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Two-Hybrid Techniques Coupled with an Integral Transformation for Caputo Time-Fractional Navier-Stokes Equations

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Abstract: In this paper, two-hybrid techniques, namely q-homotopy analysis Elzaki transform method (q-HAETM) and iterative Elzaki transform method (IETM) have been applied to obtain the numerical solutions of time-fractional Navier-Stokes equations in polar coordinate described in the Caputo sense. q-HAETM is the combination of the homotopy analysis method and Elzaki transform method, and IETM is the combination of two reliable methods, i.e. iterative method and Elzaki transform method. Two example problems are solved, and the obtained results are compared with the solutions solved by other techniques which show the efficacy and powerfulness of the methods.

Keywords: Homotopy analysis method, Elzaki transform, Caputo fractional derivative, Navier-Stokes equation, viscous fluid.

1 Introduction

Fractional calculus is an essential discipline of mathematics which offers the differentiation and integration of arbitrary order. Fractional calculus has become widespread due to its numerous applications in the field of solid mechanics, fluid dynamics, financial mathematics, social sciences and other areas of science and engineering [1] -[13]. The solutions of non-integer order differential equations are more complicated than integer-order differential equations, so computationally efficient and reliable numerical methods are to be developed to handle them. Authors have written different books [14]-[18] in which different types of studies and analyses on non-integer order calculus help the scientists better understand the concepts of fractional calculus.

In this investigation, we have focused on the unsteady one-dimensional motion of the viscous fluid through a tube. The equations of motion of a viscous fluid are the Navier-Stokes (NS) equations which may be written in [19] polar coordinate as

$$\frac{\partial}{\partial t}u(r,t) = P + \nu\left(\frac{\partial^2}{\partial r^2}u(r,t) + \frac{1}{r}\frac{\partial}{\partial r}u(r,t)\right), \tag{1}$$

where $P = -\frac{\partial p}{\rho \partial z}$ and t, u, ρ, p and ν are the time, velocity, density, pressure and kinematic viscosity of the fluid, respectively. The aforementioned Eq. (1) can be formulated by replacing the first-order derivative of velocity concerning time with the fractional derivative of order α $(0 < \alpha \leq 1)$. Hence, the operator form of the time-fractional NS equation has the form

$$D_t^{\alpha} u = P + \nu\left(\frac{\partial^2}{\partial r^2}u(r,t) + \frac{1}{r}\frac{\partial}{\partial r}u(r,t)\right), \tag{2}$$

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where D_t^{α} is the fractional derivative in the Caputo sense of order α $(0 < \alpha \leq 1)$.
Momani and Odibat [19] investigated the polar form of the NS equation using the homotopy perturbation method (HPM). The time-fractional NS equations have been deliberated by Ganji et al. [20] utilizing the Adomian decomposition method (ADM). The classical NS equation in the fractional form was first addressed and solved by El-Shahed and Salem [21] using Laplace transform (LT) and finite Hankel transforms. The same form of NS equations in polar coordinate has been solved by Kumar et al. [22] and Ragab et al. [23] using modified Laplace decomposition method and Homotopy analysis method (HAM). Recently, Xu et al. [24] has explored the solution of the NS equations with space-fractional derivative by replacing the Laplacian operator with Riesz fractional derivatives. The analytical study of three-dimensional viscous flow near an infinite rotating disk has been discussed by Rashidi and Shahmohamadi [25] using the variational iteration method, and the Pade approximation. The time-fractional NS equation in cartesian coordinate is investigated by Jena and Chakraverty [26] using hybrid techniques, namely homotopy perturbation Elzaki transform method.
The present paper aims to point out the semi-analytical techniques, namely q-HATM and IETM, that are used to solve NS equations arising in fluid mechanics. The q-HATM and IETM are the couplings of homotopy analysis method and iterative method with Elzaki transform. Their major advantage is that they can combine two different powerful techniques to get rapid convergence for the mentioned problem. Although Elzaki transform method is the effective method for solving fractional differential equations, it fails to handle nonlinear terms arising from the fractional differential equations. These difficulties may be overcome by coupling this transform with the HAM and the iterative method. Although a number of studies related to the usual integral transformations, such as Fourier, Laplace, etc. have been achieved, the number of the studies involving the Elzaki transform (ET) are quite a few in the literature. ET is a novel transform which was presented by ELzaki in 2010 [27]. It is an enhanced remodel of Sumudu and Laplace transforms. There are some differential equations which may not be solved by Sumudu and Laplace transform, but may be merely explained by ET. The ET has several properties, but the most useful property over the other integral transforms is its ‘unity’ feature, which plays an essential role [28].
The remaining parts of the paper are arranged as follows: Basic properties of fractional calculus are presented in Section 2. Some necessary definitions and theorems of ET are addressed in Section 3. q-HATM, and IETM are defined in Section 4 and Section 5, respectively. Two test problems are incorporated in Section 6 to validate the precision of the present methods. Results and discussion are illustrated in Section 7. Section 8 is devoted to conclusion.

2 Basic definitions of fractional calculus

Definition 1. *The integral operator ${}_aJ_t^{\alpha}$ of the order α of function $f(t)$ in Riemann-Liouville sense is written as [14, 15]*

$$\begin{aligned} {}_aJ_t^{\alpha} f(t) &= \tfrac{1}{\Gamma(\alpha)} \int_a^t (t-\xi)^{\alpha-1} f(\xi)\, d\xi, \\ {}_aJ_t^{0} f(t) &= f(t), \end{aligned} \tag{3}$$

where $\alpha > 0,\ t \geq a \geq 0,$ and $\Gamma(.)$ is the Gamma function.

Definition 2. *The operator D_t^{α} of $u(t)$ in the Caputo sense is well-defined as [15, 18]*

$$D_t^{\alpha} u(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{u^m(\xi)}{(t-\xi)^{\alpha-m+1}} d\xi, & m-1 < \alpha < m, \\ \frac{d^m}{dt^m} u(t), & \alpha = m. \end{cases} \tag{4}$$

Following [14], we may have

$$J^{\alpha} t^m = \frac{\Gamma(m+1)}{\Gamma(m+\alpha+1)} t^{m+\alpha}, \tag{5}$$

$$D^{\alpha} t^m = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} t^{m-\alpha}. \tag{6}$$

Definition 3.

$$D_t^{\alpha} J_t^{\alpha} \psi(t) = \psi(t), \tag{7}$$

$$J_t^{\alpha} D_t^{\alpha} \psi(t) = \psi(t) - \sum_{k=0}^{m} \psi^{(k)}\left(0^+\right) \frac{t^k}{k!}, \tag{8}$$

for $t > 0$ and $m-1 < \alpha \leq m$ [14, 15, 16, 17, 18].

3 Elzaki transformation

The Elzaki transform (ET) of the function $g(t)$ is defined as

$$\mathscr{E}[g(t)] = G(s) = s\int_0^\infty g(\tau)\, e^{\frac{-\tau}{s}}\, d\tau, \tau > 0. \tag{9}$$

The Elzaki transform is an effective integral transformation technique for solving those equations which may not be solved by other transformations [27]. Operating integration by parts in Eq. (9) to find ET of some functions is as follows [27] :

1.$\mathscr{E}\left[\frac{\partial g(x,t)}{\partial t}\right] = \frac{1}{s}G(x,s) - s\,g(x,0)$,
2.$\mathscr{E}\left[\frac{\partial^2 g(x,t)}{\partial t^2}\right] = \frac{1}{s^2}G(x,s) - g(x,0) - s\frac{\partial g(x,0)}{\partial t}$,
3.$\mathscr{E}\left[\frac{\partial g(x,t)}{\partial x}\right] = \frac{d}{dx}G(x,s)$,
4.$\mathscr{E}\left[\frac{\partial^2 g(x,t)}{\partial x^2}\right] = \frac{d^2}{dx^2}G(x,s)$.

3.1 Elzaki transform of fractional order derivatives defined in the Caputo sense

Theorem 1.[28] *If $F(p)$ is Laplace transform of $g(t)$, Elzaki transform $G(s)$ of $g(t)$ is given by*

$$G(s) = sF\left(\frac{1}{s}\right). \tag{10}$$

Theorem 2.[28] *If $G(s)$ is the Elzaki transform of $g(t)$, then*

$$\mathscr{E}[D^\alpha g(t)] = \frac{G(s)}{s^\alpha} - \sum_{k=0}^{n-1} s^{k-\alpha+2} g^{(k)}(0), n-1 < \alpha \le n. \tag{11}$$

4 q-homotopy analysis Elzaki transform method

To demonstrate the elementary idea of q-Homotopy analysis Elzaki transform method, we have considered the nonlinear non-homogenous fractional PDE as follows:

$$D_t^\alpha \psi(x,t) + R\psi(x,t) + N\psi(x,t) = \phi(x,t), \tag{12}$$

where D_t^α is the fractional derivative in the Caputo sense, R and N are the linear and nonlinear differential operator, respectively and $\phi(x,t)$ is the source term. Taking ET on both sides of Eq. (12) and using Eq. (11), we have

$$\mathscr{E}[\psi(x,t)] = \sum_{k=0}^{n-1} s^{k+2}\psi^{(k)}(x,0) + s^\alpha \mathscr{E}[\phi(x,t) - R\psi(x,t) - N\psi(x,t)]. \tag{13}$$

Define a nonlinear operator as

$$N[\xi(x,t;q)] = \mathscr{E}[\xi(x,t;q)] - \sum_{k=0}^{n-1} s^{k+2}\xi^{(k)}(x,t;q)\left(0^+\right) + s^\alpha \mathscr{E}[R\xi(x,t;q) + N\xi(x,t;q) - \phi(x,t)], \tag{14}$$

where $q \in \left[0, \frac{1}{n}\right]$ is the embedding parameter, and $\xi(x,t;q)$ is the unknown function. Constructing a homotopy we get

$$(1-nq)\mathscr{E}[\xi(x,t;q) - \psi_0(x,t)] = \hbar q H(x,t) N[\xi(x,t;q)], \tag{15}$$

where $\mathscr{E}$ denotes the Elzaki transform, $H(x,t)$ represents the non-zero auxiliary function, $\hbar \neq 0$ is an auxiliary parameter and $\psi_0(x,t)$ is the initial value of $\psi(x,t)$.

If $q=0$ and $q=\frac{1}{n}$, we obtain $\xi(x,t;0)=\psi_0(x,t)$, and $\xi\left(x,t;\frac{1}{n}\right)=\psi(x,t)$, respectively. As q increases from 0 to $\frac{1}{n}$, $\xi(x,t;q)$ varies from $\psi_0(x,t)$ to the solution of Eq. (12). Using Taylor's series expansion on $\xi(x,t;q)$, one has

$$\xi(x,t;q)=\psi_0(x,t)+\sum_{m=1}^{\infty}\psi_m(x,t)\,q^m, \tag{16}$$

where

$$\psi_m(x,t)=\frac{1}{m!}\left.\frac{\partial^m\psi(x,t;q)}{\partial q^m}\right|_{q=0}. \tag{17}$$

If R, $u_0(x,t)$, $H(x,t)$ and $\hbar$ are properly chosen, Eq. (16) converges at $q=\frac{1}{n}$. Hence, we have

$$\psi(x,t)=\psi_0(x,t)+\sum_{m=1}^{\infty}\psi_m(x,t)\left(\frac{1}{n}\right)^m. \tag{18}$$

Let us define the vector

$$\vec{\psi_m}=\{\psi_0,\psi_1,\ldots,\psi_m\}. \tag{19}$$

Differentiating Eq. (15) m-times with respect to q, setting $q=0$ and dividing by $m!$, we have the $m-th$ order deformation equation [29,30] as follows:

$$\mathscr{E}\left[\psi_m(x,t)-\chi_m\psi_{m-1}(x,t)\right]=\hbar H(x,t)\,\Re_m\left(\vec{\psi_{m-1}}(x,t)\right). \tag{20}$$

Applying the inverse Elzaki transform to both sides of Eq. (20), we obtain

$$\psi_m(x,t)=\chi_m\psi_{m-1}(x,t)+\mathscr{E}^{-1}\left[\hbar H(x,t)\,\Re_m\left(\vec{\psi_{m-1}}(x,t)\right)\right], \tag{21}$$

where

$$\Re_m\left(\vec{u_{m-1}}(x,t)\right)=\frac{1}{(m-1)!}\left.\frac{\partial^{m-1}N(\xi(x,t;q))}{\partial q^{m-1}}\right|_{q=0}, \tag{22}$$

and

$$\chi_m=\begin{cases}0, & m\le 1\,,\\ n, & m>1\,.\end{cases} \tag{23}$$

Theorem 3. *If we can find a constant $0<a<1$ such that $\|\psi_{m+1}(x,t)\|\le a\|\psi_m(x,t)\|$ for each value of m. Moreover, if the truncated series $\sum_{m=1}^{r}\psi_n(x,t)\left(\frac{1}{n}\right)^m$ is taken as approximate solution of $\psi(x,t)$, the maximum absolute truncated error is given in [31]*

$$\left\|\psi(x,t)-\sum_{m=0}^{r}\psi_m(x,t)\left(\frac{1}{n}\right)^m\right\|\le\frac{a^{r+1}}{n^r(n-a)}\psi_0(x,t). \tag{24}$$

Proof. One may see the proof of this theorem in [31].

5 Iterative Elzaki transform method

To clarify the simple idea of this technique, let us apply the Elzaki transform to both sides of Eq. (12) and we get

$$\mathscr{E}[\psi(x,t)]=\sum_{k=0}^{n-1}s^{k+2}\psi^{(k)}(x,0)+s^{\alpha}\mathscr{E}[\phi(x,t)-R\psi(x,t)-N\psi(x,t)]. \tag{25}$$

Applying inverse Elzaki transform to Eq. (25), we obtain

$$\psi(x,t)=\mathscr{E}^{-1}\left[\sum_{k=0}^{n-1}s^{k+2}\psi^{(k)}(x,0)\right]+\mathscr{E}^{-1}\left[s^{\alpha}\mathscr{E}[\phi(x,t)-R\psi(x,t)-N\psi(x,t)]\right]. \tag{26}$$

Let

$$\left.\begin{array}{c} f(x,t) = \mathscr{E}^{-1}\left[\sum_{k=0}^{n-1} s^{k+2}\psi^{(k)}(x,0)\right] + \mathscr{E}^{-1}\left[s^{\alpha}\mathscr{E}\left[\phi(x,t)\right]\right] \\ K\psi(x,t) = -\mathscr{E}^{-1}\left[s^{\alpha}\mathscr{E}\left[R\psi(x,t)\right]\right] \\ W\psi(x,t) = -\mathscr{E}^{-1}\left[s^{\alpha}\mathscr{E}\left[N\psi(x,t)\right]\right] \end{array}\right\}. \tag{27}$$

Consequently, the Eq. (26) can be rearranged as

$$\psi(x,t) = f(x,t) + K\psi(x,t) + W\psi(x,t), \tag{28}$$

where $f(x,t)$ is the known function, K and W are the linear and nonlinear operator of ψ. Hence, solution of Eq. (28) can be revised in series form as

$$\psi(x,t) = \sum_{j=0}^{\infty} \psi_j(x,t), \tag{29}$$

From Eq. (29), we get

$$K\sum_{j=0}^{\infty} \psi_j(x,t) = \sum_{j=0}^{\infty} K(\psi_j(x,t)), \tag{30}$$

the nonlinear operator W can be written as [32]

$$W\left(\sum_{j=0}^{\infty} \psi_j(x,t)\right) = W(\psi_0) + \sum_{j=0}^{\infty}\left[W\left(\sum_{i=0}^{j}\psi_i\right) - W\left(\sum_{i=0}^{j-1}\psi_i\right)\right]. \tag{31}$$

Furthermore, using Eqs. (29)-(31) in Eq. (28), we obtain

$$\sum_{j=0}^{\infty} \psi_j(x,t) = f(x,t) + \sum_{j=0}^{\infty} K(\psi_j(x,t)) + W(\psi_0) + \sum_{j=0}^{\infty}\left[W\left(\sum_{i=0}^{j}\psi_i\right) - W\left(\sum_{i=0}^{j-1}\psi_i\right)\right]. \tag{32}$$

Let us define the recurrence relation

$$\left.\begin{array}{c} \psi_0 = f(x,t), \\ \psi_1 = K(\psi_0) + W(\psi_0), \\ \psi_{m+1} = K(\psi_m) + W(\psi_0 + \psi_1 + \psi_2 + \ldots + \psi_m) - W(\psi_0 + \psi_1 + \psi_2 + \ldots + \psi_{m-1}). \end{array}\right\} \tag{33}$$

From this, we get

$$\psi_1 + \psi_2 + \psi_3 + \ldots + \psi_{m+1} = K(\psi_0 + \psi_1 + \psi_2 + \ldots + \psi_m) + W(\psi_0 + \psi_1 + \psi_2 + \ldots + \psi_m). \tag{34}$$

So the m-term approximate solution of Eq. (12) may be written as

$$\psi = \psi_0 + \psi_1 + \psi_2 + \psi_3 + \ldots + \psi_{m-1}. \tag{35}$$

One may see the convergence analysis of this method in [32].

6 Illustrative examples

To validate the efficacy of the two methods, two example problems have been investigated. Here, we have solved NS equations in polar coordinate.

*Example 1.*Let us consider the time-fractional NS equation [19] as

$$\frac{\partial^{\alpha}\psi}{\partial t^{\alpha}} = P + \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r}, 0 < \alpha \leq 1, \tag{36}$$

with initial condition (IC)

$$\psi(r,0) = 1 - r^2. \tag{37}$$

Implementation of q-HAETM

Applying the ET on Eq. (36) and simplifying, we get

$$\mathscr{E}[\psi(r,t)] = s^2\left(1-r^2\right) + s^{\alpha}\mathscr{E}\left[P + \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r}\right]. \tag{38}$$

Defining the nonlinear operator, we obtain

$$\begin{aligned} N[\xi(r,t;q)] = \mathscr{E}[\xi(r,t;q)] - \left(1-\tfrac{\chi_m}{n}\right)s^2\left(1-r^2\right) - \\ s^{\alpha}\mathscr{E}\left[P\left(\left(1-\tfrac{\chi_m}{n}\right)\right) + \tfrac{\partial^2\xi(r,t;q)}{\partial r^2} + \tfrac{1}{r}\tfrac{\partial\xi(r,t;q)}{\partial r}\right]. \end{aligned} \tag{39}$$

Using the above-mentioned procedure of q-HAETM, we have the m-th order deformation equation for $H(x,t) = 1$ as

$$\mathscr{E}[\psi_m(r,t) - \chi_m\psi_{m-1}(r,t)] = \hbar\Re_m\left(\overrightarrow{\psi_{m-1}}(r,t)\right), \tag{40}$$

and taking the inverse Elzaki transform on both sides of Eq. (40), we obtain

$$\psi_m(r,t) = \chi_m\psi_{m-1}(r,t) + \mathscr{E}^{-1}\left[\hbar\Re_m\left(\overrightarrow{\psi_{m-1}}(r,t)\right)\right], \tag{41}$$

where

$$\begin{aligned} \Re_m\left(\overrightarrow{\psi_{m-1}}(r,t)\right) = \mathscr{E}[\psi_{m-1}(r,t)] - \left(1-\tfrac{\chi_m}{n}\right)s^2\left(1-r^2\right) - \\ s^{\alpha}\mathscr{E}\left[P\left(\left(1-\tfrac{\chi_m}{n}\right)\right) + \tfrac{\partial^2\psi_{m-1}(r,t)}{\partial r^2} + \tfrac{1}{r}\tfrac{\partial\psi_{m-1}(r,t)}{\partial r}\right]. \end{aligned} \tag{42}$$

Substituting Eq. (42) into Eq. (41) and using Eq. (37), we obtain the solutions as

$$\psi_0 = 1 - r^2, (6.8)\,\psi_1 = -\hbar\frac{(P-4)\,t^{\alpha}}{\Gamma(1+\alpha)}, \tag{43}$$

$$\psi_2 = -\hbar(\hbar+1)\frac{(P-4)\,t^{\alpha}}{\Gamma(1+\alpha)}, \tag{44}$$

$$\psi_3 = -\hbar(\hbar+n)^2\frac{(P-4)\,t^{\alpha}}{\Gamma(1+\alpha)}, \tag{45}$$

continuing likewise, we can get remaining ψ_m, $m = 4, 5, 6, \ldots$. Hence, the series solution of Eq. (36) is written as

$$\psi(r,t) = \psi_0(r,t) + \sum_{m=1}^{\infty}\psi_m(r,t)\left(\frac{1}{n}\right)^m.$$

For $\hbar = -1$ and $n = 1$, the above-mentioned equation can be rewritten as

$$\psi(r,t) = 1 - r^2 + \frac{(P-4)\,t^{\alpha}}{\Gamma(1+\alpha)}, \tag{46}$$

which is the same as a result of ref. [19].

Implementation of IETM

Applying inverse Elzaki transform to Eq. (38), we get

$$\psi(r,t) = \left(1-r^2\right) + \mathscr{E}^{-1}\left[s^{\alpha}\mathscr{E}\left[P + \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r}\right]\right], \tag{47}$$

$$\psi(r,t) = \left(1-r^2\right) + \frac{Pt^{\alpha}}{\Gamma(1+\alpha)}\mathscr{E}^{-1}\left[s^{\alpha}\mathscr{E}\left[\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r}\right]\right]. \tag{48}$$

From Eq. (27), we obtain

$$f(r,t) = \left(1-r^2\right) + \frac{Pt^\alpha}{\Gamma(1+\alpha)}, \tag{49}$$

$$K\psi(r,t) = \mathscr{E}^{-1}\left[s^\alpha \mathscr{E}\left[\frac{\partial^2\psi}{\partial r^2}\right]\right], \tag{50}$$

$$W\psi(r,t) = \mathscr{E}^{-1}\left[s^\alpha \mathscr{E}\left[\frac{1}{r}\frac{\partial\psi}{\partial r}\right]\right]. \tag{51}$$

From Eq. (33), we have

$$\psi_0 = \left(1-r^2\right) + \frac{Pt^\alpha}{\Gamma(1+\alpha)}, \tag{52}$$

$$\begin{aligned} \psi_1 &= K(\psi_0) + W(\psi_0) \\ &= \mathscr{E}^{-1}\left[s^\alpha \mathscr{E}\left[\tfrac{\partial^2\psi_0}{\partial r^2}\right]\right] + \mathscr{E}^{-1}\left[s^\alpha \mathscr{E}\left[\tfrac{1}{r}\tfrac{\partial\psi_0}{\partial r}\right]\right] \\ &= -4\tfrac{t^\alpha}{\Gamma(1+\alpha)} \end{aligned} \tag{53}$$

$$\begin{aligned} \psi_2 &= K(\psi_1) + W(\psi_0+\psi_1) - W(\psi_0) \\ &= \mathscr{E}^{-1}\left[s^\alpha \mathscr{E}\left[\tfrac{\partial^2\psi_1}{\partial r^2}\right]\right] + \mathscr{E}^{-1}\left[s^\alpha \mathscr{E}\left[\tfrac{1}{r}\tfrac{\partial(\psi_1+\psi_0)}{\partial r}\right]\right] - \mathscr{E}^{-1}\left[s^\alpha \mathscr{E}\left[\tfrac{1}{r}\tfrac{\partial\psi}{\partial r}\right]\right] \\ &= 0 \end{aligned} \tag{54}$$

Computating similarly, we get

$$\psi_3 = \psi_4 = \cdots = 0. \tag{55}$$

Thus, the solution of Eq. (36) obtained by IETM is

$$\psi = \psi_0 + \psi_1 + \psi_2 + \cdots\cdots = \left(1-r^2\right) + \frac{(P-4)}{\Gamma(1+\alpha)}t^\alpha. \tag{56}$$

This solution is consistent with the solution in [23].

*Example 2.*Now, we have considered the NS Eq. of the form

$$\frac{\partial^\alpha\psi}{\partial t^\alpha} = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r}, \tag{57}$$

with IC

$$\psi(r,0) = r. \tag{58}$$

Implementation of q-HAETM

Operating ET on both sides of Eq. (57) and simplifying, we obtain

$$\mathscr{E}[\psi(r,t)] = s^2(r) + s^\alpha \mathscr{E}\left[\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r}\right]. \tag{59}$$

Considering the nonlinear operator as

$$N[\xi(r,t;q)] = \mathscr{E}[\xi(r,t;q)] - \left(1-\frac{\chi_m}{n}\right)s^2 r - s^\alpha \mathscr{E}\left[\frac{\partial^2\xi(r,t;q)}{\partial r^2} + \frac{1}{r}\frac{\partial\xi(r,t;q)}{\partial r}\right]. \tag{60}$$

and using the aforementioned procedure of q-HAETM, we have the m-th order deformation equation for $H(x,t) = 1$

$$\mathscr{E}[\psi_m(r,t) - \chi_m\psi_{m-1}(r,t)] = \hbar\Re_m\left(\vec{\psi}_{m-1}(r,t)\right). \tag{61}$$

Inverse ET on both sides of Eq. (61) reduces

$$\psi_m(r,t) = \chi_m\psi_{m-1}(r,t) + \mathscr{E}^{-1}\left[\hbar\Re_m\left(\vec{\psi}_{m-1}(r,t)\right)\right], \tag{62}$$

where

$$\Re_m\left(\overrightarrow{\psi_{m-1}}(r,t)\right)=\mathscr{E}\left[\psi_{m-1}(r,t)\right]-\left(1-\frac{\chi_m}{n}\right)s^2(r)-s^\alpha\mathscr{E}\left[\frac{\partial^2\psi_{m-1}(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_{m-1}(r,t)}{\partial r}\right]. \tag{63}$$

Putting Eq. (63) into Eq. (62) and using Eq. (58), we have

$$\psi_0=r,\psi_1=\frac{-\hbar}{r}\frac{t^\alpha}{\Gamma(1+\alpha)}, \tag{64}$$

$$\psi_2=\frac{-\hbar(\hbar+n)t^\alpha}{r\Gamma(1+\alpha)}+\frac{\hbar^2t^{2\alpha}}{r^3\Gamma(1+2\alpha)}, \tag{65}$$

$$\psi_3=\frac{-\hbar(\hbar+n)^2t^\alpha}{r\Gamma(1+\alpha)}+\frac{\hbar^2(\hbar+n)t^{2\alpha}}{r^3\Gamma(1+2\alpha)}-\frac{9\hbar^3t^{3\alpha}}{r^5\Gamma(1+3\alpha)}. \tag{66}$$

Continuing in this manner, the remaining ψ_m, $m=4,5,6,\ldots$ can be calculated. Therefore, the series solution of Eq. (57) is given as

$$\psi(r,t)=\psi_0(r,t)+\sum_{m=1}^{\infty}\psi_m(r,t)\left(\frac{1}{n}\right)^m.$$

For $\hbar=-1$ and $n=1$, the last equation can be reduced as

$$\psi(r,t)=r+\sum_{k=0}^{\infty}\frac{1^2\times3^2\times\cdots\times(2k-3)^2}{r^{2k-1}}\frac{t^{k\alpha}}{\Gamma(1+k\alpha)}. \tag{67}$$

This is the exact solution of Eq. (57). Moreover, this solution is similar to the solution solved by Momani and Odibat [19].

Implementation of IETM

Applying inverse ET to Eq. (59), we get

$$\psi(r,t)=r+\mathscr{E}^{-1}\left[s^\alpha\mathscr{E}\left[\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}\right]\right]. \tag{68}$$

From Eq. (27), we obtain

$$f(r,t)=r, \tag{69}$$

$$K\psi(r,t)=\mathscr{E}^{-1}\left[s^\alpha\mathscr{E}\left[\frac{\partial^2\psi}{\partial r^2}\right]\right], \tag{70}$$

$$W\psi(r,t)=\mathscr{E}^{-1}\left[s^\alpha\mathscr{E}\left[\frac{1}{r}\frac{\partial\psi}{\partial r}\right]\right]. \tag{71}$$

From Eq. (33), we have

$$\psi_0=r, \tag{72}$$

$$\begin{aligned}\psi_1&=K(\psi_0)+W(\psi_0)\\&=\mathscr{E}^{-1}\left[s^\alpha\mathscr{E}\left[\tfrac{\partial^2\psi_0}{\partial r^2}\right]\right]+\mathscr{E}^{-1}\left[s^\alpha\mathscr{E}\left[\tfrac{1}{r}\tfrac{\partial\psi_0}{\partial r}\right]\right]\\&=\tfrac{t^\alpha}{r\Gamma(1+\alpha)},\end{aligned} \tag{73}$$

$$\begin{aligned}\psi_2&=K(\psi_1)+W(\psi_0+\psi_1)-W(\psi_0)\\&=\mathscr{E}^{-1}\left[s^\alpha\mathscr{E}\left[\tfrac{\partial^2\psi_1}{\partial r^2}\right]\right]+\mathscr{E}^{-1}\left[s^\alpha\mathscr{E}\left[\tfrac{1}{r}\tfrac{\partial(\psi_1+\psi_0)}{\partial r}\right]\right]-\mathscr{E}^{-1}\left[s^\alpha\mathscr{E}\left[\tfrac{1}{r}\tfrac{\partial\psi_0}{\partial r}\right]\right]\\&=\tfrac{t^{2\alpha}}{r^3\Gamma(1+2\alpha)},\end{aligned} \tag{74}$$

$$\begin{aligned}\psi_3 &= K(\psi_2) + W(\psi_0+\psi_1+\psi_2) - W(\psi_0+\psi_1)\\ &= \mathscr{E}^{-1}\left[s^{\alpha}\mathscr{E}\left[\frac{\partial^2\psi_2}{\partial r^2}\right]\right] + \mathscr{E}^{-1}\left[s^{\alpha}\mathscr{E}\left[\frac{1}{r}\frac{\partial(\psi_2+\psi_1+\psi_0)}{\partial r}\right]\right] - \mathscr{E}^{-1}\left[s^{\alpha}\mathscr{E}\left[\frac{1}{r}\frac{\partial(\psi_0+\psi_1)}{\partial r}\right]\right]\\ &= \frac{9t^{3\alpha}}{r^5\Gamma(1+3\alpha)}.\end{aligned} \tag{75}$$

Proceeding in this way, the remaining ψ_m, $m = 4, 5, 6, \ldots$ can be defined. Thus, the solution of Eq. (57) obtained by IETM is

$$\psi = \psi_0+\psi_1+\psi_2+\cdots = r + \sum_{k=0}^{\infty}\frac{1^2\times 3^2\times\cdots\times(2k-3)^2}{r^{2k-1}}\frac{t^{k\alpha}}{\Gamma(1+k\alpha)}. \tag{76}$$

This solution is consistent with the solution in [23].

7 Results and discussion

Plots of the solution for Eqs. (36)–(37) for $\hbar = -1$, $n = 1$, and $\alpha = P = 1$ are depicted in Figure 1which indicates that solution obtained by q-HAETM and IETM are identical with the exact solutions. Plots of solutions for different values of $\alpha\,(= 0.2, 0.4, 0.6, 0.8, 1)$ at $n = 1$, $P = 1$, and $t = 1$ are exhibited in Figure 2. In Figure 3, different values of $\hbar\,(= -0.2, -0.5, -0.9, -1, -1.2)$ are chosen to reduce the errors. The solution obtained by the present methods is compared with the results of HPM and ADM, which are depicted in Table 1. The efficiency of q-HAETM and IETM can be improved by taking more terms from the series solution. A similar simulation has been done for the solution of Eq. (57)-(58) using the same parameters with the same values similar to the previous problem, and the obtained results are depicted in Table 2 and Figures 4-6.

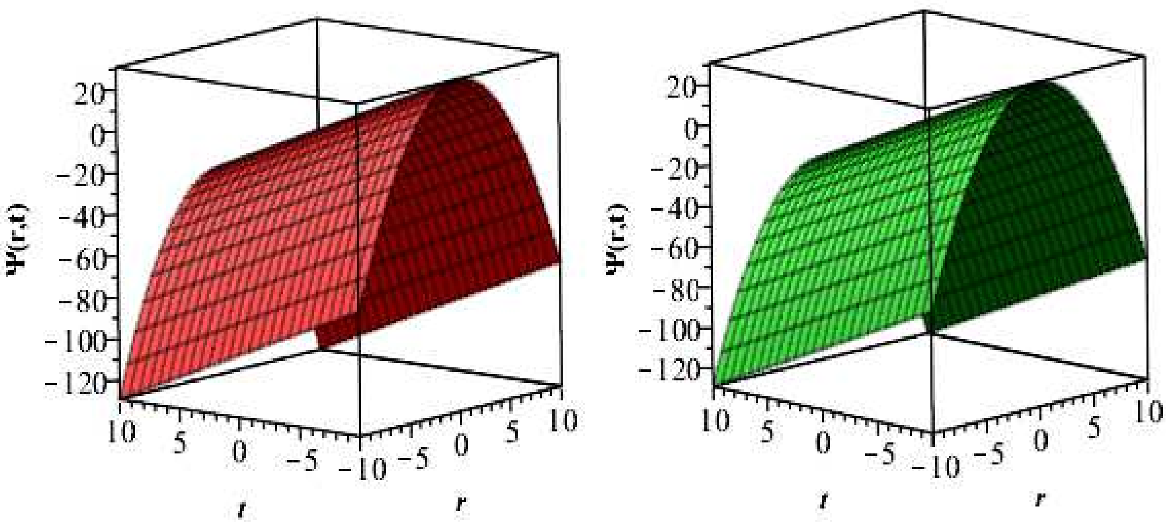


Fig. 1: Approximate solution of Ex.1 (left) and exact solution of Ex.1 at $P = 1$, $\alpha = 1$ (right).

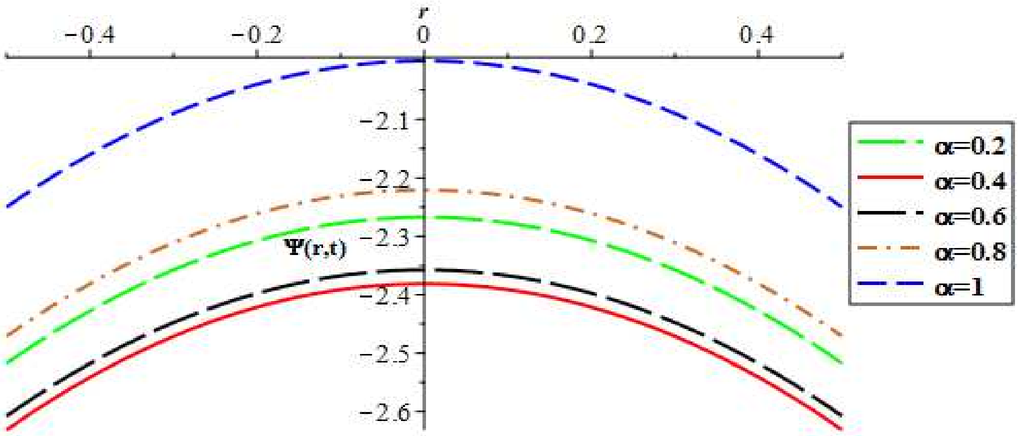


Fig. 2: The behavior of the solution of Ex. 1 at $P = 1, t = 1$ for different values of α.

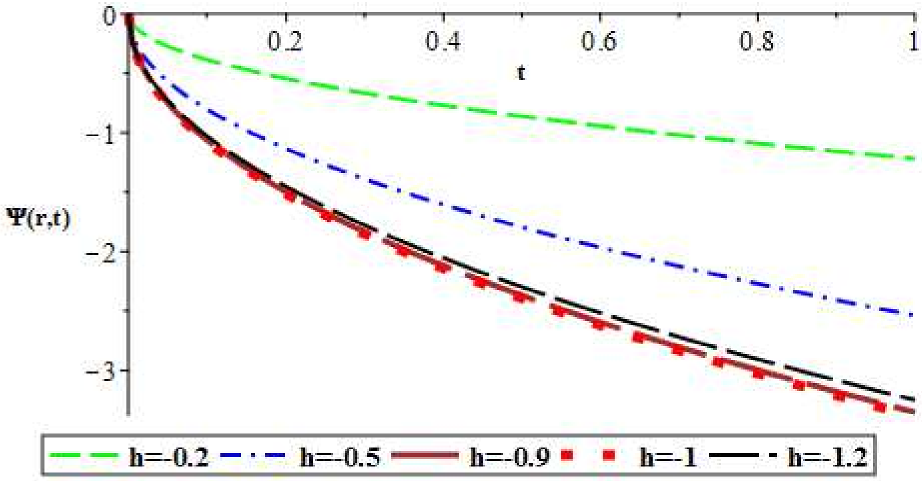
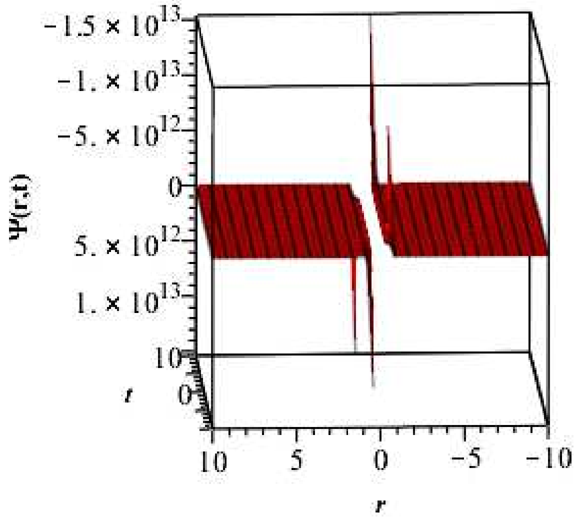


Fig. 3: Plot of the solution of Ex.1 at $P = 1, r = 1$ and $\alpha = 0.5$ for different values of $\hbar$.

Table 1: Comparison of results obtained by HPM and ADM with the present methods, i.e. q-HAETM and IETM at $\alpha = 1, \hbar = -1$ and $n = 1$ for Ex.1

r	t	HPM (Ganji et al. [20])	ADM (Momani and Odibat [19])	q-HAETM (Present)	IETM (Present)	Exact Solution
0.25	0.5	-0.5625	-0.5625	-0.5625	-0.5625	-0.5625
0.50		-0.75	-0.75	-0.75	-0.75	-0.75
0.75		-1.0625	-1.0625	-1.0625	-1.0625	-1.0625
1.00		-1.5	-1.5	-1.5	-1.5	-1.5
0.25	0.9	-1.7625	-1.7625	-1.7625	-1.7625	-1.7625
0.50		-1.95	-1.95	-1.95	-1.95	-1.95
0.75		-2.2625	-2.2625	-2.2625	-2.2625	-2.2625
1.00		-2.7	-2.7	-2.7	-2.7	-2.7



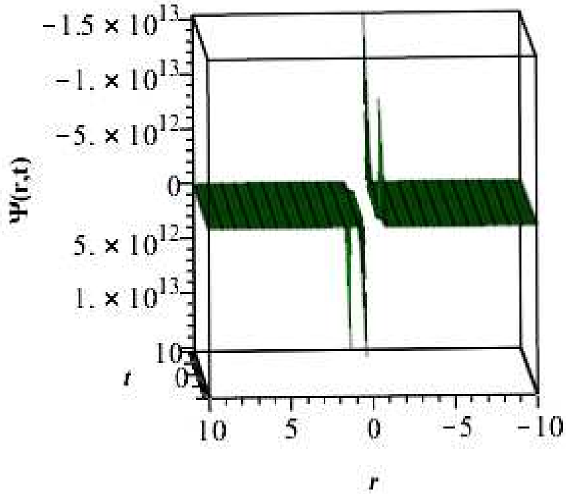


Fig. 4: 10-term approximate solution of Ex. 2 (left), exact solution of Ex.2 at $\alpha = 1$ (right).

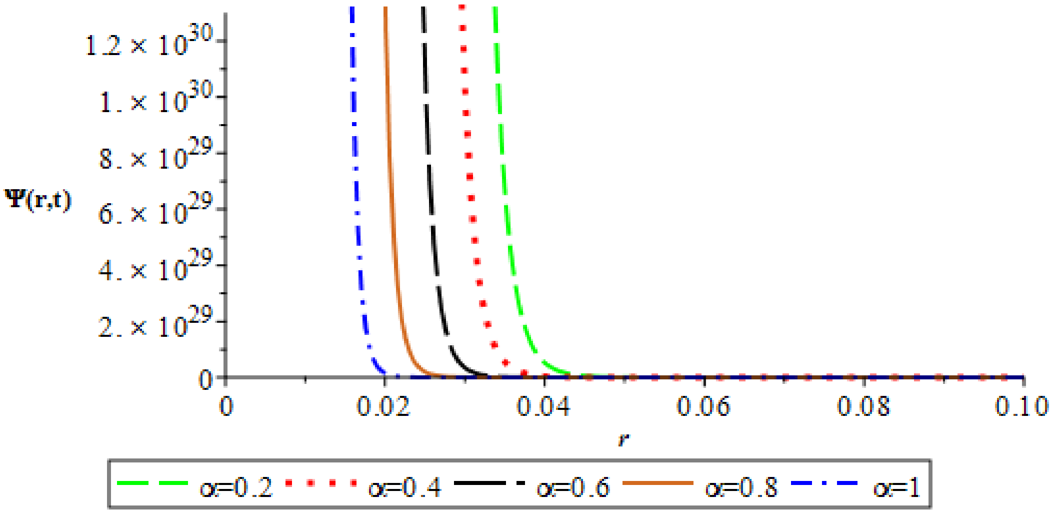


Fig. 5: The behavior of the solution of Ex.2 at $\hbar = -1$and $t = 1$for different values of α.

Table 2: Comparison of results obtained by HPM and ADM with the present methods, i.e. q-HAETM and IETM at $\alpha = 1$, $\hbar = -1$ and $n = 1$ for Ex.2.

r	t	HPM (Ganji et al. [20])	ADM (Momani and Odibat [19])	q-HAETM (Present)	IETM (Present)	Exact Solution
0.25	0.5	149791.1389	149791.1389	149791.1389	149791.1389	149791.1389
0.50		29.25075245	29.25075245	29.25075245	29.25075245	29.25075245
0.75		3.211685286	3.211685286	3.211685286	3.211685286	3.211685286
1.00		1.892327624	1.892327624	1.892327624	1.892327624	1.892327624
0.25	0.9	32438665.70	32438665.70	32438665.70	32438665.70	32438665.70
0.50		582.3037157	582.3037157	582.3037157	582.3037157	582.3037157
0.75		17.92314660	17.92314660	17.92314660	17.92314660	17.92314660
1.00		4.396399682	4.396399682	4.396399682	4.396399682	4.396399682

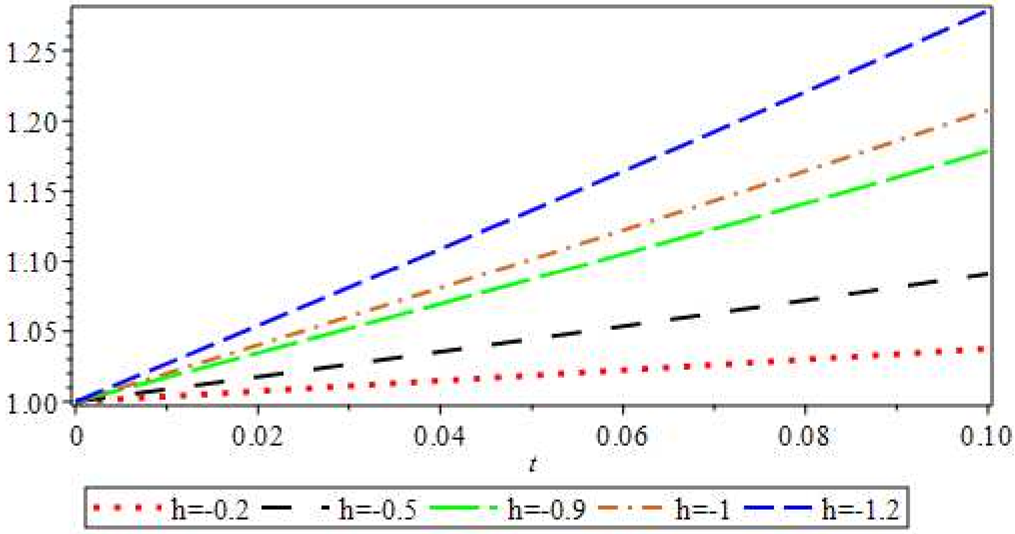


Fig. 6: Plot of the solution of Ex.2 at $r = 1$ and $\alpha = 1$ for different values of $\hbar$.

8 Conclusion

In this article, q-HAETM and IETM have been implemented to find the solution of time-fractional NS equation. The results we obtained in this study by considering the two methods are the same as the solution of Ganji et al. [20] as well as Momani and Odibat [19]. q-HATM offers a suitable way to control the convergence region of the solution by introducing the parameter $\hbar$ and $n\ (\geq 1)$. In addition, q-HATM does not require any linearization and perturbation. This helps us overcome the difficulties of truncation errors, high computer memory, and more times. The obtained results demonstrate that q-HAETM and IETM are powerful and useful tools for solving non-integer order PDEs.

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