1} - \hat{\alpha}_{EBEL_2} = \hat{\alpha}_{EBEL_2} - \hat{\alpha}_{EBEL_3} > 0$$

$$\implies \quad \hat{\alpha}_{EBEL_3} < \hat{\alpha}_{EBEL_2} < \hat{\alpha}_{EBEL_1} \qquad \square$$

Proof (ii): From (62),

$$\begin{aligned}\lim_{S_r\to\infty}\left(\hat{\alpha}_{EBEL_1} - \hat{\alpha}_{EBEL_2}\right) &= \lim_{S_r\to\infty}\left(\hat{\alpha}_{EBEL_2} - \hat{\alpha}_{EBEL_3}\right) \\ &= \lim_{S_r\to\infty}\frac{2r-1}{\lambda^2}\left[log\left(\frac{S_r+\lambda}{S_r}\right)\left(\frac{S_r}{\lambda}+\frac{1}{2}\right)-1\right] \\ &= 0\end{aligned} \tag{64}$$

Thus,

$$\lim_{S_r\to\infty}\hat{\alpha}_{EBEL_1} = \lim_{S_r\to\infty}\hat{\alpha}_{EBEL_2} = \lim_{S_r\to\infty}\hat{\alpha}_{EBEL_3} \qquad \square$$

6 Comparison of Performance of Bayes and E-Bayes Eestimators

To compare the performance, we have considered Mean Square Error (MSE) as a criterion of performance. For this, we have generated different samples through Monte Carlo Simulation using the following algorithm.

1. Set the default values of β, λ, n and r as 0.5, 0.7, (50,70,100) and (10,15,20), respectively.
2. Generate c and p from uniform hyperprior distribution and gamma hyperprior, respectively.
3. For generated values of c and p, generate α from gamma prior using equation (7).
4. For known values of β, draw a random Type-II censored sample.
5. Compute Bayes estimates and E-Bayes estimates of the unknown shape parameter given in (14), (16), (18), (21), (23), (29), (30), (32), (4.2), (35), (36), (37), (38), (39), (40) and (41), respectively.

6. Repeat the process (1 to 5) 10,000 times and compute MSE where ,

$$MSE = \frac{1}{10000}\sum_{i=1}^{r}(\hat{\alpha}_i - \alpha)^2$$

for all the estimates for different sample sizes and termination sample numbers (r). The simulation results obtained after using the above algorithm are shown in Tables 1 and 2.

1. From Table 1 and 2, we observe that E-Bayes estimates are better than Bayes estimates for all the Loss functions as they have lesser values of MSE with different combinations of n and r. Moreover, Bayes and E- Bayes under DE Loss Function can be ordered as $\hat{\alpha}_{EBDG_3} < \hat{\alpha}_{EBDG_2} < \hat{\alpha}_{EBDG_1} < \hat{\alpha}_{DG}$; under AB loss function they can be ordered as $\hat{\alpha}_{EBAB_3} < \hat{\alpha}_{EBAB_2} < \hat{\alpha}_{EBAB_1} < \hat{\alpha}_{AB}$; under ME loss function they are ordered as $\hat{\alpha}_{EBME_3} < \hat{\alpha}_{EBME_2} < \hat{\alpha}_{EBME_1} < \hat{\alpha}_{ME}$; and under E loss function they can be ordered as $\hat{\alpha}_{EBEL_3} < \hat{\alpha}_{EBEL_2} < \hat{\alpha}_{EBEL_1} < \hat{\alpha}_{EL}$.
2. Hierarchy among Bayes estimates can be ordered as $\hat{\alpha}_{AB} < \hat{\alpha}_{DE} < \hat{\alpha}_{EL} < \hat{\alpha}_{ME}$. Similarly, hierarchy observed among E-Bayes estimates is ordered as $\hat{\alpha}_{EBAB} < \hat{\alpha}_{EBDE} < \hat{\alpha}_{EBEL} < \hat{\alpha}_{EBME}$.
3. MSE of all estimates generally decrease as n and r increase. Bayes estimates get closer to E-Bayes estimates as n and r increase. This relation has been satisfied for all Bayes and E-Bayes estimates of $\alpha(\hat{\alpha}_{BS})$ and $(\hat{\alpha}_{EBS})$, respectively.

The obtained estimators have been compared on the simulated data. To further check their performance and validity, we have considered a real data set, which is a fitting case for ILD under Type-II censoring. The real data set has been taken from Lee and Wang [37].

Real data set analysis

The data set given below represents the remission times of a 128 bladder cancer patients.

0.08, 2.09, 3.48, 4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.0

We generated Type-II censored random samples from the data set, and all computations were performed using the programme developed through MatLab software. We have considered the gamma priors for α and β by assuming the value of hyperparameters as λ = 1.5 and c = 1.4. Results are shown in table 3 and 4.

Table 1: Bayes and E-Bayes Estimates of α and their MSE (within parenthesis) for different values of n and r

$Loss function \rightarrow$			DEF				ABLF			
$\alpha\downarrow$	$n\downarrow$	$r\downarrow$	$\hat{\alpha}_{DG}$	$\hat{\alpha}_{EBDG_1}$	$\hat{\alpha}_{EBDG_2}$	$\hat{\alpha}_{EBDG_3}$	$\hat{\alpha}_{AB}$	$\hat{\alpha}_{EBAB_1}$	$\hat{\alpha}_{EBAB_2}$	$\hat{\alpha}_{EBAB_3}$
0.09069	50	10	0.04392 (0.00218)	0.04457 (0.00212)	0.04455 (0.00212)	0.04452 (0.00213)	0.04541 (0.00204)	0.04612 (0.00198)	0.04609 (0.00198)	0.04607 (0.00199)
		15	0.07476 (0.00025)	0.07559 (0.00022)	0.07555 (0.00022)	0.07551 (0.00023)	0.07659 (0.00019)	0.07742 (0.00017)	0.07738 (0.00017)	0.07734 (0.00017)
		20	0.10315 (0.00018)	0.10408 (0.00017)	0.10402 (0.00017)	0.10397 (0.00017)	0.10508 (0.00024)	0.10602 (0.00023)	0.10596 (0.00023)	0.10590 (0.00023)
0.12639	70	10	0.02633 (0.01001)	0.02678 (0.00992)	0.026779 (0.00992)	0.026772 (0.00992)	0.02726 (0.00982)	0.02771 (0.00973)	0.02771 (0.00973)	0.02770 (0.00973)
		15	0.03928 (0.00758)	0.03976 (0.00750)	0.03974 (0.00750)	0.03973 (0.00750)	0.04024 (0.00742)	0.04072 (0.00733)	0.04071 (0.00734)	0.04071 (0.00734)
		20	0.07616 (0.00252)	0.07691 (0.00244)	0.07688 (0.00245)	0.07684 (0.00245)	0.07759 (0.00238)	0.07834 (0.00230)	0.07831 (0.00231)	0.07827 (0.00231)
0.27170	100	10	0.00998 (0.00019)	0.01021 (0.00018)	0.01021 (0.00018)	0.01021 (0.00018)	0.01033 (0.00018)	0.010574 (0.00017)	0.010573 (0.00017)	0.010572 (0.00017)
		15	0.01498 (7.80290e-05)	0.015227 (7.37471e-05)	0.015225 (7.376724e-05)	0.015224 (7.37873e-05)	0.015350 (7.75433e-05)	0.015596 (7.16449e-05)	0.015594 (6.75630e-05)	0.015593 (6.75827e-05)
		20	0.02112 (7.23036e-06)	0.02139 (5.86956e-06)	0.02139 (5.87816e-06)	0.02138 (5.88676e-06)	0.02152 (5.24878e-06)	0.021789 (4.09952e-06)	0.021789 (4.10684e-06)	0.021786 (4.11417e-06)

Table 2: Bayes and E-Bayes Estimates of α and their MSE (within parenthesis) for different values of n and r

$Loss function \rightarrow$			MEF				ELF			
$\alpha \downarrow$	$n \downarrow$	$r \downarrow$	$\hat{\alpha}_{ME}$	$\hat{\alpha}_{EBME_1}$	$\hat{\alpha}_{EBME_2}$	$\hat{\alpha}_{EBME_3}$	$\hat{\alpha}_{EL}$	$\hat{\alpha}_{EBEL_1}$	$\hat{\alpha}_{EBEL_2}$	$\hat{\alpha}_{EBEL_3}$
		10	0.03231 (0.00340)	0.03294 (0.00333)	0.03292 (0.00337)	0.03291 (0.00337)	0.03618 (0.00297)	0.03681 (0.00290)	0.03680 (0.00290)	0.03678 (0.00290)
0.09069	50	15	0.06105 (0.00087)	0.06184 (0.00083)	0.06181 (0.00083)	0.06178 (0.00083)	0.06562 (0.00062)	0.06643 (0.00058)	0.06639 (0.00059)	0.06635 (0.00592)
		20	0.08866 (4.14306e-06)	0.08956 (1.28038e-06)	0.08951 (1.39729e-06)	0.08946 (1.51930e-06)	0.093491 (7.81439e-06)	0.09440 (1.37624e-05)	0.09435 (1.33704e-05)	0.09429 (1.29834e-05)
		10	0.01935 (0.01145)	0.01979 (0.01136)	0.01979 (0.01136)	0.01978 (0.01136)	0.02168 (0.01096)	0.02212 (0.01087)	0.02212 (0.01087)	0.02211 (0.01087)
0.12639	70	15	0.03206 (0.00889)	0.03253 (0.00880)	0.03252 (0.008811)	0.03251 (0.008812)	0.03446 (0.008449)	0.03494 (0.008363)	0.03493 (0.008364)	0.03492 (0.008366)
		20	0.06544 (0.00371)	0.06618 (0.00362)	0.06615 (0.00362)	0.06612 (0.00363)	0.06902 (0.00329)	0.06975 (0.00320)	0.06972 (0.00321)	0.06970 (0.00321)
		10	0.007317 (0.00027)	0.007553 (0.00026)	0.007552 (0.00026)	0.007551 (0.00026)	0.008206 (0.00024)	0.008441 (0.00023)	0.008442 (0.00023)	0.008441 (0.00023)
0.27170	100	15	0.01221 (0.000134)	0.012458 (0.000128)	0.012457 (0.000128)	0.012456 (0.000128)	0.013135 (0.00011)	0.013381 (0.000108)	0.013380 (0.000108)	0.013379 (0.000108)
		20	0.01814 (3.21859e-05)	0.018407 (2.92425e-05)	0.018405 (2.92590e-05)	0.018403 (2.92755e-05)	0.01913 (2.18882e-05)	0.0191401 (1.94716e-05)	0.0191400 (1.94850e-05)	0.019139 (1.95000e-05)

Table 3: Bayes and E-Bayes Estimates of α , bias and their MSE (within parenthesis) for different values of r

$Loss function \rightarrow$		DEF				ABLF			
$\alpha \downarrow$	$r \downarrow$	$\hat{\alpha}_{DG}$	$\hat{\alpha}_{EBDG_1}$	$\hat{\alpha}_{EBDG_2}$	$\hat{\alpha}_{EBDG_3}$	$\hat{\alpha}_{AB}$	$\hat{\alpha}_{EBAB_1}$	$\hat{\alpha}_{EBAB_2}$	$\hat{\alpha}_{EBAB_3}$
0.22008	25	0.35155 0.13147 (0.01728)	0.34952 0.12944 (0.01675)	0.34812 0.12803 (0.01639)	0.34671 0.12662 (0.01603)	0.35801 0.13793 (0.01902)	0.35603 0.13594 (0.01848)	0.35459 0.13451 (0.01809)	0.35316 0.13307 (0.01770)
0.13552	20	0.23597 0.10045 (0.01009)	0.23303 0.09833 (0.00966)	0.23303 0.09750 (0.00950)	0.23221 0.09668 (0.00934)	0.24162 0.10609 (0.01125)	0.23952 0.10400 (0.01081)	0.23868 0.10315 (0.01064)	0.23784 0.10231 (0.01046)
0.40953	15	0.12400 -0.28553 (0.08152)	0.12167 -0.28786 (0.08286)	0.12135 -0.28818 (0.08305)	0.12103 -0.28850 (0.08323)	0.12822 -0.28131 (0.07913)	0.12590 -0.28363 (0.08044)	0.12557 -0.28396 (0.08063)	0.12524 -0.28429 (0.08082)
0.23685	10	0.45335 0.21650 (0.04687)	0.45069 0.21384 (0.04572)	0.44879 0.21194 (0.04491)	0.44689 0.21004 (0.04411)	0.46013 0.22328 (0.04985)	0.45749 0.22064 (0.04868)	0.45556 0.21871 (0.04783)	0.45363 0.21678 (0.04699)

1. From Table 3 and 4, it can be observed that E-Bayes estimates are better than the Bayes estimators for all the considered loss functions. Moreover, the MSE decrease with the decrease in the value of the r for all loss functions. Moreover, Bayes and E-Bayes estimators can be ordered as $\hat{\alpha}_{EBDG_3} < \hat{\alpha}_{EBDG_2} < \hat{\alpha}_{EBDG_1} < \hat{\alpha}_{DG}$, for DE loss function, under AB loss function, they can be ordered as $\hat{\alpha}_{EBAB_3} < \hat{\alpha}_{EBAB_2} < \hat{\alpha}_{EBAB_1} < \hat{\alpha}_{AB}$; under ME loss function they can be ordered as $\hat{\alpha}_{EBME_3} < \hat{\alpha}_{EBME_2} < \hat{\alpha}_{EBME_1} < \hat{\alpha}_{ME}$; and under Entropy loss function, they can be ordered as $\hat{\alpha}_{EBEL_3} < \hat{\alpha}_{EBEL_2} < \hat{\alpha}_{EBEL_1} < \hat{\alpha}_{EL}$.
2. Hierarchy among Bayes estimates can be ordered as $\hat{\alpha}_{ME} < \hat{\alpha}_{EL} < \hat{\alpha}_{DE} < \hat{\alpha}_{AB}$. Similarly, hierarchy observed among E-Bayes estimates is $\hat{\alpha}_{EBME} < \hat{\alpha}_{EBEF} < \hat{\alpha}_{EBDE} < \hat{\alpha}_{EBAB}$.

Table 4: Bayes and E-Bayes Estimates of α, Bias and their MSE (within parenthesis) for different values of r

$Loss function \rightarrow$		MEF				ELF			
$\alpha \downarrow$	$r \downarrow$	$\hat{\alpha}_{ME}$	$\hat{\alpha}_{EBME_1}$	$\hat{\alpha}_{EBME_2}$	$\hat{\alpha}_{EBME_3}$	$\hat{\alpha}_{EL}$	$\hat{\alpha}_{EBEL_1}$	$\hat{\alpha}_{EBEL_2}$	$\hat{\alpha}_{EBEL_3}$
0.40953	25	0.08940 -0.32009 (0.10245)	0.08993 -0.31960 (0.10214)	0.08969 -0.31984 (0.10224)	0.08945 -0.32007 (0.10244)	0.09994 -0.30959 (0.09584)	0.10051 -0.30902 (0.09549)	0.10024 -0.30928 (0.09565)	0.09998 -0.30955 (0.09582)
0.23685	20	0.40253 0.16568 (0.02745)	0.39967 0.16281 (0.02651)	0.39798 0.16113 (0.02596)	0.39630 0.15944 (0.02542)	0.41947 0.18262 (0.03335)	0.41667 0.17982 (0.03233)	0.41492 0.17807 (0.03170)	0.41316 0.17631 (0.03108)
0.2208	15	0.30306 0.08297 (0.00688)	0.30075 0.08067 (0.00650)	0.29954 0.07946 (0.00631)	0.29833 0.07825 (0.00612)	0.31922 0.09914 (0.00982)	0.31701 0.09693 (0.00939)	0.31573 0.09565 (0.00914)	0.31445 0.09437 (0.00890)
0.13552	10	0.19364 0.05812 (0.00337)	0.19133 0.05581 (0.00311)	0.19066 0.05513 (0.00304)	0.18999 0.05446 (0.00296)	0.20775 0.07223 (0.00521)	0.20551 0.06998 (0.00489)	0.20478 0.06926 (0.00479)	0.20406 0.06853 (0.00469)

7 Conclusion

In this article, Bayesian and E-Bayesian methods have been used to obtain the estimates of the unknown parameter of inverse Lomax distribution for different loss functions such as De Groot Loss Function, Al-Bayyati Loss function, Entropy Loss Function and Minimum Expected Loss Function.
It was observed that when we increasing sample size n and termination number r, the performance of both estimators improves in terms of a decrease in MSE. It was seen that E-Bayes estimates with the decreasing loss function are superior to E-Bayes estimates with constant and increasing loss functions. So, it is concluded that parameter to estimate the parameters of ILD, E-Bayesian estimation is a good choice over the Bayesian estimation technique.

Conflict of Interest

The authors declare that they have no conflict of interest.

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