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A Generalized Gronwall Inequality for Caputo Fractional Dynamic Delta Operator

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Abstract: In the present paper we obtain the generalized Gronwall type inequality using the Caputo fractional delta operator. In addition we detect the existence of solution the of Cauchy's type problem on fractional dynamic equations using dynamic delta operator. Applying the obtained inequality, we investigate the properties of solution on fractional dynamic equations.

Keywords: Gronwall inequality, fractional dynamic, delta operator.

1 Introduction

Fractional calculus is an important tool which generalizes the differential and integral calculus of arbitrary order. It is possible to define the differentiation and integration for non-integer order. Fractional calculus is more suitable for modeling the real world problems in various branches of science and engineering. In 1989, Stefan Hilger introduced time scale calculus; a unification of the differential and difference calculus [1]. Since then, several authors have addressed the properties and applications of dynamic equations on time scales [2]. Basic information on time scale calculus can be found in [3,4].

In [5,6,7,8,9,10], the authors investigated the Gronwall type inequality and some other inequalities as well as their applications to fractional differential equations using various fractional operators. Recently, the authors have explored in [11,12] inequalities on convex functions.

On the other hand the fractional calculus and time scale calculus and obtained results on the existence as well as properties of fractional differential equations on time scales [13,14,15,16,17,18,19]. Results of obtaining fractional time scale can be used in certain applications where the system is continuous and discrete and its behavior is dynamic in nature.

The basic theory on fractional dynamic calculus and equations on time scales is involved in [20,21,2]. These types of problems have applications in studying the properties of various processes in materials [22]. Inspired by above-mentioned pieces of literature in this paper, we detect the estimates on Gronwall type inequality and obtain the existence of the solution of Cauchy's Type problem on fractional dynamic equations on time scales. Using the obtained inequality we study the properties of Cauchy's Type such as the continuous dependence of solution. I organise the manuscript as follows. In section 2 several basic definitions and theorems utilised in this manuscript are presented. Section 3 deal with the Gronwall type inequality within Caputo fractional delta operator. Section 4 presents the existence and uniqueness of the investigated Cauchy problem. In section 5 we obtain the results for continuous dependence of the solution.

2 Preliminaries

This section comprises some basic definitions and theorems used in our subsequent discussions.

$\mathbb{T}$ denotes any time scale which has a topology inherited from standard topology on $\mathbb{T}$. C_{rd} denotes the set of all rd-continuous functions.

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In [23] we construct the metric space where $[t_0,\infty)_{\mathbb{T}} = I_{\mathbb{T}}$. Now consider the space function $C_{rd}(I_{\mathbb{T}},\mathbb{R}^n)$ such that $\sup_{t\in I_{\mathbb{T}}} \frac{v(x)}{e_\eta(x,x_0)} < \infty$ where $\eta > 0$. This space is denoted by $C^{\eta}_{rd}(I_{\mathbb{T}},\mathbb{R}^n)$.

We couple the space $C^{\eta}_{rd}(I_{\mathbb{T}},\mathbb{R}^n)$ by suitable metric

$$m^{\infty}_{\eta}(u,v) = \sup_{t\in I_T} \frac{|u(t)-v(t)|}{e_\eta(t,t_0)},$$

where the norm is defined as

$$|u|^{\infty}_{\eta} = \sup_{x\in I_{\mathbb{T}}} \frac{|u(t)|}{e_\eta(t,t_0)}.$$

More properties of m^{∞}_{η} and $|.|^{\infty}_{\eta}$ can be found in [23].

We define delta power function as

Definition 2.1 [2] Let $\alpha \in \mathbb{R}$, we define the generalized delta power function h_α on $\mathbb{T}$ as follows:

$$h_\alpha(t,t_0) = L^{-1}\left(\frac{1}{z^{\alpha+1}}\right)(t), \quad t \geq t_0,$$

for all $z \in C\backslash\{0\}$ such that L^{-1} exists, $t \geq t_0$. The fractional generalized delta power function $h_\alpha(x,y)$ on T, $t \geq s \geq t_0$ which is defined as the shift of $h_\alpha(t,t_0)$, i.e.

$$h_\alpha(t,s) = \widehat{h_\alpha(.,t_0)}(t,s), \quad t,s \in T, \quad t \geq s \geq t_0.$$

We define the Riemann-Liouville Fractional delta integral and Riemann Liouville Fractional delta derivative as follows: Suppose $\alpha \geq 0$ and $[-\overline{\alpha}]$ denotes the integral part of $-\alpha$.

Definition 2.2 [2, 16] For a function $u : \mathbb{T} \to \mathbb{R}$ the Riemann Liouville fractional delta integral of order α defined by

$$I^0_{\Delta,x_0} f(t) = f(t),$$

$$\begin{aligned}\left(I^0_{\Delta,t_0} f\right)(t) &= (h_{\alpha-1}(.,t_0) * f)(t) \\ &= \int_{t_0}^{t} \widehat{h_{\alpha-1}(.,t_0)}(t,\sigma(u)) f(u)\Delta u \\ &= \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(u)) f(u)\Delta u.\end{aligned}$$

Definition 2.3 [2, 16] Let $\alpha \geq 0$, $m = -\overline{[}-\alpha]$, $f : \mathbb{T} \to \mathbb{R}$. For $s,t \in \mathbb{T}^{k^m}$, $s < t$, the Riemann-Liouville fractional delta derivative of order α is defined by the expression

$$D^{\alpha}_{\Delta,s} f(t) = D^m_\Delta I^{m-\alpha}_{\Delta,s} f(x), \quad t \in \mathbb{T},$$

if it exists. For $\alpha < 0$ we define

$$D^{\alpha}_{\Delta,s} f(t) = I^{-\alpha}_{\Delta,s} f(t), \quad t,s \in T, \quad t > s,$$

$$I^{\alpha}_{\Delta,s} f(t) = D^{-\alpha}_{\Delta,s} f(t), \quad t,s \in T^{k^m}, \quad t > s, \quad r = \overline{[-\alpha]} + 1.$$

The Caputo Fractional delta derivative is defined as follows:

Definition 2.4 [2] Let $t \in \mathbb{T}$. Caputo fractional delta derivative of order $\alpha \geq 0$ using the Riemann-Liouville fractional delta derivative is defined as :

$${}^{C}D^{\alpha}_{\Delta,t_0} f(t) = D^{\alpha}_{\Delta,t_0}\left(f(t) - \sum_{k=0}^{m-1} h_k(t,t_0) f^{\Delta^k}(t_0)\right), \quad t > t_0,$$

where $m = \overline{[\alpha]} + 1$ if $\alpha \notin \mathbb{N}$, $m = \overline{[\alpha]}$ if $\alpha \in \mathbb{N}$.

3 Gronwall type inequality

Now give the Gronwall type inequality using Caputo fractional delta operator and we prove this by iteration. Suppose $\alpha \geq 0$ and $[-\overline{\alpha}]$ denotes the integral part of $-\alpha$.

Theorem 3.1 Let $\alpha > 0$, $y, u : \mathbb{T} \to \mathbb{R}$ be two non-negative integrable functions and v be non negative, non decreasing and rd-continuous function, $v(t) \leq B$ be a constant. If

$$y(t) \leq u(t) + v(t) \int_{t_0}^{t} h_{\alpha-1}(t, \sigma(\tau)) \Delta\tau, \tag{1}$$

for $t \in I_{\mathbb{T}}$, then

$$y(t) \leq u(t) + \int_{t_0}^{t} \left[\sum_{k=1}^{\infty} (v(t))^k h_{k\alpha-1}(t, \sigma(\tau)) u(\tau) \right] \Delta\tau, \tag{2}$$

for $t \in I_{\mathbb{T}}$.

Proof Define a function Q by

$$Q\psi(t) = v(t) \int_{t_0}^{t} h_{\alpha-1}(t, \sigma(\tau)) \psi(\tau) \Delta\tau, \tag{3}$$

then we get

$$y(t) \leq u(t) + Qy(t). \tag{4}$$

Taking iteration of (4) consecutively we get for $n \in N$

$$y(t) \leq \sum_{k=0}^{n-1} Q^k u(t) + Q^n u(t). \tag{5}$$

Now we prove by induction hypotheses that if ψ is non negative function then

$$Q^k \psi(t) \leq \int_{t_0}^{t} (v(t))^k h_{k\alpha-1}(t, \sigma(s)) \psi(s) \Delta s. \tag{6}$$

If $k = 1$ the result is obvious. Suppose the formula is valid for $k \in N$ then we have

$$\begin{aligned} &Q^{k+1}\psi(t) \\ &= Q.Q^k \psi(t) \\ &\leq v(t) \int_{t_0}^{t} h_{\alpha-1}(t, \sigma(\tau)) \left[\int_{t_0}^{\tau} (v(\tau))^k h_{k\alpha-1}(\tau, \sigma(s)) \psi(s) \Delta s \right] \Delta\tau. \end{aligned} \tag{7}$$

We have v non decreasing $v(\tau) \leq v(t)$ for $\tau \leq t$, from (7)

$$\begin{aligned} &Q^{k+1}\psi(t) \\ &\leq (v(t))^{k+1} \int_{t_0}^{t} \left[\int_{t_0}^{\tau} h_{\alpha-1}(t, \sigma(\tau)) h_{k\alpha-1}(\tau, \sigma(s)) \Delta\tau \right] \psi(s) \Delta s. \end{aligned} \tag{8}$$

From [21] and properties of the inner integral we have

$$\int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau))\, h_{k\alpha-1}(\tau,\sigma(s))\,\Delta\tau = h_{(k+1)\alpha-1}(\tau,\sigma(s)). \tag{9}$$

Then from (8) we get

$$Q^{k+1}\psi(t) \le (v(t))^{k+1}\int_{t_0}^{t} h_{(k+1)\alpha-1}(\tau,\sigma(s))\,\psi(s)\,\Delta s. \tag{10}$$

This proves that

$$Q^{n}\psi(t) \le \int_{t_0}^{t} (v(t))^{k}\, h_{k\alpha-1}(\tau,\sigma(s))\,\psi(s)\,\Delta s. \tag{11}$$

Now, we indicate that $\psi^n y(t) \to 0$ as $n \to \infty$.
Since $g(t)$ is rd-continuous and there exists $B > 0$ such that $g(t) \le B$ then we have

$$Q^{n}y(t) \le \int_{t_0}^{t} B^{N} h_{N\alpha-1}(\tau,\sigma(s))y(s)\,\Delta s, \tag{12}$$

where $Q^n y \to 0$ as $n \to \infty$.
Therefore we have from (5)

$$y(t) \le \sum_{k=0}^{\infty} Q^k f(t). \tag{13}$$

Thus we get

$$\begin{aligned} y(t) &\le \sum_{k=0}^{\infty} Q^k f(t) \\ &\le u(t) + \int_{t_0}^{t} \sum_{k=1}^{\infty} (v(t))^{k} h_{k\alpha-1}(\tau,\sigma(s))\, f(t)\Delta\tau, \end{aligned} \tag{14}$$

for $t \in I_{\mathbb{T}}$, which is required inequality.

4 Existence and uniqueness

Now we consider the Cauchy's type of problem with Caputo fractional delta derivative, suppose $\alpha > 0$

$${}^{C}D^{\alpha}_{\Delta,t_0}u(t) = f(t,u(t)), \quad t \in I_{\mathbb{T}}, \tag{15}$$

with the initial condition

$${}^{C}D^{\alpha}_{\Delta,t_0}u(t_0) = \overline{w}, \tag{16}$$

where $f : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$ is a given function and $0 < \alpha < 1$.

Let $L_\Delta[t_0,a)$ denotes the space of Δ Lebesgue summable function in $[t_0,a)$. Define the space

$$L^{\alpha}_{\Delta}[t_0,\alpha) = \left\{ y \in L_\Delta[t_0,a) : D^{\alpha}_{\Delta,t_0}y \in L_\Delta[t_0,a) \right\}.$$

Then from Theorem 52, [16], (15) and (16) are equivalent to

$$u(t) = wh_{\alpha-1}(t,t_0) + \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau))\, f(\tau,u(\tau))\,\Delta\tau. \tag{17}$$

The next theorem addresses the existence of solution.

Theorem 4.1 Let $L \geq 0$ be a constant. Suppose the function f in (15) is rd-continuous and satisfies

$$|f(x_1,x_2) - f(x_1,\overline{x_2})| \leq L|x_2 - \overline{x_2}|, \tag{18}$$

and let

$$p_1 = \sup_{t\in I_T} \frac{1}{e_\eta(t,t_0)} \left| wh_{\alpha-1}(t,t_0) + \int_{t_0}^{t} h_{\alpha-1}(t,t_0) f(\tau,0)\Delta\tau \right| < \infty. \tag{19}$$

If $\frac{L}{\eta} < 1$ then equation (15) has a unique solution $u \in C_{rd}^{\eta}(I_{\mathbb{T}},\mathbb{R}^n)$.

Proof. Let $u \in C_{rd}^{\eta}(I_{\mathbb{T}},\mathbb{R}^n)$ and define operator G by

$$(Gu)(t) = wh_{\alpha-1}(t,t_0) + \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) f(\tau,u(\tau))\Delta\tau, \tag{20}$$

for $t \in I_{\mathbb{T}}$.

We prove that G maps $C_{rd}^{\eta}(I_{\mathbb{T}},\mathbb{R}^n)$ into itself and is a contraction map. From (20) we have

$$\begin{aligned}(Gu)(t) &= wh_{\alpha-1}(t,t_0) + \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) f(\tau,u(\tau))\Delta\tau \\ &\quad - \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) f(\tau,0)\Delta\tau + \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) f(\tau,0)\Delta\tau. \end{aligned}\tag{21}$$

From (21) we have

$$\begin{aligned}|Gu|_\eta^\infty &= \sup_{t\in I_{\mathbb{T}}} \frac{|(Gu)(t)|}{e_\eta(t,t_0)} \\ &= \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\eta(t,t_0)} \Bigg| wh_{\alpha-1}(t,t_0) + \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) f(\tau,u(\tau))\Delta\tau \\ &\quad - \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) f(\tau,0)\Delta\tau + \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) f(\tau,0)\Delta\tau \Bigg| \\ &\leq \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\eta(t,t_0)} \left| wh_{\alpha-1}(t,t_0) + \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) f(\tau,0)\Delta\tau \right| \\ &\quad + \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\eta(t,t_0)} \Bigg| \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) f(\tau,u(\tau))\Delta\tau \\ &\quad - \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) f(\tau,0)\Delta\tau \Bigg| \\ &= p_1 + \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\eta(t,t_0)} \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) |f(\tau,u(\tau)) - f(\tau,0)|\Delta\tau \\ &= p_1 + \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\eta(t,t_0)} \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) L(u(\tau))\Delta\tau \\ &= p_1 + L|u|_\eta^\infty \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\eta(t,t_0)} \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) e_\eta(\tau,t_0)\Delta\tau \end{aligned}$$

$$
\begin{aligned}
&= p_1 + L|u|_\eta^\infty \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\eta(t,t_0)} I_{t_0}^{\Delta}(e_\eta(\tau,t_0))\Delta\tau \\
&\le p_1 + L|u|_\eta^\infty \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\eta(t,t_0)} \left(\frac{e_\eta(t,t_0)-1}{\eta}\right) \\
&\le p_1 + L|u|_\eta^\infty \frac{1}{\eta}\left(1-\frac{1}{e_\eta(t,t_0)}\right) \\
&= p_1 + \frac{L}{\eta}|u|_\eta^\infty \\
&< \infty.
\end{aligned} \tag{22}
$$

This proves that G maps $C_{rd}^{\eta}(I_{\mathbb{T}},\mathbb{R}^n)$ into itself.

Now we prove that G is a contraction map.

Let $x,y \in C_{rd}^{\eta}(I_{\mathbb{T}},\mathbb{R}^n)$. Then from (7) and by hypotheses we get

$$
\begin{aligned}
m_\eta^\infty(Gx,Gy) &= \sup_{t\in I_{\mathbb{T}}} \frac{|(Gx)(t)-(Gy)(t)|}{e_\eta(t,t_0)} \\
&= \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\eta(t,t_0)} \left| \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) f(\tau,x(\tau))\Delta\tau \right. \\
&\quad \left. - \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) f(\tau,y(\tau))\Delta\tau \right| \\
&\le \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\eta(t,t_0)} \left| \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) L \frac{|x(\tau)-y(\tau)|}{e_\eta(t,t_0)} e_\eta(t,t_0) \right| \Delta\tau \\
&= \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\eta(t,t_0)} \left| \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) m_\eta^\infty(x,y) e_\eta(t,t_0)\Delta\tau \right| \\
&= \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\eta(t,t_0)} L m_\eta^\infty(x,y) \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) e_\eta(t,t_0)\Delta\tau \\
&= L m_\eta^\infty(x,y) \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\eta(t,t_0)} \left(\frac{e_\eta(t,t_0)-1}{\eta}\right) \\
&\le \frac{L}{\eta} m_\eta^\infty(x,y).
\end{aligned} \tag{23}
$$

Since $\frac{L}{\eta} < 1$. Thus, G has a unique fixed point in $C_{rd}^{\eta}(I_{\mathbb{T}},\mathbb{R}^n)$ from Banach Fixed point theorem. The fixed point of G is a solution of equation (15). This completes the proof of theorem.

5 Continuous dependence

In this section, we obtain the results for continuous dependence of solution of (4.1). Now, consider the equation (15) and the corresponding equation

$$ {}^{C}D_{\Delta,t_0}^{\alpha} v(t) = \overline{f}(t,v(t)), \ t \in I_{\mathbb{T}}, \tag{24} $$

with initial condition

$$ {}^{C}D_{\Delta,t_0}^{\alpha} v(t) = \overline{w}, \tag{25} $$

where $f : I_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ and $\overline{w}$ is a given constant.

Now we give the theorem which deals with continuous dependence of solution of (15).

Theorem 5.1 Suppose the function f in (15) is rd-continuous and satisfies the condition (18). Let $v(t)$ be solution of (24) and

$$\begin{aligned} H(t) = & \left| w h_{\alpha-1}(t,t_0) - \overline{w} h_{\alpha-1}(t,t_0) \right| \\ & + \left| \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) f(\tau,v(\tau)) \Delta\tau \right. \\ & \left. - \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) \overline{f}(\tau,v(\tau)) \Delta\tau \right|, \end{aligned} \tag{27}$$

where f and $\overline{f}$ are functions in (15) and (24). Then, the solution $u(t), t \in I_{\mathbb{T}}$ of (15) dependence on functions on right hand side of (15) and

$$|u(t) - v(t)| \le H(t) + \int_{t_0}^{t} \left[\sum_{k=1}^{\infty} L^k h_{k\alpha-1}(t,\sigma(\tau)) H(\tau) \right] \Delta\tau, \tag{28}$$

for $t \in I_{\mathbb{T}}$.

Proof. The solutions of the equation $(15)-(16)$ and $(24)-(25)$ are

$$u(t) = w h_{\alpha-1}(t,t_0) + \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) f(\tau,u(\tau)) \Delta\tau \tag{29}$$

and

$$v(t) = \overline{w} h_{\alpha-1}(t,t_0) + \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) f(\tau,v(\tau)) \Delta\tau, \tag{30}$$

respectively.

We have

$$\begin{aligned} |u(t) - v(t)| \le & \left| w h_{\alpha-1}(t,t_0) - \overline{w} h_{\alpha-1}(t,t_0) \right| \\ & + \left| \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) f(\tau,u(\tau)) \Delta\tau \right. \\ & \left. - \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) f(\tau,v(\tau)) \Delta\tau \right| \\ & + \left| \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) f(\tau,v(\tau)) \Delta\tau \right. \\ & \left. - \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) \overline{f}(\tau,v(\tau)) \Delta\tau \right| \\ \le & H(t) + \int_{t_0}^{t} h_{\alpha-1}(t,\sigma(\tau)) L |u(\tau) - v(\tau)| \Delta\tau. \end{aligned} \tag{31}$$

Now, the application of Theorem (3.1) to equation (31) provides the required inequality (30).

References

[1] S. Hilger, Analysis on measure chain-a unified approach to continuous discrete calculus, *Res. Math.* **18**, 18-56 (1990).
[2] S. G. Georgiev, *Fractional dynamic calculus and fractional dynamic equations on time scales*, Springer, 2017.
[3] M. Bohner and A. Peterson, *Dynamic equations on time scales*, Birkhauser Boston/Berlin, (2001).
[4] M. Bohner and A. Peterson, *Advances in dynamic equations on time scales*, Birkhauser Boston/Berlin, (2003).
[5] Y. Adjabi, F. Jarad and T. Abdeljawad, On generalized fractional operators and a Gronwall type inequality with applications, *Filomat* **31**(17), 5457-5473 (2017).
[6] R. Almedia, A Gronwall inequality for a general Caputo fractional operator, *Math. Ineq. Appl.* **20**(4), 1089-1105 (2017).
[7] R. Almeida, A Caputo fractional derivative of a function with respect to another function, *Commun. Nonlin. Sci. Numer. Simul.* **44**, 460-481 (2017).
[8] H. Khan, C. Tunç, D. Baleanu, A. Khan and A. Alkhazzan, Inequalities for n-class of functions using the Saigo fractional integral operator *RACSAM*, **113**(2407) (2019).
[9] M. Z. Sarikaya, Gronwall type inequality for conformable fractional integral, *Konuralp J. Math.* **4**(2), 217-222 (2016).
[10] H. Ye, J. Gao and Y. Ding, A generalized Gronwall inequality and its application to a fractional differential equation, *J. Math. Anal. Appl.* **328**, 1075-1081 (2007).
[11] S. Jain, K. Mehrez, D. Baleanu and P. Agarwal, Certain Hermite–Hadamard inequalities for logarithmically convex functions with applications *Mathematics* **7**(2), 163(2019).
[12] D. Nie, S Rashid, A. Akdemir, D. Baleanu and J. Liu, On some new weighted inequalities for differentiable exponentially convex and exponentially quasi-convex functions with applications *Mathematics* **7**(8), 727(2019).
[13] A. Ahamadkhanlu and M. Jahanshahi, On the existence and uniqueness of solution of initial value problem for fractional order differential equations on time scales, *Bull. Iran. Math. Soc.* **38**(1), 241-252 (2012).
[14] N. Benkhettou, A. Hammoudi and D. Torres, Existence and uniqueness of solution for a fractional Riemann- Liouville initial value problem on time scales, *J. King Saud Univ. Sci.* **28**(1), 87-92 (2016).
[15] R. A. Yan, S. R. Sun and Z. L. Han Existence of solutions of boundary value problems for Caputo fractional differential equations on time scales, *Bull. Iran. Math. Soc.* **42**(2), 247-262 (2016).
[16] J. Zhu and Y. Zhu, Fractional Cauchy problem with Riemann-Liouville fractional delta derivative on time scales, *Abstr. Appl. Anal.* **2013** Art. Id 401596, 1-19 (2013).
[17] J. Zhu and L. Wu, Fractional Cauchy problem with Caputo nabla derivative on time scales, *Abstr. Appl. Anal.* **2015**, Art. Id 486054, 1-23 (2015).
[18] M. M. A. Khater, R. A. M. Attia, A. Abdel-Aty, M. A. Abdou, H. Eleuch, D. Lu, Analytical and semi-analytical ample solutions of the higher-order nonlinear Schrödinger equation with the non-Kerr nonlinear term, *Results in Physics* **16**, 103000 (2020).
[19] X. Zhang and C. Zhu, Cauchy problem for a class of fractional differential equations on time scales, *Int. J. Comput. Math.* **91**(3), 527–538 (2014).
[20] G. Anastassiou, *Frontiers in time scales and inequalities*, World Scientific Publishing Company, (2015).
[21] G. Anastassiou, Principle of delta fractional calculus on time scales and inequalities, *Math. Comput. Model.* **52**, 556-566 (2010).
[22] D. Baleanu, K. Diethelm, E. Scalas and J. Trujillo, *Fractional calculus models and numerical methods*, *World Scientific Publishing Company*, (2016).
[23] T. Kulik and C. C. Tisdell, Volterra integral equations on time scales: basic qualitative and quantitative results with applications to initial value problems on unbounded domains, *Int. J. Differ. Equ.* **3**(1), 103-133 (2008).