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Impulsive Fractional Differential Equation of Order $\alpha \in (2,3)$

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Abstract: In this paper, we study the uniqueness solution for impulsive fractional differential equation of order $\alpha \in (2,3)$ existence by using Banach fixed point theorem, Schauder fixed point theorem and we present an example to illustrate the uniqueness result.

Keywords: Fractional order differential equation, fractional differential equations, impulsive conditions, fixed point theorem.

1 Introduction

Differential equations study with varying impulsive effects can be narrated with a contrasting description that will talk about evolutionary processes that are nature-inspired. The varying detailed process about relevant development in fractional calculus and functional differential equations with a state-dependent delay has been studied thoroughly in the references [1–12]. Feckan et al. [7] talk about a counter-example to show the importance of error in the formula of the solution to the impulsive Cauchy problems for varying differential equations with fractional order $q \in (0,1)$. Feckan et al. [7] correctly found that method and establish existence by using a fixed point theorem. Wang et.al [8] established an agreeable conditional context for the continuance of the solutions by applying fixed point theorem first for linear and nonlinear impulsive conditions. Recently, Liu et al. [13] studied and talked about the existence and uniqueness of solutions for nonlinear singular multi-term impulsive fractional differential equations. The present work on which we are working was motivated by the papers [7, 8, 13]. This research work will talk about the study of linear impulsive fractional functional integro differential equations:

$$D_t^{\alpha} y(t) = J_t^{3-\alpha} f(t, y_t), t \in J = [0,T], t \neq k, \tag{1}$$

$$\triangle y(t_k) = x_k, \triangle y'(t_k) = z_k, \triangle y''(t_k) = p_k; k = 1,2,3,4,....,m, \tag{2}$$

$$y(t) = \phi(t), y'(t) = \varphi(t), y''(t) = \eta(t); t \in [-d,0], \tag{3}$$

where y' denotes the first order derivative and y'' denotes the second order derivative of y with respect to t and D_t^{α} is denoted Caputo's derivative of order $\alpha \in (2,3)$. $f : J \times PC_0 \longrightarrow X$, is given continuous function and PC_0 is a abstract phase space with yt be the element of PC_0 defined by $yt(\theta) = y(t+\theta), \theta \in [-d;0]$.

We have impulsive point $0 = t_0 < t_1 < .. < t_m < t_{m+1} = T$, and $x_k; z_k; p_k \in R$.

The presented research work is concerned with the existence result for impulsive fractional functional integro differential equations with varying state-dependent delay subject to certain initial conditions. To the best of the author's knowledge, this work is a new state of the art. The state of artwork has been divided into four sections. First part talks about the introduction and the second part talks about some percussive preliminaries. The section three presents the main result of this manuscript and the section fourth contains an illustrative example.

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2 Preliminaries

Let $(X;\|.\|x)$ be a complex Banach space of function with the norm. To circumvent the repetitive iterations, of some definitions, we will be adopting some basic preliminaries from [14] such as Riemann-Liouville fractional operator, Caputo's derivative, phase space $PC_0;PC_0^{'}$ and others preliminary.

Definition 21*Caputo's derivative of order $\alpha > 0$ with lower limit a, for a function $f:[a;\infty] \to R$ such that $f \in C^n([a;\infty];X)$, is defined as*

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^n(s)\, ds = J_t^{n-\alpha} f^n(s), t > a, \tag{4}$$

where $a \geq 0; n-1 < \alpha < n; n \in N$.

Definition 22*The Riemann- Liouville fractional integral operator of order $\alpha > 0$ with lower limit a, for a continuous function $f:[a;\infty) \to R$ is defined by*

$$_aJ_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds, t > a, \tag{5}$$

where $a \geq 0$ and $\Gamma(.)$ is Gamma function.

Lemma 1.*A piecewise continuous differential function $y(t):[-d,T] \to X$ is a solution of the system (1)-(3) if and only if it satisfied the following integral equation*

$$y(t) = \begin{cases} \phi(0) + \varphi(0)t + \eta(0)\frac{t^2}{2} + \int_0^t (t-s) f(s,y_t)\, ds, \ t \in [0,t_1], \\ \phi(0) + \varphi(0)t + \eta(0)\frac{(t-t_1)^2}{2} + \sum_{0<t_k<t} x_k + \\ \sum_{0<t_k<t} (t-t_k) z_k + \sum_{0<t_k<t} \frac{(t-t_k)^2}{2} E_k + \\ \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s) f(s,y_t)\, ds + \\ \sum_{0<t_k<t} (t-t_k) \int_{t_{k-1}}^{t_k} f(s,y_t)\, ds + \\ \sum_{0<t_k<t} \frac{(t-t_k)^2}{2} \int_{t_{k-1}}^{t_k} f(s,y_t)\, ds + \\ \int_{t_k}^t (t-s) f(s,y_t)\, ds \qquad t \in [t_k, t_{k+1}]. \end{cases} \tag{6}$$

Proof: If $t \in (0,t_1]$ then by the standard procedure the solution of (1). We get,

$$y(t) = a_0 + b_0 t + \frac{c_0 t^2}{2} + \int_0^t (t-s)^2 f(s,y_t) ds \tag{7}$$

using initial condition $y(0) = \phi(0), wegeta_0 = \phi(0)$, then (7) become

$$y(t) = \phi(0) + b_0 t + \frac{c_0 t^2}{2} + \int_0^t (t-s) f(s,y_t) ds. \tag{8}$$

On differentiating (6) with respect to t and by initial condition $y'(0) = \varphi(0)$, we get $b_0 = \varphi(0)$, then the equation (8) become

$$\phi(0) + \varphi_0 t + \frac{c_0 t^2}{2} + \int_0^t (t-s) f(s,y_t) ds \tag{9}$$

Now again differentiating equation (6) with respect to t and by initial condition $y"(0) = \eta(0)$

then equation (9) become

$$\phi(0) + \varphi_0 t + \frac{\eta(0) t^2}{2} + \int_0^t (t-s) f(s,y_t) ds. \tag{10}$$

.

If $t \in (t_1,t_2)$, then the solution of equation (1),we have

$$y(t) = a_1 + b_1(t-t_1) + \frac{c_1(t-t_1)^2}{2} + \int_{t_1}^t (t-s) f(s,y_t) ds. \tag{11}$$

. By impulsive condition $y(t_1^+)-y(t_1^-)=x_1$, the equation (11)written as $y(t_1^+)-y(t_1^-)=a_1$. Similarly by impulsive condition $y(t_1^+)=x_1$, the equation (8)written as

$$y(t_1^-)=\phi(0)+\varphi_0 t+\frac{\eta(0)t^2}{2}+\int_0^t (t-s)f(s,y_t)ds \tag{12}$$

from equation (11) and (12), we get

$$a_1=\phi(0)+\varphi_0 t+\frac{\eta(0)t^2}{2}+x_1+\int_0^t (t_1-s)f(s,y_t)ds. \tag{13}$$

Hence equation (9) can we written as

$$y(t)=\phi(0)+\varphi_0 t+\frac{\eta(0)t^2}{2}+x_1+b_1(t-t_1)+\frac{c_1(t-t_1)^2}{2}+\int_0^t (t_1-s)f(s,y_t)ds+\int_{t_1}^t (t-s)f(s,y_t)ds. \tag{14}$$

On differentiating equation (13) with respect to t and by impulsive condition $y'(t_1^+)-y'(t_1^-)=z_1$ we get $y'(t_1^+)=b_1$ on differentiating equation (6) with respect to t and by impulsive condition $y'(t_1^+)-y'(t_1^-)=z_1$, we get

$$y'(t_1^-)=\varphi(0)+\eta(0)t+\int_0^{t_1} f(s,y_t)ds. \tag{15}$$

From equations (14) and (15), we get

$$b_1=\varphi(0)+\eta(0)t+z_1+\int_0^{t_1} f(s,y_t)ds. \tag{16}$$

Hence equation (13) can be written as

$$\begin{aligned} y(t)=\phi(0)+\varphi_0 t_1+\frac{\eta(0)t_1^2}{2}+x_1+z_1(t-t_1)+\frac{c_1(t-t_1)^2}{2}+ \\ (t-t_1)\int_0^{t_1} f(s,y_t)ds)+\int_0^t (t_1-s)f(s,y_t)ds+\int_{t_1}^t (t-s)f(s,y_t)ds. \end{aligned} \tag{17}$$

Now double differentiating equation (13) with respect to t and by impulsive condition $y''(t_1^+)-y''(t_1^-)=E_1$, we get $y''(t_1^+)=C_1$. On double differentiating equation (6) with respect to t and by impulsive condition $y''(t_1^+)-y''(t_1^-)=E_1$, we get

$$y''(t_1^+)=\eta(0)+\int_0^{t_1} f(s,y_t)ds. \tag{18}$$

. From equations (17) and (18)

$$C_1=E_1+\eta(0)+\int_0^{t_1} f(s,y_t)ds \tag{19}$$

hence equation (17) become

$$\begin{aligned} y(t)=\phi(0)+\varphi_0 t+\frac{\eta(0)(t-t_1)^2}{2}+x_1+z_1(t-t_1)+\frac{E_1(t-t_1)}{2}+\int_{t_1}^t (t-s)f(s,y_t)ds+ \\ (t-t_1)\int_0^{t_1} f(s,y_t)ds)+\frac{(t-t_1)^2}{2}\int_0^t f(s,y_t)ds+\int_{t_1}^t (t-s)f(s,y_t)ds. \end{aligned} \tag{20}$$

Repeating the process in this way,the solution $y(t)$ for$t\in(t_k,t_{k+1}]$ can be written as

$$\begin{aligned} y(t)=\phi(0)+\varphi_0 t+\frac{\eta(0)(t-t_1)^2}{2}+\sum_{0<t_{k<t}} x_k+\sum_{0<t_{k<t}} z_k(t-t_k)+\sum_{0<t_{k<t}}\frac{E_k(t-t_k)}{2}+ \\ \sum_{0<t_{k<t}}\int_{t_{k-1}}^{t_k}(t_k-s)f(s,y_t)ds+\sum_{0<t_{k<t}}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)f(s,y_t)ds)+ \\ \sum_{0<t_{k<t}}\frac{(t-t_k)^2}{2}\int_{t_{k-1}}^{t_k} f(s,y_t)ds+\int_{t_k}^t (t-s)f(s,y_t)ds. \end{aligned} \tag{21}$$

Summarizing the result (21), it is clear that the solution given in (21) satisfies the system (1)-(3). This completes the proof of the lemma.

3 Existence result

To prove the result we have obtained, we shall assume the function $\rho : [0,T]_X PC_0 \to [-d,T]$ is continuous for the expected informative analysis. The following assumptions need to be kept in mind

1.$(k_1) f : JXPC_0 \to X$ is jointly continuous function and there exist positive constant B_{f_1} such that

$$f(t_1,\tau) - f(t,\zeta)||_X \leq Bf_1||\tau - \zeta||PC_0.$$

2.$(K_2 f)$ is continuous and there exist positive constant N_1such that $||f(t_1,\tau)|| \leq N_1$ for every $\tau \in PC_0$.

By following the existence result, now it seems that there is no doubt to state the existence theorem based on the contraction principle.

Theorem 1.*Suppose that the assumption (K_1) satisfied and $\delta = \frac{T^2}{2}(2+2n+nT)B_{f_1} < 1$. Then the problem (1) to (3) has a unique solution on J.*

Proof: Consider the space $PC_0'' = y \in PC_0'' : y(0) = \phi(0)$ and $y(t) = \phi(t), t \in [-d,0]$,

$$P_y t = \phi(0) + \varphi_0 t + \frac{\eta(0)(t-t_1)^2}{2} + \int_0^t (t-s) f(s,y_t) ds, t \in [0,t], \tag{22}$$

$$\begin{aligned}\phi(0) + \varphi_0 t + \frac{\eta(0)(t-t_1)^2}{2} + \sum_{0<t_{k<t}} x_k + \sum_{0<t_{k<t}} z_k(t-t_k) + \sum_{0<t_{k<t}} (t-t_k) \int_{t_{k-1}}^{t_k} f(s,y_t)ds) + \\ \sum_{0<t_{k<t}} \int_{t_{k-1}}^{t_k} (t_k - s) f(s,y_t)ds + \sum_{0<t_{k<t}} \frac{(t-t_k)^2}{2} \int_{t_{k-1}}^{t_k} f(s,y_t)ds + \int_{t_k}^t (t-s) f(s,y_t)ds.\end{aligned} \tag{23}$$

To show this , Let us consider $y, y^* \in PC_0''$ for $t \in [0,t_1]$ then

$$||P_y t - P_{y^*}(t)||_\times \leq \int_0^t (t-s)||f(s,y_t) - f(s,y_t^*)||_\times ds \leq \frac{T^2}{2}(B_{f_1})||y-y^*||_\times \tag{24}$$

for $t \in (t_k, t_{k+1})$ we have

$$\begin{aligned}||P_y t - P_{y^*}(t)||_\times \leq \sum_{0<t_{k<t}} \int_{t_{k-1}}^{t_k} (t_k - s)||f(s,y_t) - f(s,y_t^*)||ds \\ + \sum_{0<t_{k<t}} (t-t_k) \int_{t_{k-1}}^{t_k} ||f(s,y_t) - f(s,y_t^*)||ds + \sum_{0<t_{k<t}} \frac{(t-t_k)^2}{2} \int_{t_{k-1}}^{t_k} ||f(s,y_t) - f(s,y_t^*)||ds \\ + \int_{t_k}^t (t-s)||f(s,y_t) - f(s,y_t^*)|| \\ \leq \frac{T^2}{2}((B_{f_1} + nT^2(B_{f_1}) + \frac{nT^3}{2}(B_{f_1}) + \frac{T^2}{2}(B_{f_1})||y-y^*||_\times \\ \leq T^2((B_{f_1}) + nT^2(B_{f_1}) + \frac{nT^3}{2}(B_{f_1}))||y-y^*||_\times \\ \leq \frac{T^2}{2}(2+2n+nT)||y-y^*||_\times(B_{f_1})\end{aligned} \tag{25}$$

for all $t \in [0,T]$

$$||P_y(t) - P_{y^*}(t)||_\times \leq \delta||y-y^*||_\times \tag{26}$$

since $\delta < 1$, its implies that P is contraction mapping and P has a unique fixed point $y \in PC_0''$.

It means that the system (1) -(3) has a unique solution. This complete the proof of the theorem.

Theorem 2.*If the assumption K_2 hold then problem (1) -(3) have at least one solution.*

Proof: Now we show that P is continuous, for this purpose, we consider a sequence $y^n \to y$,then

$$\begin{aligned}||P_{y^n}-P_y||_\times \leq \sum(t-t_1)\int_0^t (f(s,y_t^n)-(f(s,y_t))ds+\sum\int_0^t (t_1-s)(f(s,y_t^n)-(f(s,y_t))ds \\ +\frac{(t-t_1)^2}{2}\int_0^t (f(s,y_t^n)-(f(s,y_t))ds+\int_0^t (t-s)(f(s,y_t^n)-(f(s,y_t))ds.\end{aligned} \tag{27}$$

Since the function f is continuous so $||P_{y^n}-P_y||_\times \to 0$ as $n\to\infty$, implies P is continuous.

(ii) To confirm P-map bounded set into the bounded set. We have provided

$$\begin{aligned}||P_y||_\times \leq (t-t_1)\int_0^t ||f(t,y_s)||ds+\int_0^t ||t-s||.||f(t,y_s)||ds+||\frac{(t-t_1)^2}{2}||\int_0^t ||f(s,y_t)||ds+ \\ \int_0^t ||t-s||.||f(t,y_s)||ds \leq NT+\frac{T^2}{2}N+\frac{T^2}{2}N+\frac{T^2}{2}N+\leq NT+\frac{3T^2}{2}N=K(Constant).\end{aligned} \tag{28}$$

This condition is for bounded set.

(iii) Next, we shall show that P is family of equicontinous functions.
Let $l_1,l_2\in[0,T]$ such that $0\leq l_1\leq l_2\leq T$. Then

$$\begin{aligned}||P_y(l_2)-P_y(l_1)||_\times \leq \varphi(0)(l_2-l_1)+\frac{\eta(0)}{2}((l_2-t_1)^2-(l_1-t_1)^2)+ \\ \sum_{0<t_k<t}(l_2-t_k)-(l_1-t_k)z_k+\sum_{0<t_k<t}E_k[(l_2-t_k)^2-(l_1-t_k)^2]+ \\ \sum_{0<t_k<t}(l_2-t_k)-(l_1-t_k)\int_{t_{k-1}}^{t_k} f(s,y_t)ds+\sum_{0<t_k<t}[\frac{(l_2-t_k)^2}{2}- \\ \frac{(l_1-t_k)^2}{2}]\int_{t_{k-1}}^{t_k} f(s,y_t)ds+\int_{t_k}^{l_2}[(l_2-s)-(l_1-s)]f(s,y_t)ds \\ -\int_{t_k}^{l_1}[(l_2-s)-(l_1-s)]f(s,y_t)ds \leq \varphi(0)(l_2-l_1)+\frac{\eta(0)}{2}[(l_2^2-l_1^2)-2t_1(l_2-l_1)] \\ +\sum_{0<t_k<t}(l_2-l_1)z_k+\sum_{0<t_k<t}\frac{E_k}{2}[(l_2^2-l_1^2)-2t_k(l_2-l_1)]+\sum_{0<t_k<t}(l_2-l_1) \\ \int_{t_{k-1}}^{t_k} f(s,y_t)ds+\sum_{0<t_k<t}\frac{(l_2^2-l_1^2)-2t_k(l_2-l_1)}{2}\int_{t_{k-1}}^{t_k} f(s,y_t)ds+ \\ \int_{l_1}^{l_2}(l_2-l_1)f(s,y_t)ds.\end{aligned} \tag{29}$$

if $l_2\to l_1$ Then $||P_y(l_2)-P_y(l_1)||\to 0$.
So we can conclude by Arzela Ascoli's theorem that P is a family of equicontinous map. Finally, it follows from the Schauder's fixed point theorem.

4 Example

Let us talk about the following non- linear impulsive fractional functional integral boundary value problem.

$$D_t^{\frac{5}{2}}(y(t))=\frac{1}{\Gamma(3-\alpha)}\int_0^t \frac{(t-s)^{1-\alpha}.e^{-t}y(1-\sigma(||y||))}{(16+e^t)(1+y(t-\sigma(||y||)))}, \tag{30}$$

$$y(t)=\phi(t),y'(t)=\varphi(t),y''(t)=\zeta(t)t\in[-d,0], \tag{31}$$

$$\Delta_y|_{t=\frac{1}{2}} = \frac{x_{\frac{1}{2}}}{7}, \Delta_{y'}|_{t=\frac{1}{2}} = \frac{z_{\frac{1}{2}}}{16}, \Delta_{y"}|_{t=\frac{1}{2}} = \frac{p_{\frac{1}{2}}}{25}. \tag{32}$$

For the phase space, by setting $f(t,\phi) = \frac{e^{-t}.\phi}{(16+e^t)(1+\phi)}$.
The equations (30) to (32) depicted above can be written in the abstract form as (1) -(3). With this, we can easily verify all assumptions (K_1) and (K_2). Therefore, system (30) to (32) has uniqueness or at least one solution.

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