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Study of Memory Effect in an Economic Order Quantity Model for Completely Backlogged Demand During Shortage

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Abstract: The most commonly developed inventory models are the classical economic order quantity model, is governed by the integer order differential equations. We want to come out from the traditional thought i.e. classical order inventory model where the memory phenomena are absent. Here, we want to incorporate the memory effect that is based on the fact economic agents remember the history of changes of exogenous and endogenous variables. In this paper, we have proposed and solved a fractional order economic order quantity model with constant demand rate where the demand is fully backlogged during shortage time. Finally, a numerical example has been illustrated for this model to show the memory dependency of the system. The numerical example clears that for the considered system the profit is maximum in long memory affected system compared to the low memory affected or memory less system.

Keywords: Fractional order derivative, long memory effect and short memory effect, fractional order inventory model.

1 Introduction

Recently, fractional order integration and differentiation have been applied to the real world problem for its memory property. But in this system growth of any processes is slower compared to the ordinary differential system. Fractional order integration and differentiation are generalizations of integer-order integration and differentiation with including nth derivatives and fractional n-fold integrals[1,2]. Usually, authors use Riemann-Liouville (RL) integration and Caputo fractional order derivative to develop different problems [3,4,5,6,7,8,9,1]. Recently Baleanu and his collaborators developed and used the new non-local fractional derivative with non-singular kernel in different physical problems[10, 11,12]. The fractional order derivative has different type of significance in different applications. In physical problem it is used as the roughness parameter of the surface [13], in biological and financial system it is the indicator of memory [3, 4,5,6,7,8,9,1] etc.

The memory means the dependence of the process not only on the current state of the processes but also on the past history of the process. Like the application in physics and biology it is recently using in economic analysis[4,5,6]. However, there are some areas of operation research where the application of fractional calculus has been started to apply in the last few years. Our main interest is to include the application fractional calculus in the inventory models.

It is well known that rate of change of integer orders of a function at any particular point is determined by the property of the function in the infinitely small neighborhood of the considered point. Hence, the integer order rate of change of any function/system is assumed as the instant rate of change of the marginal output, when the input level changes. Therefore, the dynamic memory effect is not present in classical calculus. Thus the classical calculus is not able to include the previous state of the system [8,9]. In the fractional derivative, the rate of change is affected by all points of

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the considered interval, so it is able to incorporate the previous /memory effects of any system. Here, the order of fractional derivative will be treated as an index of memory. So, the fractional order system can remove amnesia from any system [14].

Application of fractional order derivative is suitable to formulate and analyze the real-life phenomenon which has memory effects. M. Saedian et al. studied fractional order susceptible-infected-recovered (SIR) epidemic model to investigate the existence of memory effect in the biological system and to incorporate this effect they have used the memory kernel function[3]. They have established in their paper[3] that the real epidemic process is clearly sustained by a non-Markovian dynamics.

Tarasova et al. have developed many research articles[4,5,6,7] using the concept of the memory effect of fractional order derivative and integration. In the paper[6] they gave an idea of economic interpretation of fractional derivatives using the Caputo derivative. In [7] they discussed elasticity for the economic process with memory using fractional differential calculus. They defined that generalization of point price elasticity of demand to the case of the processes with memory. In these generalizations, they take into account dependence of demand not only from the current price (price at the current time) but also changes of price for some time interval.

Pakhira et al. [8,9,15,16,17,18] developed some memory dependent inventory models including the fractional order rate of change of the inventory level and fractional order effect of different costs. The inclusion of the memory effect in the inventory model is necessary to handle practical business policy. In this paper, our aim is to develop a memory dependent economic order quantity model where demand is completely backlogged during shortage[19]. Here, Caputo fractional order derivative and Riemann-Liouville fractional order integration have been used to develop the economic order quantity model where fully backlogged is permitted during shortage time. To formulate the fractional order differential equation model we have used the memory kernel concept as develop in [3]. Here different costs are established using fractional order integration. Finally the effects of inclusion of memory parameter are investigated using numerical examples.

Our analysis clears that when the integral memory index is absent but the differential memory index is present, the minimized total average cost is gradually decreasing with gradually increasing memory effect but optimal ordering interval is gradually increasing. The numerical value of the minimized total average cost with the presence of the differential memory index is low compared with presence of the integral memory index. It is clear from the numerical results that total order quantity needs to be adjusted for short past experience effect but not for long experience.

The rest part of the paper is organized as follows: in the section 2, review of fractional calculus has been presented. In the section 3, model formulation has been discussed. In the section-3.4 fractional order inventory model formulation with memory kernel has been given. In the section 3.5 fractional order model Analysis has been presented. In the section 4, numerical examples are given and finally in the section 5, some conclusions are cited.

2 Review of fractional calculus

2.1 Euler gamma function

Euler's gamma function is one of the best tools in fractional calculus which was proposed by the Swiss mathematicians Leonhard Euler (1707-1783).The gamma function is continuous extension from the factorial notation. The gamma function is denoted and defined by the formulae.

$$\Gamma(x) = \int_0^{\propto} t^{x-1} e^{-t} dt \qquad (1)$$

where $x > 0$ $\Gamma(x)$ is extended for all real and complex numbers. It has the basic properties[1].
$\Gamma(x+1) = \Gamma(x)$,$\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$,$\Gamma(-\frac{7}{6}) = -\frac{6}{7}\left(\Gamma\left(\frac{1}{6}\right)\right)$

2.2 Riemann-Liouville fractional derivative[1]

Left Riemann-Liouville fractional derivative of any continuous function $f(x)$ of order α where $\alpha \in (0,1]$ is defined as

$$ {}_aD_x^{\alpha}(f(x)) = \frac{1}{\Gamma(m-\alpha)}\left(\frac{d}{dx}\right)^m \int_a^x (x-\tau)^{m-\alpha-1} f(\tau)d\tau \tag{2} $$

where $x > a$
Right Riemann-Liouville fractional derivative of order is defined as follows

$$ {}_xD_b^{\alpha}(f(x)) = \frac{1}{\Gamma(m-\alpha)}\left(-\frac{d}{dx}\right)^m \int_x^b (\tau-x)^{m-\alpha-1} f(\tau)d\tau \tag{3} $$

where $x > 0$

In terms of Riemman-Liouville definition the fractional derivative any constant is not equal to zero which creates a difficulty between ordinary calculus and fractional calculus. To overcome this difference M. Caputo [20] gave the new definition which is given below.

2.3 Caputo fractional order derivative

Left Caputo fractional derivative [20] for the function $f(x)$ which has continuous, bounded derivatives in$[a,b]$ is denoted and defined as follows

$$ {}_a^CD_x^{\alpha}(f(x)) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-\tau)^{m-\alpha-1} f^m(\tau)d\tau \tag{4} $$

where $0 \le m-1 < \alpha < m$.
Right Caputo fractional derivative for the function $f(x)$ which has continuous and bounded derivatives in $[a,b]$ is defined as follows

$$ {}_x^CD_b^{\alpha}(f(x)) = \frac{1}{\Gamma(m-\alpha)} \int_x^b (\tau-x)^{m-\alpha-1} f^m(\tau)d\tau \tag{5} $$

where $0 \le m-1 < \alpha < m$. In terms of Caputo definition the derivative of any constant A is zero i.e.${}_a^CD_x^{\alpha}(A) = 0$.

2.4 Fractional Laplace transforms method

The Laplace transform [1] of the function $f(x)$ is defined as

$$ F(s) = L(f(t)) = \int_0^{\infty} e^{-st} f(t)dt \tag{6} $$

where $s > 0$, is called the transform parameter. The Laplace transformation [1] of n^{th} order derivative is defined as

$$ L(f^n(t)) = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^k(0) \tag{7} $$

where $f^n(t)$denotesn^{th}derivative of the function $f$with respect to t and for non – integer m it is defined in generalized form [1,2]as,

$$ L(f^{\alpha}(t)) = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^k f^{\alpha-k-1}(0) \tag{8} $$

where,$(n-1) < \alpha \le n$

Table 1: Different symbols and items for the EOQ models

(i)$D(t)$: Demand rate.	(ii)Q :Total order quantity.
(iii)P :Per unit cost.	(iv)C_1t^{α} :Inventory holding cost per unit.
(v)C_3 :Ordering cost or setup cost per order.	(vi)$I(t)$:Stock level or inventory level.
(vii)T :Ordering interval.	(viii)$HOC_{\alpha,\beta}(T)$:Inventory holding cost with fractional effect.
(ix)$T^*_{\alpha,\beta}$:Optimal ordering interval with fractional effect.	(x)$TOC^{av}_{\alpha,\beta}$:Total average cost during the total time interval.
(xi)$TOC^*_{\alpha,\beta}$:Minimized total average cost with fractional effect.	(xii)$(B,.),(\Gamma,.)$:Beta function and gamma function respectively.
(xiii)$SOC_{\alpha,\beta}(T)$:Shortage cost with fractional effect.	(xiv)$POC_{\alpha,\beta}(T)$:Total purchasing cost with fractional effect.
(xv)C_2 :Shortage cost per unit per unit time.	

3 Model formulation

In this paper, the classical and fractional order EOQ models are formulated on the basis of the following assumptions:
(i)Time horizon is infinite.
(ii)Lead time is zero.
(iii)Demand rate is throughout the interval i.e. in $0 \leq t \leq t_1$ and $t_1 \leq t \leq T$ both interval.
(iv)Shortage is allowed in this model.
(v)There is permitted complete backlogging during shortage time.

3.1 Notations

The notations are used to develop the model are given in Table-1.

3.2 Classical inventory model

Here the positive inventory level $I_1(t)$ and the negative inventory level $I_2(t)$ both reduce due to constant demand γ during the time interval $[0,t_1]$ and $[t_1,T]$ respectively. The classical or memory less inventory system with constant demand can be governed by the following system of integer order differential equations :

$$\frac{d(I_1(t))}{dt} = -\gamma \tag{9}$$

for $0 \leq t \leq t_1$.

$$\frac{d(I_2(t))}{dt} = -\gamma \tag{10}$$

for $t_1 \leq t \leq T$
with $I_1(t_1) = 0, I_2(t_1) = 0$

The classical model is followed from [19,21]. In the next section we shall establish the fractional order inventory control model using the memory kernel method [3].

3.3 Fractional order inventory model formulation with memory kernel

To establish the influence of memory effects, the differential equation (9-10) can be written using the kernel functions in the following form:

$$\frac{d(I_1(t))}{dt} = -\int_0^t K(t-t')\gamma dt' \tag{11}$$

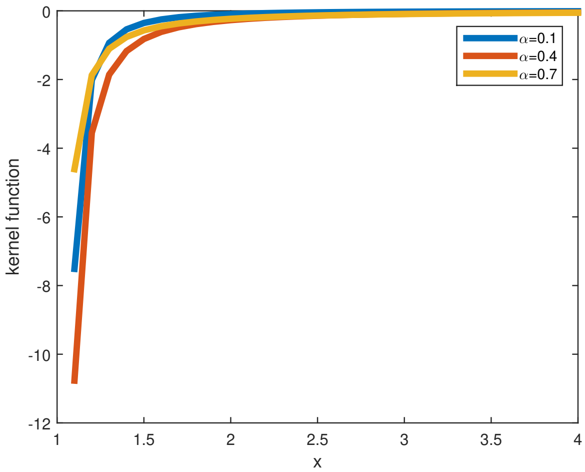

Fig. 1: Plot of memory kernel function $k(x-\xi) = \frac{(x-\xi)^{\alpha-2}}{\Gamma(\alpha-1)}$ for$\xi = 3$ different values ofα .

$$\frac{d(I_2(t))}{dt} = -\int_0^t K(t-t')\gamma dt' \tag{12}$$

in which $k(t-t')$ is the kernel function. This type of kernel guarantees the existence of scaling features as it is often intrinsic in most natural phenomena. Thus, to generate the fractional order $k(t-t') = \frac{(t-t')^{\alpha-2}}{\Gamma(\alpha-1)}$ where $0 < \alpha \leq 1$ and $\Gamma(\alpha)$ denotes the gamma function.

Using the definition of fractional derivative[1,2] we can re-write the Equation(11-12) to the form of fractional differential equations with the Caputo-type derivative in the following form

$$\frac{d(I_1(t))}{dt} = -{}_0D_t^{-(\alpha-1)}(\gamma) \tag{13}$$

$$\frac{d(I_2(t))}{dt} = -{}_0D_t^{-(\alpha-1)}(\gamma) \tag{14}$$

Now, applying fractional Caputo derivative of order$(\alpha-1)$ on both sides of the(13-14) and using the fact the Caputo fractional order derivative and fractional order integral are inverse operators, the following fractional differential equations can be obtained for the model

$${}_0^CD_t^{\alpha}(I_1(t)) = -\gamma \tag{15}$$

$${}_0^CD_t^{\alpha}(I_2(t)) = -\gamma \tag{16}$$

along with boundary condition $I_1(t_1) = 0, I_2(t_1) = 0$ Here, α controls the strength of memory. when $\alpha \to 1$, memory of the system becomes weak and the small value of α (close to 0.1) indicates long memory of the system.

It is clear from the figure that the pick of the curve gradually decreases depending on α. Here, we define memory effect in two steps(i) long memory effect,(ii) low memory effect.

Long and Short memory effect:

The strength of memory is controlled by the order of fractional derivative or fractional integration.If order of fractional derivative or fractional integrationis in the range $(0, 0.5)$ then the system has long memory effect and in$[0.5, 1)$ then is short memory effect.

3.4 Analysis of the fractional order economic order quantity model

Fractional order inventory model is governed by the two fractional order differential equation as follows

$$\frac{d^{\alpha}(I_1(t))}{dt^{\alpha}} = -\gamma \tag{17}$$

for $0 \le t \le t_1$, with $I_1(t_1) = 0$

$$\frac{d^{\alpha}(I_2(t))}{dt^{\alpha}} = -\gamma \tag{18}$$

for $t_1 \le t \le T$, with $I_2(t_1) = 0$

The inventory level can be found integrating the system (17-18) and using the boundary condition which gives

$$I_1(t) = \frac{\gamma}{\Gamma(1+\alpha)}\left(t_1^{\alpha} - t^{\alpha}\right) \tag{19}$$

$$I_2(t) = \frac{\gamma}{\Gamma(1+\alpha)}\left(t_1^{\alpha} - t^{\alpha}\right) \tag{20}$$

Since the inventory level decreases with respect to time t, so the maximum positive inventory level will occur at $t = 0$ which gives

$$M = I_1(0) = \frac{\gamma t_1^{\alpha}}{\Gamma(1+\alpha)} \tag{21}$$

Here, the maximum backorder units are

$$S = -I_2(T) = \frac{\gamma}{\Gamma(1+\alpha)}\left(T^{\alpha} - t_1^{\alpha}\right) \tag{22}$$

The order size during the total ordering interval $[0,T]$, is denoted by Q and defined as

$$Q = M + S = \frac{\gamma T^{\alpha}}{\Gamma(1+\alpha)} \tag{23}$$

In reality, holding cost depends on time. It is not constant in the entire cycle of the system. Due to that reason, the inventory holding cost per unit is assumed as a function of time in the form $C_1 t^{\alpha}$.

The inventory holding cost with memory effect is denoted by $HOC_{\alpha,\beta}$[8] and defined as

$$HOC_{\alpha,\beta}(T) = \frac{C_1}{\Gamma(\beta)}\int_0^{t_1}(t_1-t)^{\beta-1}t^{\alpha}I_1(t)\,dt = \frac{C_1\gamma t_1^{2\alpha+\beta}\left(B(\alpha+1,\beta) - B(2\alpha+1,\beta)\right)}{\Gamma(\beta)\Gamma(\alpha+1)} \tag{24}$$

β is considered as integral memory index.

Shortage cost with fractional effect is denoted by$SOC_{\alpha,\beta}$ and defined as follows

$$\begin{aligned} SOC_{\alpha,\beta}(T) &= -\frac{C_2}{\Gamma(\beta)}\int_{t_1}^{T}(T-t)^{\beta-1}I_2(t)\,dt \\ &= \frac{C_2\gamma T^{\alpha+\beta}}{\Gamma(\beta)\Gamma(\alpha+1)}\left(\frac{1}{\alpha+1} - \frac{\beta-1}{\alpha+2}\right) + \left(\frac{C_2\gamma t_1^{\alpha+1}}{\Gamma(\beta)\Gamma(\alpha+1)} - \frac{C_2\gamma t_1^{\alpha+1}}{\Gamma(\beta)(\alpha+1)\Gamma(\alpha+1)}\right)T^{\beta-1} \\ &+ \left(\frac{C_2\gamma t_1^{\alpha+2}(\beta-1)}{\Gamma(\beta)(\alpha+2)\Gamma(\alpha+1)} - \frac{C_2(\beta-1)\gamma t_1^{\alpha+2}}{2\Gamma(\beta)\Gamma(\alpha+1)}\right)T^{\beta-2} + \left(\frac{C_2(\beta-1)\gamma t_1^{\alpha}}{\Gamma(\beta)2\Gamma(\alpha+1)} - \frac{C_2\gamma t_1^{\alpha}}{\Gamma(\beta)\Gamma(\alpha+1)}\right)T^{\beta} \end{aligned} \tag{25}$$

where C_2 is the shortage cost per unit per unit time.

The purchasing cost for fractional order model is denoted by $POC_{\alpha,\beta}$ and defined as

$$POC_{\alpha,\beta}(T) = PXQ = P\left(\frac{\gamma T^{\alpha}}{\Gamma(\alpha+1)}\right) \tag{26}$$

Therefore, total average cost for the fractional order inventory model is

$$TOC^{av}_{\alpha,\beta}(T) = \frac{HOC_{\alpha,\beta}(T) + SOC_{\alpha,\beta}(T) + POC_{\alpha,\beta}(T) + C_3}{T}$$
$$= AT^{\alpha+\beta-1} + B_1T^{\beta-1} + CT^{\beta-2} + DT^{\beta-3} + ET^{\alpha-1} + FT^{-1} \tag{27}$$

where $A = \frac{C_2\gamma}{\Gamma(\beta)\Gamma(\alpha+1)}\left(\frac{1}{(\alpha+1)} - \frac{(\beta-1)}{(\alpha+2)}\right)$, $B_1 = \frac{C_2\gamma(\beta-1)t_1^{\alpha}}{2\Gamma(\beta)\Gamma(\alpha+1)} - \frac{C_2\gamma t_1^{\alpha}}{\Gamma(\beta)\Gamma(\alpha+1)}$,, $E = \frac{P\gamma}{\Gamma(\alpha+1)}$, $C = \frac{C_2\gamma t_1^{\alpha+1}}{\Gamma(\beta)\Gamma(\alpha+1)} - \frac{C_2\gamma t_1^{\alpha+1}}{\Gamma(\beta)(\alpha+1)\Gamma(\alpha+1)}$, $D = \frac{C_2\gamma(\beta-1)t_1^{\alpha+2}}{(\alpha+2)\Gamma(\beta)\Gamma(\alpha+1)} - \frac{C_2\gamma(\beta-1)t_1^{\alpha+2}}{2\Gamma(\beta)\Gamma(\alpha+1)}$ and $F = \frac{C_1\gamma t_1^{2\alpha+\beta}(B(\alpha+1,\beta) - B(2\alpha+1,\beta))}{\Gamma(\beta)\Gamma(\alpha+1)} + C_3$.

We consider three different cases of the total average cost depending on the different values of the memory indexes (i) $0 < \alpha \leq 1, 0 < \beta \leq 1$ (ii)$\beta = 1.0, 0 < \alpha \leq 1$ (iii) $\alpha = 1.0, 0 < \beta \leq 1$.

(i)Case-1: we consider both memory exist i.e. differential memory index and integral memory indexes both fractional i.e.$0 < \alpha \leq 1, 0 < \beta \leq 1$.

Here, the inventory model can be written as follows

$$\begin{cases} MinTOC^{av}_{\alpha,\beta}(T) = AT^{\alpha+\beta-1} + B_1T^{\beta-1} + CT^{\beta-2} + DT^{\beta-3} + ET^{\alpha-1} + FT^{-1} \\ SubjecttoT \geqslant 0 \end{cases} \tag{28}$$

Where
$A = \frac{C_2\gamma}{\Gamma(\beta)\Gamma(\alpha+1)}\left(\frac{1}{(\alpha+1)} - \frac{(\beta-1)}{(\alpha+2)}\right)$, $D = \frac{C_2\gamma(\beta-1)t_1^{\alpha+2}}{(\alpha+2)\Gamma(\beta)\Gamma(\alpha+1)} - \frac{C_2\gamma(\beta-1)t_1^{\alpha+2}}{2\Gamma(\beta)\Gamma(\alpha+1)}$, $E = \frac{P\gamma}{\Gamma(\alpha+1)}$, $B_1 = \frac{C_2\gamma(\beta-1)t_1^{\alpha}}{2\Gamma(\beta)\Gamma(\alpha+1)} - \frac{C_2\gamma t_1^{\alpha}}{\Gamma(\beta)\Gamma(\alpha+1)}$, $F = \frac{C_1\gamma t_1^{2\alpha+\beta}(B(\alpha+1,\beta) - B(2\alpha+1,\beta))}{\Gamma(\beta)\Gamma(\alpha+1)} + C_3$ and $C = \frac{C_2\gamma t_1^{\alpha+1}}{\Gamma(\beta)(\alpha+1)} - \frac{C_2\gamma t_1^{\alpha+1}}{\Gamma(\beta)(\alpha+1)\Gamma(\alpha+1)}$.

(a)Primal geometric programming
To solve (28)analytically, the primal geometric programming method has been applied. The dual form of(28) has been introduced by the dual variable (w).The corresponding primal geometric programming problem has been constructed in the following:

$$Maxd(w) = \left(\frac{A}{w_1}\right)^{w_1}\left(\frac{B_1}{w_2}\right)^{w_2}\left(\frac{C}{w_3}\right)^{w_3}\left(\frac{D}{w_4}\right)^{w_4}\left(\frac{E}{w_5}\right)^{w_5}\left(\frac{F}{w_6}\right)^{w_6} \tag{29}$$

Normalized condition is as

$$w_1 + w_2 + w_3 + w_4 + w_5 + w_6 = 1 \tag{30}$$

Orthogonal condition becomes as

$$(\alpha+\beta-1)w_1 + (\beta-1)w_2 + (\beta-2)w_3 + (\beta-3)w_4 + (\alpha-1)w_5 - w_6 = 0 \tag{31}$$

and the primal-dual relations are given as follows

$$\begin{cases} AT^{\alpha+\beta-1} = w_1 d(w) \\ B_1 T^{\beta-1} = w_2 d(w) \\ CT^{\beta-2} = w_3 d(w) \\ DT^{\beta-3} = w_4 d(w) \\ ET^{\alpha-1} = w_5 d(w) \\ FT^{-1} = w_6 d(w) \end{cases} \tag{32}$$

Using the above primal-dual relation the followings are given by

$$\begin{cases} \left(\dfrac{B_1 w_1}{A w_2}\right) & = \left(\dfrac{C w_2}{B_1 w_3}\right)^{\alpha} \\ \left(\dfrac{B_1 w_1}{A w_2}\right)^{\left(\frac{1}{\alpha}\right)} & = \dfrac{D w_3}{C w_4} \\ \left(\dfrac{B_1 w_1}{A w_2}\right)^{\left(\frac{\beta-\alpha-2}{\alpha}\right)} & = \dfrac{E w_4}{D w_5} \\ \dfrac{B_1 w_1}{A w_2} & = \dfrac{F w_5}{E w_6} \end{cases} \tag{33}$$

along with

$$T = \frac{C w_2}{B_1 w_3} \tag{34}$$

Solving (30),(31),and (33) the critical value$w_1^*, w_2^*, w_3^*, w_4^*, w_5^*, w_6^*$ of the dual variables $w_1, w_2, w_3, w_4, w_5, w_6$can be obtained and finally the optimum valueT^* of T has been calculated from the equation of (34) substituting the critical values. Now the minimized total average cost $TOC^*_{\alpha,\beta}$ has been calculated by substitutingT^* in (28) analytically. The minimized total average cost and the optimal ordering interval is evaluated from (28) numerically.

(ii)Case-2: In this case we consider the rate of change of inventory level is fractional but integral memory index is absent i.e.($\beta = 1.0, 0 < \alpha \leq 1$) .

In this case, the inventory model is

$$\begin{cases} MinTOC^{av}_{\alpha,1}(T) = AT^{\alpha} + B_1 T^0 + CT^{-1} + DT^{-2} + ET^{\alpha-1} \\ Subject\ to\ T \geqslant 0 \end{cases} \tag{35}$$

where $A = \dfrac{C_2\gamma}{(\alpha+1)\Gamma(\alpha+1)}, C = \dfrac{C_2\gamma t_1^{\alpha+1}}{\Gamma(\alpha+1)} - \dfrac{C_2\gamma t_1^{\alpha+1}}{(\alpha+1)\Gamma(\alpha+1)} + \dfrac{C_1\gamma t_1^{2\alpha+1}\left(B(\alpha+1,1) - B(2\alpha+1,1)\right)}{\Gamma(\alpha+1)} + C_3,$
$B_1 = -\dfrac{C_2\gamma t_1^{\alpha}}{\Gamma(\alpha+1)}, D = 0$ and $E = \dfrac{P\gamma}{\Gamma(\alpha+1)}$.

Using the similar analogy as previous, the minimized total average cost and optimal ordering interval has been solved from (35).

(iii)Case-3: Here, we consider presence of integral memory only i.e. rate of change of inventory level is unit but integral memory index present i.e.$\alpha = 1.0, 0 < \beta \leq 1$.

Then, the inventory model is as,

$$\begin{cases} MinTOC^{av}_{1,\beta}(T) = AT^{\beta} + B_1 T^{\beta-1} + CT^{\beta-2} + DT^{\beta-3} + ET^0 + FT^{-1} \\ Subject\ to\ T \geqslant 0 \end{cases} \tag{36}$$

where $A = \dfrac{C_2\gamma}{2\Gamma(\beta)\Gamma(2)} - \dfrac{C_2(\beta-1)\gamma}{3\Gamma(\beta)\Gamma(2)}, B_1 = \dfrac{C_2\gamma(\beta-1)t_1}{2\Gamma(\beta)\Gamma(2)} - \dfrac{C_2\gamma t_1}{\Gamma(\beta)\Gamma(2)}, C = \dfrac{C_2\gamma t_1^2}{\Gamma(\beta)\Gamma(2)} - \dfrac{C_2\gamma t_1^2}{2\Gamma(\beta)\Gamma(2)}, D =$
$\dfrac{C_2(\beta-1)\gamma t_1^3}{3\Gamma(\beta)\Gamma(2)} - \dfrac{C_2\gamma(\beta-1)t_1^3}{2\Gamma(\beta)\Gamma(2)}, E = \dfrac{P\gamma}{\Gamma(2)}$ and $F = \dfrac{C_1\gamma t_1^{2+\beta}\left(B(2,\beta) - B(3,\beta)\right)}{\Gamma(\beta)\Gamma(2)} + C_3.$

Using the similar way of case-1, the minimized total average cost and the optimal ordering interval has been solved from (36).

Table 2: Optimal ordering interval$T^*_{\alpha,\beta}$ and minimized total average cost $TOC^*_{\alpha,\beta}$ for $\beta = 1.0, 0 < \alpha \leq 1$

α	β	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$
0.1	1.0	167.3164	4.1316
0.2	1.0	81.5273	9.4674
0.3	1.0	51.6472	16.6348
0.4	1.0	35.8339	25.9524
0.5	1.0	25.6802	37.5886
0.6	1.0	18.3754	51.4457
0.7	1.0	12.7205	66.9596
0.8	1.0	8.1426	82.7853
0.9	1.0	4.4270	96.2934
1.0	1.0	1.8715	103.1555

Table 3: Optimal ordering interval$T^*_{\alpha,\beta}$ and minimized total average cost$TOC^*_{\alpha,\beta}$ for $\alpha = 1.0, 0 < \beta \leq 1$

		$t_1 = 1.3456$		$t_1 = .3456$	
α	β	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$
1.0	0.1	3.5850	100.5354	0.4144	100.1621
1.0	0.2	2.7919	101.0677	0.4062	100.2666
1.0	0.3	2.4492	101.5542	0.4001	100.3269
1.0	0.4	2.2540	101.9791	0.3956	100.3545
1.0	0.5	2.1278	102.3364	0.3921	100.3588
1.0	0.6	2.0403	102.6250	0.3895	100.3472
1.0	0.7	1.9772	102.8469	0.3875	100.3253
1.0	0.8	1.9307	103.0059	0.3861	100.2975
1.0	0.9	1.8964	103.1070	0.3853	100.2668
1.0	1.0	1.8715	103.1555	0.3843	100.2355

4 Numerical example

(a) To illustrate numerical results of the developed fractional order inventory model, we have considered the empirical values of the various parameters in proper units as $P = 20, C_3 = 250, C_1 = 2.5, C_2 = 1.2, \gamma = 5, t_1 = 1.3456$ and required solution has been made using Matlab minimization method.

Now, when we fix the system in absence of differential memory index (see table-3) and allow the integral memory index then the minimized total average cost gradually decreases with gradually increasing memory effect for shortage started later (here for$t_1 = 1.3456$). But if shortage started quickly (for$t_1 = 0.3456$) then we see that the minimized total average cost becomes maximum at$\beta = 0.5$ then it gradually decreases below and above. In this case the optimal ordering interval is very low compare to the previous case. If the shortage start later then we see that minimized total average cost is high in long memory affected system and gradually decreases i.e. profit increases when integral memory increases. In this case the length of optimal ordering interval is long compare to the quickly shortage starting system but it not highly long as in the case of long differential memory system. moderate compared the short stock period. Hence, from the concept of real market of inventory system and mathematical results conclude that for moderate i.e. (here long stock period) stock period is appropriate for the business.

In Table-4 we have presented the optimal ordering interval and minimized total average cost considering both memory effects simultaneously. It is clear from the Table-4 that the optimal ordering interval is long for this long memory affected system. The large optimal ordering interval implies that there may be some demurrage of inventory though the profit is high. Thus the memory affected model will be more realistic in the range $(\alpha, \beta \in [0.6, 1.0])$In reality if we consider more memory i.e. previous experiences then system will be disturbed due to its high restriction.

In the next we will investigate the effect of differential memory index on positive inventory level with respect to time.

Table 4: Optimal ordering interval$T^*_{\alpha,\beta}$ and minimized total average cost$TOC^*_{\alpha,\beta}$ for(a) $\beta = 0.7, 0 < \alpha \le 1$,(b)$\alpha = 0.7, 0 < \beta \le 1$

α	β	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$
0.1	0.7	$1.0000x10^4$	0.4734
0.2	0.7	$1.0000x10^4$	1.6247
0.3	0.7	$1.0000x10^4$	4.4020
0.4	0.7	925.7379	9.6625
0.5	0.7	335.5743	18.0839
0.6	0.7	167.6143	30.7290
0.7	0.7	81.9894	48.2179
0.8	0.7	36.2569	69.8566
0.9	0.7	11.2350	91.6450
1.0	0.7	1.9772	102.8469

(a)

α	β	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$
0.7	0.1	$1.0000x10^4$	7.0451
0.7	0.2	$1.0000x10^4$	7.4491
0.7	0.3	$1.0000x10^4$	8.8091
0.7	0.4	$1.00000x10^4$	12.9857
0.7	0.5	$3.1723x10^4$	24.1489
0.7	0.6	327.3122	37.2889
0.7	0.7	81.9894	48.2179
0.7	0.8	33.8214	56.5084
0.7	0.9	18.8792	62.5678
0.7	1.0	12.7205	66.9596

(b)

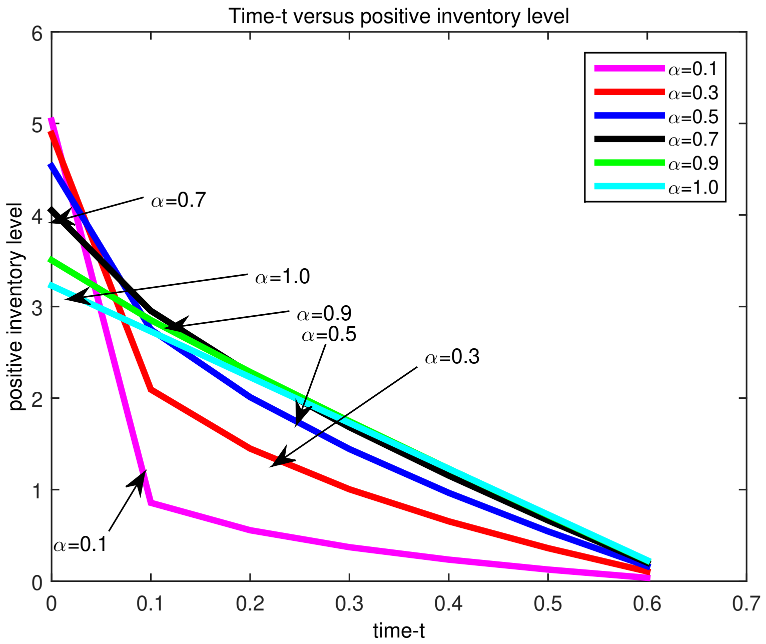

Fig. 2: Positive inventory level versus time-t for different values of α .

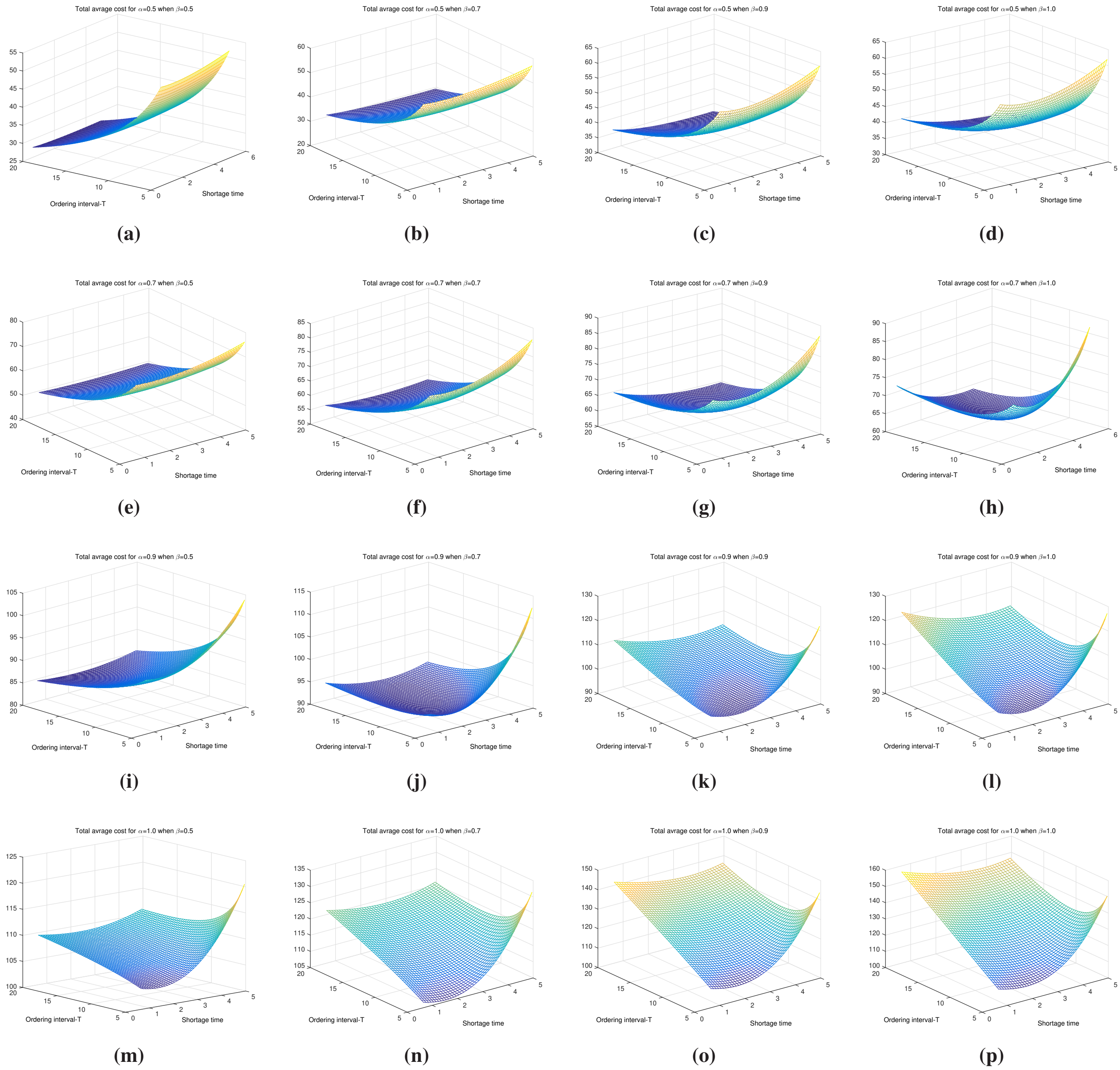

Fig. 3: Total average cost versus ordering interval-T and shortage time-t_1 for different values of α, β:(a-d)$\alpha = 0.5$ and $\beta = 0.5, 0.7, 0.9, 1.0$,(e-h)$\alpha = 0.7$, and $\beta = 0.5, 0.7, 0.9, 1.0$,(i-l) $\alpha = 0.9$ and $\beta = 0.5, 0.7, 0.9, 1.0$,(m-p)$\alpha = 1.0$, and $\beta = 0.5, 0.7, 0.9, 1.0$.

In figure-2 we have presented the positive inventory level with respect to t different values of the differential memory index (α). It is clear from the figure that the positive inventory level changes linearly for low memory affected system but for long memory system then it initially falls rapidly and then decreases with slow rate

The graphical presentations of total average cost as a function of ordering interval (T) and shortage time (t_1) for different values of memory indices are presented in figure $3(a-p)$. It is clear figures that the nature of that total average cost is monotonic increasing function both the memory effect is high. But with the decrease of memory effect the average cost

Table 5: Optimal ordering interval and minimized total average cost for $\alpha = 0.1$ and$\beta = 1.0$

.

Parameter	Parameter Change(%)	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$
P	+50%	249.7255	4.5664
	10%	183.7947	4.2320
	-10%	150.8405	4.0218
	-50%	84.9788	3.4339
C_3	+50%	167.3164	4.1316
	10%	167.3164	4.1316
	-10%	167.3164	4.1316
	-50%	167.3164	4.1316
γ	+50%	167.3164	6.1974
	10%	167.3164	4.5447
	-10%	167.3164	3.7184
	-50%	167.3164	2.0658
t_1	+50%	168.7113	3.8707
	10%	167.5878	4.0709
	-10%	167.0494	4.1982
	-50%	166.0350	4.5593
C_1	+50%	168.0579	4.1358
	10%	167.4647	4.1324
	-10%	167.1680	4.1307
	-50%	166.5743	4.1273
C_2	+50%	111.8959	5.5682
	10%	152.2021	4.4345
	-10%	185.7889	3.8192
	-50%	333.5486	2.4455

function become convex downwards. Hence the total minimized average cost will be minimized for intermediate value of the ordering interval and shortage time but in other cases the average cost function will be minimum at one end of the interval.

4.1 Sensitivity analysis

The sensitivity analysis has been performed by changing each of the values of the considered parameter by $+50\%, +10\%, -10\%, -50\%$ into show effects on the optimal ordering interval $T^*_{\alpha},_1$ and minimized total average cost $TOC^*_{\alpha},_1$ taking one parameters$P, C_3, C_1, C_2, \gamma, t_1$ at a time and keeping the remaining parameters fixed. The corresponding sensitivity estimations are placed in Table- 5 and 6 for the differential memory index $\alpha = 0.1 \& \alpha = 0.9$ respectively i.e. in long memory effect or short memory effect.

It is clear from the Table-5 and Table-6 that in long memory effect shortage cost per unit per unit timeC_2 constant demand rate $\gamma$are the most sensitive parameters but for low memory affected system the important parameters are per unit cost (P) and constant demand rate γ.

Table 6: Optimal ordering interval and minimized total average cost for $\alpha = 0.9$ and $\beta = 1.0$

.

Parameter	Parameter Change(%)	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$
P	+50%	5.9717	140.4055
	10%	4.7268	105.2241
	-10%	4.1331	87.3026
	-50%	3.0353	50.6686
C_3	+50%	4.4270	96.2934
	10%	4.4270	96.2934
	-10%	4.4270	96.2934
	-50%	4.4270	96.2934
γ	+50%	4.4270	144.4401
	10%	4.4270	105.9227
	-10%	4.4270	86.6641
	-50%	4.4270	48.1467
t_1	+50%	5.4876	96.0662
	10%	4.6139	96.1332
	-10%	4.2549	96.5165
	-50%	3.7372	98.0877
C_1	+50%	4.1603	96.8519
	10%	4.4646	96.4069
	-10%	4.3890	96.1789
	-50%	4.2315	95.7104
C_2	+50%	3.3274	98.3645
	10%	4.1237	96.8128
	-10%	4.8004	95.6972
	-50%	7.8473	92.0014

5 Conclusion

In this paper, we have developed a fractional order economic order quantity model using the concept of memory dependency with the assumption that the demand is completely backlogged during the shortage time. To investigate the memory dependent EOQ model here we have introduced two type memory indexes as (i) differential memory index and (ii) integral memory index. Presence of differential memory index and in absence of integral memory index, profit becomes highest in long memory effect but optimal ordering interval is long. So, long memory is not practical in real life, the business man should consider moderate memory. It is clearly established from the numerical examples that in presence of integral memory index and in absence of differential memory index the profit gradually increases with increasing value of the memory effect. The sensitivity analysis shows that the parameter per unit cost of the total order quantity is the most sensitive parameter for the market studies in low memory effect but in long memory the most sensitive are parameters are shortage cost per unit and demand rate are the most sensitive parameter. In long memory effect, profit becomes high compared to the low memory effect.Here, per unit cost is less sensitive in long memory affected system compare to low memory affected system. Hence, total order quantity needs to be adjusted for short past experience effect but not for long experience. For future research, the model can be extended for partial backlogging demand during shortage. This work will help the business policy maker to include the level of past experience in the marketing system.

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Conflict of Interest

The authors declare that they have no conflict of interest.

References

[1] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, John Wiley & Sons, New York, NY, USA, 1993.
[2] I. Podubly, *Fractional differential equations*,Mathematics in Science and Engineering, Academic Press, San Diego, Calif,USA.198, 1999.
[3] M. Saeedian, M. Khalighi, N. Azimi-Tafreshi, G. R. Jafari and M. Ausloos, Memory effects on epidemic evolution: The susceptible-infected-recovered epidemic model, *Phys. Rev. E95*,022409(2017).
[4] V. V. Tarasova and V. E. Tarasov, Memory effects in hereditary Keynesian model,*Probl. Mod. Sci. Edu.***38**(80),38–44 (2016).
[5] V. E.Tarasov and V. V. Tarasova, Long and short memory in economics: fractional-order difference and differentiation, *IRA-Int. J. Manag. Soc. Sci.* **5**(2),327-334 (2016).
[6] V. E. Tarasov and V. V.Tarasova, Economic interpretation of fractional derivatives,*Progr. Fract. Differ. Appl.* **3**(1),1-6 (2017).
[7] V. E. Tarasov and V. V. Tarasova, Elasticity for economic processes with memory: fractional differential calculus approach, *Fract. Differ. Calc.* **6**(2), 219-232 (2016).
[8] R. Pakhira, U. Ghosh and S. Sarkar,Study of Memory Effects in an inventory model using fractional calculus, *Appl. Math. Sci.* **12**(17),797-824 (2018).
[9] R. Pakhira, U. Ghosh and S. Sarkar, Application of memory effects in an inventory model with linear demand and no shortage, *Int. J. Res. Adv. Tech.* **6**(8) (2018).
[10] A. Alsaedi, D. Baleanu, S. Etemad and S. Rezapour, On coupled systems of time-fractional differential problems by using a new fractional derivative, *J. Funct. Space.* 8 pages, Article ID 4626940(2016).
[11] A. Atangana and D. Baleanu, Caputo–Fabrizio derivative applied to ground water flow within a confined aquifer, *J. Eng. Mech.* **143**(5), Article Number: D4016005 (2016).
[12] A. Atangana and D.Baleanu, New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. *Therm Sci.* **20**(2), 763-769 (2016).
[13] T. Das, U. Ghosh, S. Sarkar and S. Das, Time independent fractional Schrodinger equation for generalized Mie-type potential in higher dimension framed with Jumarie type fractional derivative, *J. Math. Phys.***59**, 022111 (2018).
[14] M. Du, Z. Wang and H. Hu, Measuring memory with the order of fractional derivative,*Sci. Rep.* **3**:3431,DOI:10:1038,srep03431 (2013).
[15] R. Pakhira, U. Ghosh and S. Sarkar, Study of memory effect in an inventory model with linear demand and salvage value, *Int. J. Appl. Eng. Res.* **13**(20), 14741-14751 (2018).
[16] R. Pakhira, U. Ghosh and S. Sarkar, Study of Memory Effect in Inventory Model with Quadratic Type Demand Rate and Salvage Value,*Applied Mathematical Sciences*.,13(5),209-223(2019).
[17] R. Pakhira, U. Ghosh and S. Sarkar, Study of memory effect in an economic order quantity model with quadratic type demand rate, *Comput. Meth. Sci. Tech.* **25**(2), 71–80 (2019).
[18] R. Pakhira, U. Ghosh and S. Sarkar, Study of memory effect in an inventory model with price dependent demand, *J. Appl. Econ. Sci.* **2**(64), 360–367 (2019).
[19] R. P. Tripath and N. Sang, EOQ model for constant demand rate with completely backlogged and shortages, *Appl.Comput. Math.***1**:6 (2012).
[20] M. Caputo, Linear models of dissipation whose frequency independent,*Geophys. J. Royal Astr. Soc.***13**(5), 529-539 (1967).
[21] V. K. Mishra, L. S. Singh and R. Kumar, An inventory model for deteriorating items with time-dependent demand and time-varying holding cost under partial backlogging, *J. Ind. Engin. Int.* **9**:4 (2013).