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Some Problems for the Degenerate Mixed Type Equation Involving Caputo and Atangana-Baleanu Operators Fractional Order

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Abstract: The present study investigates existence and uniqueness of solution of local problems for the degenerate parabolic-hyperbolic type equations with loaded terms involving a trace of solution in Atangana-Baleanu operators. Because, the trace of solution (i.e. $u(x,0)$) is in Atangana-Baleanu operators and hyperbolic type equation degenerates on the line $y=0$, we use some properties and estimations of hyper-geometric functions on the proofing process of the uniqueness and existence of solution of the investigated problem.

Keywords: Parabolic-hyperbolic equation, loaded terms, Caputo fractional derivative, Atangana-Baleanu fractional operator, hypergeometric functions, existence and uniqueness of solution.

1 Introduction

Because of the significant role of mathematics and mathematical physics, engineering, bioengineering, neuroscience, economy, control theory, dynamical systems pollution and combustion science (see[1,2,3,4,5,6,7,8,9,10,11]), it is very important to study fractional differential and integral-differential operators. As we know, that Mittag-Leffler function is more appropriate for expressing nature than power function. In 2016 Atangana and Baleanu proposed a new fractional derivative with non-local and no-singular kernel that is based upon the generalized Mittag-Leffler function [12]. The authors presented some useful properties of the new derivative and applied it to solve the fractional heat transfer model. Other works was conducted and applied Atangana-Baleanu operators to the model of groundwater flowing within an unconfined aquifer[13],to free convection flow of generalized Gasson fluid [14].

It should be noted that the partial differential equation (PDE)s fractional order more interesting with their intensive applications to physical phenomena, such as diffusion and wave processes [15,16,17], viscoelasticity [18,19]. In several papers (see [20,21,22,23] and others) the authors considered some classes initial and boundary value problem(BVP)s for PDEs with Riemann-Liouville and Caputo fractional derivatives. Fundamental solutions of PDEs fractional order were found. Moreover, Greens function of the first and second BVPs for such equations. Many papers addressed BVPs for the mixed type equations involving the Caputo and the Riemann-Liouville fractional differential operators, see for instance [24,25,26,27,28]. In addition, some classical methods which are adopted to solve some FDEs involving well known operators as Riemann-Liouville, Caputo, Erdelyi-Kober, Atangana-Baleanu. Applying classical methods we can get classical solutions significant for practice. In addition, other works as [29,30,31] and [32] present application of the fractional calculus to investigations of degenerating mixed type PDEs and hyperbolic equations. On the other hand, numerous papers(see [33,34,35,36]) proved the existence of approximate solutions for some fractional integro-differential equations involving the Caputo-Fabrizio and other derivatives. The approximate solution of an equation is equivalent to the approximate fixed point of an appropriate operator. Using numerical methods, one can obtain approximations of the unknown exact solution.

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The present article aims to find exact and classical solution of some problems (which differ with conditions on the parabolic domain) for a parabolic-hyperbolic type equation fractional order including Atangana-Baleanu operators, and to deduce some conclusions related with statement of the problem. The organisation of the manuscript is given below. In section 2 we present the statement of the investigated problems and necessary functional relations. Section 3 deals with the investigation of the problem I. Section 4 discuss the uniqueness of the solution for the problem II. The existence of the solution of the problem II is depicted in section 5. Finally, the conclusion part is illustrated in section 6.

2 Statement of the problem and necessary functional relations

This section involves statement of the problem. We consider equation:

$$0=\begin{cases} u_{xx}-{}_C D_{oy}^{\alpha}u+\sum\limits_{k=1}^{n} p_k(x,y)^{AB}I_x^{\beta_k}u(x,0), & \text{at } y>0 \\ (-y)^m u_{xx}-u_{yy}+\sum\limits_{k=1}^{n} q_k(x,y)^{AB}I_\eta^{\gamma_k}u(\eta,0), & \text{at } y<0 \end{cases} \tag{1}$$

with Caputo derivatives (see [2]):

$${}_C D_{oy}^{\alpha}u=\frac{1}{\Gamma(1-\alpha)}\int\limits_0^y (y-t)^{-\alpha}u_t(x,t)dt, \tag{2}$$

and Atangana-Baleanu integral operators [12]:

$${}^{AB}I_x^{\alpha}u(x)=\frac{1-\alpha}{B(\alpha)}u(x)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int\limits_0^x (x-t)^{\alpha-1}u(t)dt, \tag{3}$$

where $B(\alpha)$ can be any normalization function satisfying the conditions $B(0)=B(1)=1$, $\alpha,\ \beta_k,\ \gamma_k,\ m=const,\ m>0$, $0<\alpha,\ \beta_k,\gamma_k<1\ (k=1,2,...n)$; $\eta=x+\frac{2}{m+2}(-y)^{\frac{m+2}{2}}$.

Let Ω be a domain bounded with segments: $A_1A_2=\{(x,y):\ x=1,\ 0<y<h\}$, $B_1B_2=\{(x,y):\ x=0,\ 0<y<h\}$, $B_2A_2=\{(x,y):\ y=h,\ 0<x<1\}$ at the $y>0$, and by the characteristics:

$$A_1C:\ x+(1-2\delta)(-y)^{\frac{1}{(1-2\delta)}}=1;\quad B_1C:\ x-(1-2\delta)(-y)^{\frac{1}{1-2\delta}}=0$$

of Eq. 1 at $y<0$, where $\delta=\frac{m}{2(m+2)}$, $A_1\,(1;0)\,, A_2\,(1;h)\,, B_1\,(0;0)\,, B_2\,(0;h)\,, C\left(\frac{1}{2};-\left(\frac{m+2}{4}\right)^{1-2\delta}\right)$.

Introduce designations: $\Omega^+=\Omega\cap(y>0)$, $\Omega^-=\Omega\cap(y<0)$, $I_1=\left\{x:\ 0<x<\frac{1}{2}\right\}, I_2=\{y:0<y<h\}$.

Problem I. Find a solution $u(x,y)$ of Eq. 1 from the class:

$$V_1=\left\{u(x,y):\ u(x,y)\in C(\bar{\Omega})\cap C^2(\Omega^-),\ u_x\in C\left(\overline{\Omega}^+\setminus\{y=h\}\right),\ u_{xx}\in C\left(\Omega^+\right),\ {}_C D_{oy}^{\alpha}u\in C\left(\Omega^+\right)\right\}$$

with boundary:

$$u_x(1,y)=\varphi_1(y),\ \ 0\leq y<h, \tag{4}$$

$$u_x(0,y)=\varphi_2(y),\ \ 0\leq y<h, \tag{5}$$

$$u\left(x,-\left(\frac{x}{1-2\delta}\right)^{1-2\delta}\right)=\psi(x),\ \ x\in I_1. \tag{6}$$

and discontinuous matching conditions:

$$\lim_{y\to+0}y^{1-\alpha}u_y(x,y)=\lambda_1(x)u_y(x,-0)+$$

$$+\lambda_2(x)\int\limits_0^x r(t)u(t,0)dt+\lambda_3(x)u_x(x,0)+\lambda_4(x)u(x,0)+\lambda_5(x),\ \ (x,0)\in A_1B_1 \tag{7}$$

where $\varphi_i(y)\,(i=1,2)$, $\psi(x)$, $r(x)$, $\lambda_j(x)\,(j=1,2...5)$ are given functions, such that $\sum\limits_{j=1}^{4}\lambda_j^2(x)\neq 0$.

Problem II. Find a solution $u(x,y)$ of Eq. 1 from the class:

$$V_2=\left\{u(x,y):u(x,y)\in C(\bar{\Omega})\cap C^2(\Omega^-),\,u_{xx}\in C\left(\Omega^+\right),{}_CD_{oy}^{\alpha}u\in C\left(\Omega^+\right)\right\}$$

satisfies all conditions of the Problem I, in addition to 4 and 5, which changed to

$$u(1,y)=\varphi_3(y),\ \ 0\leq y\leq h, \tag{8}$$

$$u(0,y)=\varphi_4(y),\ \ 0\leq y\leq h, \tag{9}$$

where $\varphi_i(y)\,(i=3,4)$ are given functions. It is known [37], that the Riemann function for Eq. (1) at $y<0$ (on $\xi=x-(1-2\delta)(-y)^{\frac{1}{1-2\delta}}$ and $\eta=x+(1-2\delta)(-y)^{\frac{1}{1-2\delta}}$ coordinate)is defined with the Gauss hypergeometric function:

$$R(\xi_0,\eta_0;\xi,\eta)=\frac{(\eta-\xi)^{2\delta}}{(\eta-\xi_0)^{\delta}(\eta_0-\xi)^{\delta}}H\left(\delta,\delta,1;\frac{(\xi_0-\xi)(\eta-\eta_0)}{(\eta-\xi_0)(\eta_0-\xi)}\right). \tag{10}$$

where

$$H(a_1,a_2,a_3;z)=\frac{\Gamma(a_3)}{\Gamma(a_2)\Gamma(a_3-a_2)}\int\limits_0^1 x^{a_2-1}(1-x)^{a_3-a_2-1}(1-zx)^{-a_1}dx,$$

$0<\mathrm{Re}\,a_2<\mathrm{Re}\,a_3,\quad |\arg(1-z)|<\pi.$

In fact, the solution of the Cauchy problem for equation (1) in the domain Ω^- with initial dates

$$u(x,0)=\tau(x),\ \ 0\leqslant x\leqslant 1;\ \ u_y(x,-0)=\nu^-(x),\ \ 0<x<1 \tag{11}$$

is given by formula [37]:

$$\begin{aligned} u(\xi,\eta)=&k_1\int\limits_{\xi}^{\eta}(s-\xi)^{-\delta}(\eta-s)^{-\delta}\nu^-(s)\,ds-k_2(\eta-\xi)^{1-2\delta}\int\limits_{\xi}^{\eta}(s-\xi)^{\delta-1}(\eta-s)^{\delta-1}\tau^-(s)\,ds\\ &+\sum_{k=1}^{n}\int\limits_{\xi}^{\eta}{}^{AB}I_s^{\gamma_k}\tau(s)ds\int\limits_{s}^{\eta}\tilde{q}_k(s,z)\frac{(\eta-\xi)^{-2\delta}}{(\eta-s)^{\delta}}(z-\xi)^{\delta}H\left(\delta,\delta,1;\frac{(s-\xi)(\eta-z)}{(\eta-s)(z-\xi)}\right)dz, \end{aligned} \tag{12}$$

where $k_1=\frac{(2-4\delta)^{2\delta-1}\Gamma(2-2\delta)}{\Gamma^2(1-\delta)}$, $k_2=\frac{\Gamma(2\delta)}{\Gamma^2(\delta)}$, $\tilde{q}_k(s,z)=q_k\left(\frac{z+s}{2},-\left(\frac{s-z}{2(1-2\delta)}\right)^{1-2\delta}\right)$.

Considering 6, 3 and using some properties of Riemann-Liuoville operators (see[2]) from 12, we will get

$$\begin{aligned} \nu^-(\eta)=&\frac{k_2\Gamma(\delta)}{k_1\Gamma(1-\delta)}D_{0\eta}^{1-2\delta}\tau(\eta)-\frac{2\eta^{\delta}}{k_1\Gamma(1-\delta)}D_{0\eta}^{1-\delta}\eta^{-2\delta}\int\limits_0^{\eta}\sum_{k=1}^{n}\frac{1-\gamma_k}{B(\gamma_k)}\frac{\tau(t)dt}{(\eta-t)^{\delta}}\int\limits_t^{\eta}\frac{\tilde{q}_k(t,z)}{z^{\delta}}H\left(\delta,\delta,1;\frac{t(\eta-z)}{z(\eta-t)}\right)dz\\ &-\frac{2\eta^{\delta}}{k_1\Gamma(1-\delta)}D_{0\eta}^{1-\delta}\eta^{-2\delta}\int\limits_0^{\eta}dt\sum_{k=1}^{n}\frac{\gamma_k}{B(\gamma_k)\Gamma(\gamma_k)}\int\limits_0^{t}\frac{\tau(s)}{(t-s)^{1-\gamma_k}}ds\int\limits_t^{\eta}\frac{\tilde{q}_k(t,z)}{(\eta-t)^{\delta}z^{\delta}}H\left(\delta,\delta,1;\frac{t(\eta-z)}{z(\eta-t)}\right)dz\\ &+\frac{2}{k_1\Gamma(1-\delta)}\eta^{\delta}D_{0\eta}^{1-\delta}\psi\left(\frac{\eta}{2}\right), \end{aligned} \tag{13}$$

where $D_{ax}^{\alpha}f$ $(\alpha\in R^+)$ is the Riemann-Liouville fractional derivative which looks like [2]:

$$(D_{ax}^{\alpha}f)x=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n\int\limits_a^x\frac{f(t)}{(x-t)^{\alpha-n+1}}dt,\,n=[\alpha]+1,\,x>a. \tag{14}$$

On the other hand, considering designations 11 and $\lim_{y\to+0} y^{1-\alpha}u_y(x,y)=\nu^+(x)$, $0<x<1$ from 7 we obtain

$$\nu^+(x)=\lambda_1(x)\nu^-(x)+\lambda_2(x)\int_0^x r(t)\tau(t)dt+\lambda_3(x)\tau'(x)+\lambda_4(x)\tau(x)+\lambda_5(x). \tag{15}$$

For further supposes, from equation 1 at $y\to+0$ considering

$$\lim_{y\to0} D_{0y}^{\alpha-1}f(y)=\Gamma(\alpha)\lim_{y\to0} y^{1-\alpha}f(y)$$

we get

$$\tau''(x)-\Gamma(\alpha)\nu^+(x)+\sum_{k=1}^n p_k(x,0)^{AB}I_x^{\beta_k}\tau(x)=0,\ \ 0<x<1. \tag{16}$$

3 Investigation of the problem I

Taking $\tau'(0)=\varphi_2(0)$ and $\tau(0)=\psi(0)$ into account, twice integrating from 0 to x the equation 16, we will get

$$\tau(x)=\Gamma(\alpha)\int_0^x(x-t)\nu^+(t)dt-\sum_{k=1}^n\int_0^x(x-t)p_k(t,0)^{AB}I_t^{\beta_k}\tau(t)ds+x\varphi_2(0)+\psi(0).$$

Furthermore, using 3 and owing to 15 after some calculations, we receive

$$\tau(x)=\Gamma(\alpha)\int_0^x(x-t)\lambda_1(t)\nu^-(t)dt+\Gamma(\alpha)\int_0^x r(s)\tau(s)ds\int_s^x(x-t)\lambda_2(t)dt-\Gamma(\alpha)\int_0^x(\lambda_3(t)(x-t))'\tau(t)dt$$
$$+\Gamma(\alpha)\int_0^x(x-t)\lambda_4(t)\tau(t)dt+\Gamma(\alpha)\int_0^x(x-t)\lambda_5(t)dt+\sum_{k=1}^n\frac{1-\beta_k}{B(\beta_k)}\int_0^x(x-t)p_k(t,0)\tau(t)ds$$
$$+\sum_{k=1}^n\frac{\beta_k}{B(\beta_k)\Gamma(\beta_k)}\int_0^x\tau(s)ds\int_s^x\frac{(x-t)p_k(t,0)}{(t-s)^{1-\beta_k}}dt+x\varphi_2(0)+\psi(0)-\lambda_3(0)\psi(0)x. \tag{17}$$

On the other hand, owing to 14 and functional relation 13, we will rewrite as:

$$\nu^-(\eta)=\frac{k_2\Gamma(\delta)}{k_1\Gamma(1-\delta)}\frac{d}{d\eta}\int_0^\eta\frac{\tau(t)}{(\eta-t)^{1-2\delta}}dt-\frac{2\eta^\delta}{k_1\Gamma(1-\delta)}\frac{d}{d\eta}\int_0^\eta(\eta-t)^{\delta-1}t^{-2\delta}dt\int_0^t\sum_{k=1}^n\frac{1-\gamma_k}{B(\gamma_k)}\tau(s)ds$$
$$\times\int_s^t\frac{\tilde q_k(s,z)}{(t-s)^\delta z^\delta}H\left(\delta,\delta,1;\frac{s(t-z)}{z(t-s)}\right)dz-\frac{2\eta^\delta}{k_1\Gamma(1-\delta)}\frac{d}{d\eta}\int_0^\eta(\eta-t)^{\delta-1}t^{-2\delta}dt\int_0^t ds$$
$$\times\sum_{k=1}^n\frac{\gamma_k}{B(\gamma_k)\Gamma(\gamma_k)}\int_0^s\frac{\tau(y)}{(s-y)^{1-\gamma_k}}dy\int_s^t\frac{\tilde q_k(s,z)}{(t-s)^\delta z^\delta}H\left(\delta,\delta,1;\frac{s(t-z)}{z(t-s)}\right)dz+\frac{2}{k_1\Gamma(1-\delta)}\eta^\delta D_{0\eta}^{1-\delta}\psi\left(\frac{\eta}{2}\right), \tag{18}$$

Considering 18, we will calculate following integral $J=\Gamma(\alpha)\int_0^x(x-t)\lambda_1(t)\nu^-(t)dt$:

$$J=\frac{k_2\Gamma(\alpha)\Gamma(\delta)}{k_1\Gamma(1-\delta)}\int_0^x(x-\eta)\lambda_1(\eta)d\left[\int_0^\eta\frac{\tau(t)}{(\eta-t)^{1-2\delta}}dt\right]-\frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int_0^x(x-\eta)\lambda_1(\eta)\eta^\delta$$
$$\times d\left[\int_0^\eta(\eta-t)^{\delta-1}t^{-2\delta}dt\int_0^t\sum_{k=1}^n\frac{1-\gamma_k}{B(\gamma_k)}\tau(s)ds\int_s^t\frac{\tilde q_k(s,z)}{(t-s)^\delta}z^\delta H\left(\delta,\delta,1;\frac{s(t-z)}{z(t-s)}\right)dz\right]$$

$$-\frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int_0^x(x-\eta)\lambda_1(\eta)\eta^\delta d\left[\int_0^\eta(\eta-t)^{\delta-1}t^{-2\delta}dt\int_0^t ds\right.$$

$$\left.\times\sum_{k=1}^n\frac{\gamma_k}{B(\gamma_k)\Gamma(\gamma_k)}\int_0^s\frac{\tau(y)}{(s-y)^{1-\gamma_k}}dy\int_s^t\frac{\tilde{q}_k(s,z)}{(t-s)^\delta z^\delta}H\left(\delta,\delta,1;\frac{s(t-z)}{z(t-s)}\right)dz\right]$$

$$+\frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int_0^x(x-\eta)\lambda_1(\eta)\eta^\delta D_{0\eta}^{1-\delta}\psi\left(\frac{\eta}{2}\right)d\eta.$$

Integrating by parts after some simplifications, we finally get:

$$\begin{aligned}J=&\frac{k_2\Gamma(\alpha)\Gamma(\delta)}{k_1\Gamma(1-\delta)}\int_0^x\tau(t)dt\int_t^x\frac{(\lambda_1(\eta)-(x-\eta)\lambda_1'(\eta))}{(\eta-t)^{1-2\delta}}d\eta\\&+\frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int_0^x\tau(s)ds\int_s^x\left((x-\eta)\lambda_1(\eta)\eta^\delta\right)'d\eta\int_s^\eta\frac{(\eta-t)^{\delta-1}t^{-2\delta}}{(t-s)^\delta}dt\sum_{k=1}^n\frac{1-\gamma_k}{B(\gamma_k)}\int_s^t\frac{\tilde{q}_k(s,z)}{z^\delta}H\left(\delta,\delta,1;\frac{s(t-z)}{z(t-s)}\right)dz\\&-\frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int_0^x\tau(y)dy\int_y^x\left((x-\eta)\lambda_1(\eta)\eta^\delta\right)'d\eta\int_y^\eta(\eta-t)^{\delta-1}t^{-2\delta}dt\sum_{k=1}^n\frac{\gamma_k}{B(\gamma_k)\Gamma(\gamma_k)}\int_y^t\frac{(s-y)^{\gamma_k-1}}{(t-s)^\delta}ds\\&\times\int_s^t\frac{\tilde{q}_k(s,z)}{z^\delta}H\left(\delta,\delta,1;\frac{s(t-z)}{z(t-s)}\right)dz+\frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int_0^x(x-\eta)\lambda_1(\eta)\eta^\delta D_{0\eta}^{1-\delta}\psi\left(\frac{\eta}{2}\right)d\eta.\end{aligned}\tag{19}$$

Substituting 19 to 17, we will have:

$$\begin{aligned}\tau(x)=&\Gamma(\alpha)\int_0^x r(s)\tau(s)ds\int_s^x(x-t)\lambda_2(t)dt-\Gamma(\alpha)\int_0^x(\lambda_3(t)(x-t))'\tau(t)dt+\Gamma(\alpha)\int_0^x(x-t)\lambda_4(t)\tau(t)dt\\&+\sum_{k=1}^n\frac{1-\beta_k}{B(\beta_k)}\int_0^x(x-t)p_k(t,0)\tau(t)ds+\frac{k_2\Gamma(\alpha)\Gamma(\delta)}{k_1\Gamma(1-\delta)}\int_0^x\tau(t)dt\int_t^x\frac{(\lambda_1(\eta)-(x-\eta)\lambda_1'(\eta))}{(\eta-t)^{1-2\delta}}d\eta\\&+\frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int_0^x\tau(s)ds\int_s^x\left((x-\eta)\lambda_1(\eta)\eta^\delta\right)'d\eta\int_s^\eta\frac{(\eta-t)^{\delta-1}t^{-2\delta}}{(t-s)^\delta}dt\sum_{k=1}^n\frac{1-\gamma_k}{B(\gamma_k)}\int_s^t\frac{\tilde{q}_k(s,z)}{z^\delta}H\left(\delta,\delta,1;\frac{s(t-z)}{z(t-s)}\right)dz\\&-\frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int_0^x\tau(y)dy\int_y^x\left((x-\eta)\lambda_1(\eta)\eta^\delta\right)'d\eta\int_y^\eta(\eta-t)^{\delta-1}t^{-2\delta}dt\\&\times\sum_{k=1}^n\frac{\gamma_k}{B(\gamma_k)\Gamma(\gamma_k)}\int_y^t\frac{(s-y)^{\gamma_k-1}}{(t-s)^\delta}ds\int_s^t\frac{\tilde{q}_k(s,z)}{z^\delta}H\left(\delta,\delta,1;\frac{s(t-z)}{z(t-s)}\right)dz\\&+\sum_{k=1}^n\frac{\beta_k}{B(\beta_k)\Gamma(\beta_k)}\int_0^x\tau(s)ds\int_s^x\frac{(x-t)p_k(t,0)}{(t-s)^{1-\beta_k}}dt+\Gamma(\alpha)\int_0^x(x-t)\lambda_5(t)\\&+\frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int_0^x(x-\eta)\lambda_1(\eta)\eta^\delta D_{0\eta}^{1-\delta}\psi\left(\frac{\eta}{2}\right)d\eta+x\varphi_2(0)+\psi(0)(1-\lambda_3(0)x).\end{aligned}\tag{20}$$

Hence, we will rewrite 20 as the second kind Volterra integral equation:

$$\tau(x)=\int_0^x K(x,s)\tau(s)ds+f(x)\tag{21}$$

where

$$K(x,s)=\Gamma(\alpha)r(s)\int_s^x (x-t)\lambda_2(t)dt-\Gamma(\alpha)\lambda_3'(s)(x-s)+\Gamma(\alpha)\lambda_3(s)+\Gamma(\alpha)(x-s)\lambda_4(s)$$
$$+\sum_{k=1}^n \frac{1-\beta_k}{B(\beta_k)}(x-s)p_k(s,0)+\frac{k_2\Gamma(\alpha)\Gamma(\delta)}{k_1\Gamma(1-\delta)}\int_s^x \frac{(\lambda_1(t)-(x-t)\lambda_1'(t))}{(t-s)^{1-2\delta}}dt$$
$$+\frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int_s^x \left[(\delta x-(3\delta+1)\eta)\eta^{\delta-1}\lambda_1(\eta)+\lambda_1'(\eta)\eta^{\delta}(x-\eta)\right]d\eta\int_s^\eta \frac{(\eta-t)^{\delta-1}t^{-2\delta}}{(t-s)^{\delta}}dt$$
$$\times\sum_{k=1}^n \frac{1-\gamma_k}{B(\gamma_k)}\int_s^t \frac{\tilde{q}_k(s,z)}{z^{\delta}}H\left(\delta,\delta,1;\frac{s(t-z)}{z(t-s)}\right)dz$$
$$-\frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int_s^x [(\delta x-(\delta+1)\eta)\eta^{\delta-1}\lambda_1(\eta)+\lambda_1'(\eta)\eta^{\delta}(x-\eta)]d\eta\int_s^\eta (\eta-t)^{\delta-1}t^{-2\delta}dt$$
$$\times\sum_{k=1}^n \frac{\gamma_k}{B(\gamma_k)\Gamma(\gamma_k)}\int_s^t \frac{(y-s)^{\gamma_k-1}}{(t-y)^{\delta}}dy\int_y^t \frac{\tilde{q}_k(y,z)}{z^{\delta}}H\left(\delta,\delta,1;\frac{y(t-z)}{z(t-y)}\right)dz$$
$$+\sum_{k=1}^n \frac{\beta_k}{B(\beta_k)\Gamma(\beta_k)}\int_s^x \frac{(x-t)p_k(t,0)}{(t-s)^{1-\beta_k}}dt, \tag{22}$$

$$f(x)=x\varphi_2(0)+\psi(0)(1-\lambda_3(0)x)+\Gamma(\alpha)\int_0^x (x-t)\lambda_5(t)dt$$
$$+\frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int_0^x (x-\eta)\lambda_1(\eta)\eta^{\delta}D_{0\eta}^{1-\delta}\psi\left(\frac{\eta}{2}\right)d\eta. \tag{23}$$

Lemma 1. *Let valid* $0<\alpha,\ \beta_k,\ \gamma_k<1$ *and conditions*

$$p_k(x,y)\in C\left(\overline{\Omega}^+\right)\cap C^2\left(\Omega^+\right),\ q_k(x,y)C\left(\overline{\Omega}^-\right)\cap C^1\left(\Omega^-\right),\ k=1,2,...,n; \tag{24}$$

$$\varphi_i(y)\in C[0,h]\cap C^1(0,h),\ \ (i=1,2),\ \ \psi(x)\in C(\bar{I}_1)\cap C^2(I_1); \tag{25}$$

$$\lambda_j(x)\in C[0,1]\cap C^1(0,1),\ (j=1,2,5),\ \lambda_3(x)\in C[0,1]\cap C^3(0,1),\ \lambda_4(x)\in C[0,1]\cap C^2(0,1), \tag{26}$$

then the integral equation 21 has a unique solution.

Proof. Based on the class of given functions 24-26, from 22 we have

$$|K(x,s)|\le c_1|x-s|+c_2|(x-s)^{2\delta}|+const|A_1(x,s)|+const|A_2(x,s)|+const|A_3(x,s)|+const|A_4(x,s)|, \tag{27}$$

where $A_1(x,s)=\int_s^x \frac{x-t}{(t-s)^{1-2\delta}}dt$, $A_2(x,s)=\sum_{k=1}^n\int_s^x \frac{(x-t)}{(t-s)^{1-\beta_k}}dt$,

$$A_3(x,s)=\int_s^x (x-\eta)d\eta\int_s^\eta (\eta-t)^{\delta-1}t^{-2\delta}dt\sum_{k=1}^n \frac{\gamma_k}{B(\gamma_k)\Gamma(\gamma_k)}\int_s^t \frac{(y-s)^{\gamma_k-1}}{(t-y)^{\delta}}dy\int_y^t \frac{\tilde{q}_k(y,z)}{z^{\delta}}H\left(\delta,\delta,1;\frac{y(t-z)}{z(t-y)}\right)dz. \tag{28}$$

$$A_4(x,s)=\int_s^x (x-\eta)d\eta\int_s^\eta \frac{(\eta-t)^{\delta-1}t^{-2\delta}}{(t-s)^{\delta}}dt\sum_{k=1}^n \frac{1-\gamma_k}{B(\gamma_k)}\int_s^t \frac{\tilde{q}_k(s,z)}{z^{\delta}}H\left(\delta,\delta,1;\frac{s(t-z)}{z(t-s)}\right)dz. \tag{29}$$

Applying replacement $t-s=z(x-s)$, we will get: $A_1(x,s)=(x-s)^{1+2\delta}\int\limits_0^1 z^{2\delta-1}(1-z)dz$, $A_2(x,s)=\sum\limits_{k=1}^{n}(x-s)^{1+\beta_k}\int\limits_0^1 z^{\beta_k-1}(1-z)dz$, which follows, that

$$|A_1(x,s)|\leq const\cdot(x-s)^{1+2\delta},\, |A_2(x,s)|\leq const\cdot(x-s)^{1+\beta^*},\, \beta^*=max\{\beta_1,...,\beta_n\}. \tag{30}$$

Due to estimation of hypergeometric function [37]:

$$H(a_1,a_2,a_3;z)\leq\begin{cases}c_1, \quad if \quad a_3-a_1-a_2>0, \quad 0\leq z\leq 1\\ c_2(1-z)^{a_3-a_1-a_2}, \quad if \quad a_3-a_1-a_2<0, \quad 0<z<1\\ c_3\left(1+|\ln(1-z)|\right), \quad if \quad a_3-a_1-a_2=0\end{cases} \tag{31}$$

taking 24-26 into account and making some evaluation from 28, we get

$$|A_3(x,s)|\leq const\cdot\left|\int\limits_s^x(x-\eta)d\eta\int\limits_s^\eta(\eta-t)^{\delta-1}t^{-2\delta}dt\sum_{k=1}^{n}\frac{\gamma_k}{B(\gamma_k)\Gamma(\gamma_k)}\int\limits_s^t\frac{(y-s)^{\gamma_k-1}}{(t-y)^{\delta}}dy\right|$$

Considering replacement $y-s=z(t-s)$, we obtain

$$|A_3(x,s)|\leq const\cdot\left|\int\limits_s^x(x-\eta)d\eta\sum_{k=1}^{n}\frac{\gamma_k}{B(\gamma_k)\Gamma(\gamma_k)}\int\limits_s^\eta(\eta-t)^{\delta-1}t^{-2\delta}(t-s)^{1+\gamma_k-\delta}dt\right|.$$

Moreover, introducing the same subsequent replacements $t-s=z(\eta-s)$ and $\eta-s=z(x-s)$ consistently, making some estimations we will get

$$|A_3(x,s)|\leq const|(x-s)^{3+\gamma^*}|,\; \gamma^*=max\{\gamma_1,...,\gamma_n\}. \tag{32}$$

Based on the same estimations, we will infer that

$$|A_4(x,s)|\leq const(x-s)^2. \tag{33}$$

Hence, taking 30, 32 and 33 into account, from 27 we conclude that

$$|K(x,s)|\leq c_1|x-s|+c_2|(x-s)^{2\delta}|+c_3|(x-s)^{1+2\delta}|+c_4|(x-s)^{1+\beta^*}|+c_5|(x-s)^{3+\gamma^*}|+c_6(x-s)^2, \tag{34}$$

where $c_i=const,\; i=1,2,...6$.

Easy to spot, that

$$|f(x)|\leq c_7|x|+c_8x^2+c_9|x^{2\delta+2}|,\; (c_i=const, i=7,8,9). \tag{35}$$

Based on 34 and 35, we will infer that

$$|K(x,s)|\leq const, K(x,s)\in C\left([0,1]\times[0,x]\right)\cup C^{2,1}_{x,s}\left((0,1)\times(0,x)\right) \tag{36}$$

$$|f(x)|\leq const, f(x)\in C[0,1]\cup C^2(0,1). \tag{37}$$

Adopting the theory of Volterra type integral equations and considering 36 and 37 we will conclude the equation 21 unique solvability. Besides the solution of the equation 21 we can write via resolvent-kernel as

$$\tau(x)=\int\limits_0^x R(x,s)f(s)ds+f(x)$$

where $R(x,s)$ is resolvent-kernel of $K(x,s)$. Lemma 1 is proved.

Finding $\tau(x)$, we will determine $\nu^-(x)$ from the functional relation 18. Hence, a solution of the investigated problem is restored in the domain Ω^- as a solution of the Cauchy problem (see 12). Solution of the **Problem I** is restored in the domain Ω^+ as a solution of the second BVP for Eq. 1(see [22]):

$$u(x,y)=\int_0^y G(x,y,0,\eta)\varphi_1(\eta)d\eta-\int_0^y G(x,y,1,\eta)\varphi_2(\eta)d\eta+\int_0^1 G_0(x-\xi,y)\tau(\xi)d\xi$$
$$-\int_0^y\int_0^1 G(x,y,0,\eta)\sum_{k=1}^n p_k(\xi,0)^{AB}I_\xi^{\beta_k}\tau(\xi)d\xi d\eta$$

here

$$G_0(x-\xi,y)=\frac{1}{\Gamma(1-\alpha)}\int_0^y \eta^{-\alpha}G(x,y,\xi,\eta)d\eta,$$
$$G(x,y,\xi,\eta)=\frac{(y-\eta)^{\alpha/2-1}}{2}\sum_{n=-\infty}^{\infty}\left[e_{1,\alpha/2}^{1,\alpha/2}\left(-\frac{|x-\xi+2n|}{(y-\eta)}^{\alpha/2}\right)+e_{1,\alpha/2}^{1,\alpha/2}\left(-\frac{|x+\xi+2n|}{(y-\eta)}^{\alpha/2}\right)\right]$$

is the Green's function of the second boundary problem for Eq. 1 in the domain Ω^+ with the Riemanne-Liouville fractional differential operator instead of the Caputo ones [24], [22],

$$e_{1,\delta}^{1,\delta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(\delta-\delta n)}$$

is the Wright type function [23].

4 Uniqueness of the solution for the problem II

Uniqueness and existence of solution of the **Problem II** will be proved using the same methods as in works [26,31,32]. Assuming $\lambda_5(x)\equiv 0$ and considering 15 from 16, we have

$$\tau''(x)-\Gamma(\alpha)\lambda_1(x)\nu^-(x)+\Gamma(\alpha)\lambda_2(x)\int_0^x r(t)\tau(t)dt+$$
$$+\Gamma(\alpha)\lambda_3(x)\tau'(x)+\Gamma(\alpha)\lambda_4(x)\tau(x)+\sum_{k=1}^n p_k(x,0)^{AB}I_x^{\beta_k}\tau(x)=0,\ \ 0<x<1 \tag{38}$$

In addition, we multiply to $\tau(x)$ equation 38 and integrate from 0 to 1:

$$\int_0^1\tau''(x)\tau(x)dx-\Gamma(\alpha)\int_0^1\tau(x)\lambda_1(x)\nu^-(x)dx+\Gamma(\alpha)\int_0^1\tau(x)\lambda_2(x)dx\int_0^t r(s)\tau(s)dt+ \tag{39}$$
$$+\Gamma(\alpha)\int_0^1\tau(x)\lambda_3(x)\tau'(x)dx+\Gamma(\alpha)\int_0^1\lambda_4(x)\tau^2(x)+\sum_{k=1}^n\int_0^1\tau(x)p_k(x,0)^{AB}I_x^{\beta_k}\tau(x)dx=0 \tag{40}$$

Considering $\tau(0)=\varphi_4(0)$, $\tau(1)=\varphi_3(0)$ easy to spot, that $\int_0^1\tau''(x)\tau(x)dx\leq 0$. Moreover, we can prove that

$$\Gamma(\alpha)\int_0^1\tau(x)\lambda_2(x)dx\int_0^t r(s)\tau(s)dt\leq 0,\ \text{if}\ \left(\frac{\lambda_2(x)}{r(x)}\right)'\leq 0,\ \frac{\lambda_2(0)}{r(0)}\leq 0,$$

$$\Gamma(\alpha)\int_0^1 \tau(x)\lambda_3(x)\tau'(x)dx \le 0, \text{ if } \lambda_3'(x) > 0$$

$$\sum_{k=1}^{n}\int_0^1 \tau(x)p_k(x,0)^{AB}I_x^{\beta_k}\tau(x)dx \le 0, \text{ if } p_k(x,0) \le 0,\ p_k'(x,0) \le 0$$

In [31] was proved that

$$\Gamma(\alpha)\int_0^1 \tau(x)\lambda_1(x)\nu^-(x)dx \ge 0, \text{at } q_k(x,y) \le 0,\ \lambda_1(x) \ge 0.$$

To conclude, we will formulate this

Lemma 2. *If conditions* $0<\alpha,\ \beta_k,\ \gamma_k<1,\ (k=1,2,...,n)$ *and*

$$\left(\frac{\lambda_2(x)}{r(x)}\right)' \le 0,\ \frac{\lambda_2(0)}{r(0)} \le 0, \lambda_3'(x) > 0,$$

$$p_k(x,0) \le 0,\ p_k'(x,0) \le 0,\ q_k(x,y) \le 0,\ \lambda_1(x) \ge 0$$

*hold, then the solution of the **Problem II.** unique.*

5 Existence of the solution of the problem II

Theorem 1. *If all conditions of the **Lemma** 2. (24) are satisfied and*

$$\varphi_i(y) \in C[0,h] \cap C^1(0,h);\ (i=3,4)\quad \psi(x) \in C^1\left(\overline{I_1}\right) \cap C^2(I_1), \tag{41}$$

*then the solution of the **Problem II** exists.*

Proof. Eq.(16) is written as

$$\tau''(x) = f_2(x), \tag{42}$$

where

$$f_2(x) = \Gamma(\alpha)\nu^+(x) - \sum_{k=1}^{n} p_k(x,0)^{AB}I_x^{\beta_k}\tau(x).$$

Solution of Eq. 42 satisfying $\tau(0)=\varphi_4(0),\ \tau(1)=\varphi_3(0)$ has a form:

$$\tau(x) = \int_0^x (x-t)f_2(t)dt - x\int_0^1 (1-t)f_2(t)dt + \varphi_4(0)(1-x) + x\varphi_3(0). \tag{43}$$

We will investigate integral $J_1 = \Gamma(\alpha)\int_0^x (x-t)\nu^+(t)dt - x\int_0^1 (1-t)\nu^+(t)dt$. Substituting 15 into J_1, we get

$$\begin{aligned} J_1 = {} & \Gamma(\alpha)\int_0^x (x-t)\lambda_1(t)\nu^-(t)dt + \Gamma(\alpha)\int_0^x (x-t)\lambda_2(t)dt\int_0^t r(s)\tau(s)ds \\ & + \Gamma(\alpha)\int_0^x (x-t)\lambda_3(t)\tau'(t)dt + \Gamma(\alpha)\int_0^x (x-t)\lambda_4(t)\tau(t)dt + \Gamma(\alpha)\int_0^x (x-t)\lambda_5(t)dt \\ & - x\int_0^1 (1-t)\lambda_1(t)\nu^-(t)dt - x\int_0^1 (1-t)\lambda_2(t)dt\int_0^t r(s)\tau(s)ds - x\int_0^1 (1-t)\lambda_3(t)\tau'(t)dt \\ & - x\int_0^1 (1-t)\lambda_4(t)\tau(t)dt - x\int_0^1 (1-t)\lambda_5(t)dt. \end{aligned}$$

Considering 18 and integrating by parts after some calculations, we will get:

$$J_{11}=\Gamma(\alpha)\int\limits_0^x (x-t)\lambda_1(t)\nu^-(t)dt=\frac{k_2\Gamma(\delta)\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int\limits_0^x ((x-\eta)\lambda_1(\eta))'d\eta\int\limits_0^\eta \frac{\tau(t)}{(\eta-t)^{1-2\delta}}dt$$
$$-\frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int\limits_0^x \left((x-\eta)\lambda_1(\eta)\eta^\delta\right)'d\eta\int\limits_0^\eta \tau(s)ds\int\limits_s^\eta \frac{(\eta-t)^{\delta-1}t^{-2\delta}}{(t-s)^\delta}dt$$
$$\times\sum_{k=1}^n\frac{1-\gamma_k}{B(\gamma_k)}\int\limits_s^t\frac{\tilde{q}_k(s,z)}{z^\delta}H\left(\delta,\delta,1;\frac{s(t-z)}{z(t-s)}\right)dz$$
$$-\frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int\limits_0^x \left((x-\eta)\lambda_1(\eta)\eta^\delta\right)'d\eta\int\limits_0^\eta (\eta-t)^{\delta-1}t^{-2\delta}dt\int\limits_0^t ds$$
$$\times\sum_{k=1}^n\frac{\gamma_k}{B(\gamma_k)\Gamma(\gamma_k)}\int\limits_0^s\frac{\tau(y)}{(s-y)^{1-\gamma_k}}dy\int\limits_s^t\frac{\tilde{q}_k(s,z)}{(t-s)^\delta z^\delta}H\left(\delta,\delta,1;\frac{s(t-z)}{z(t-s)}\right)dz$$
$$+\frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int\limits_0^x (x-\eta)\lambda_1(\eta)\eta^\delta D_{0\eta}^{1-\delta}\psi\left(\frac{\eta}{2}\right)d\eta, \qquad (44)$$

$$J_{12}=\Gamma(\alpha)\int\limits_0^1 (1-t)\lambda_1(t)\nu^-(t)dt=\frac{k_2\Gamma(\delta)\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int\limits_0^1 ((1-\eta)\lambda_1(\eta))'d\eta\int\limits_0^\eta \frac{\tau(t)}{(\eta-t)^{1-2\delta}}dt$$
$$-\frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int\limits_0^1 \left((1-\eta)\lambda_1(\eta)\eta^\delta\right)'d\eta\int\limits_0^\eta \tau(s)ds\int\limits_s^\eta \frac{(\eta-t)^{\delta-1}t^{-2\delta}}{(t-s)^\delta}dt$$
$$\times\sum_{k=1}^n\frac{1-\gamma_k}{B(\gamma_k)}\int\limits_s^t\frac{\tilde{q}_k(s,z)}{z^\delta}H\left(\delta,\delta,1;\frac{s(t-z)}{z(t-s)}\right)dz$$
$$-\frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int\limits_0^1 \left((1-\eta)\lambda_1(\eta)\eta^\delta\right)'d\eta\int\limits_0^\eta (\eta-t)^{\delta-1}t^{-2\delta}dt\int\limits_0^t ds$$
$$\times\sum_{k=1}^n\frac{\gamma_k}{B(\gamma_k)\Gamma(\gamma_k)}\int\limits_0^s\frac{\tau(y)}{(s-y)^{1-\gamma_k}}dy\int\limits_s^t\frac{\tilde{q}_k(s,z)}{(t-s)^\delta z^\delta}H\left(\delta,\delta,1;\frac{s(t-z)}{z(t-s)}\right)dz$$
$$+\frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int\limits_0^1 (1-\eta)\lambda_1(\eta)\eta^\delta D_{0\eta}^{1-\delta}\psi\left(\frac{\eta}{2}\right)d\eta, \qquad (45)$$

Taking into account 44, 45 and substituting J_1 to 43 after some simplification, we will get second kind Fredholm integral equation:

$$\tau(x)=\int\limits_o^1 K_1(x,t)\tau(t)dt+F(x) \qquad (46)$$

where

$$K_1(x,t)=\begin{cases}K_{11}(x,t);\ 0\leqslant t\leqslant x,\\ K_{12}(x,t);\ x\leqslant t\leqslant 1.\end{cases} \qquad (47)$$

$$
\begin{aligned}
K_{11}(x,t) &= \Gamma(\alpha) r(t) \int_t^x (x-s)\lambda_2(s)ds - \Gamma(\alpha)\left[((x-t)\lambda_3(t))' + (x-t)\lambda_4(t)\right] \\
&- xr(t)\int_t^x (1-s)\lambda_2(s)ds + x\left[((1-t)\lambda_3(t))' - (1-t)\lambda_4(t)\right] \\
&- \frac{1-\alpha}{B(\alpha)}(x-t)\sum_{k=1}^n p_k(x,0) - \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\sum_{k=1}^n p_k(x,0)\int_t^x \frac{(x-s)}{(s-t)^{1-\alpha}}ds \\
&+ \frac{1-\alpha}{B(\alpha)}(1-t)\sum_{k=1}^n p_k(x,0) - \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\sum_{k=1}^n p_k(x,0)\int_t^x \frac{(1-s)}{(s-t)^{1-\alpha}}ds \\
&+ \frac{k_2\Gamma(\delta)\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int_t^x \frac{((x-\eta)\lambda_1(\eta))'}{(\eta-t)^{1-2\delta}}d\eta - \frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int_t^x \left((x-\eta)\lambda_1(\eta)\eta^{\delta}\right)' d\eta \int_t^{\eta} \frac{(\eta-s)^{\delta-1}s^{-2\delta}}{(s-t)^{\delta}}ds \\
&\times \sum_{k=1}^n \frac{1-\gamma_k}{B(\gamma_k)}\int_t^s \frac{\tilde{q}_k(t,z)}{z^{\delta}} H\left(\delta,\delta,1;\frac{t(s-z)}{z(s-t)}\right)dz \\
&- \frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int_t^x \left((x-\eta)\lambda_1(\eta)\eta^{\delta}\right)' d\eta \int_t^{\eta}(\eta-y)^{\delta-1}y^{-2\delta}dy \\
&\times \sum_{k=1}^n \frac{\gamma_k}{B(\gamma_k)\Gamma(\gamma_k)}\int_t^y \frac{(s-t)^{\gamma_k-1}}{(y-s)^{\delta}}ds \int_s^y \frac{\tilde{q}_k(s,z)}{z^{\delta}} H\left(\delta,\delta,1;\frac{s(y-z)}{z(y-s)}\right)dz \\
&+ \frac{k_2\Gamma(\delta)\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int_t^1 \frac{((1-\eta)\lambda_1(\eta))'}{(\eta-t)^{1-2\delta}}d\eta - \frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int_t^1 \left((1-\eta)\lambda_1(\eta)\eta^{\delta}\right)' d\eta \int_t^{\eta} \frac{(\eta-s)^{\delta-1}s^{-2\delta}}{(s-t)^{\delta}}ds \\
&\times \sum_{k=1}^n \frac{1-\gamma_k}{B(\gamma_k)}\int_t^s \frac{\tilde{q}_k(t,z)}{z^{\delta}} H\left(\delta,\delta,1;\frac{t(s-z)}{z(s-t)}\right)dz \\
&- \frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int_t^1 \left((1-\eta)\lambda_1(\eta)\eta^{\delta}\right)' d\eta \int_t^{\eta}(\eta-y)^{\delta-1}y^{-2\delta}dy \\
&\times \sum_{k=1}^n \frac{\gamma_k}{B(\gamma_k)\Gamma(\gamma_k)}\int_t^y \frac{(s-t)^{\gamma_k-1}}{(y-s)^{\delta}}ds \int_s^y \frac{\tilde{q}_k(s,z)}{z^{\delta}} H\left(\delta,\delta,1;\frac{s(y-z)}{z(y-s)}\right)dz \qquad (48)
\end{aligned}
$$

$$K_{12}(x,t)=\frac{1-\alpha}{B(\alpha)}(1-t)\sum_{k=1}^{n}p_k(x,0)-\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\sum_{k=1}^{n}p_k(x,0)\int_t^x\frac{(1-s)}{(s-t)^{1-\alpha}}ds$$
$$-xr(t)\int_t^x(1-s)\lambda_2(s)ds+x\left[((1-t)\lambda_3(t))'-(1-t)\lambda_4(t)\right]+\frac{1-\alpha}{B(\alpha)}(1-t)\sum_{k=1}^{n}p_k(x,0)$$
$$-\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\sum_{k=1}^{n}p_k(x,0)\int_t^x\frac{(1-s)}{(s-t)^{1-\alpha}}ds$$
$$+\frac{k_2\Gamma(\delta)\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int_t^1\frac{((1-\eta)\lambda_1(\eta))'}{(\eta-t)^{1-2\delta}}d\eta-\frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int_t^1\left((1-\eta)\lambda_1(\eta)\eta^{\delta}\right)'d\eta\int_t^{\eta}\frac{(\eta-s)^{\delta-1}s^{-2\delta}}{(s-t)^{\delta}}ds$$
$$\times\sum_{k=1}^{n}\frac{1-\gamma_k}{B(\gamma_k)}\int_t^s\frac{\tilde{q}_k(t,z)}{z^{\delta}}H\left(\delta,\delta,1;\frac{t(s-z)}{z(s-t)}\right)dz$$
$$-\frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int_t^1\left((1-\eta)\lambda_1(\eta)\eta^{\delta}\right)'d\eta\int_t^{\eta}(\eta-y)^{\delta-1}y^{-2\delta}dy$$
$$\times\sum_{k=1}^{n}\frac{\gamma_k}{B(\gamma_k)\Gamma(\gamma_k)}\int_t^y\frac{(s-t)^{\gamma_k-1}}{(y-s)^{\delta}}ds\int_s^y\frac{\tilde{q}_k(s,z)}{z^{\delta}}H\left(\delta,\delta,1;\frac{s(y-z)}{z(y-s)}\right)dz \quad (49)$$

$$F(x)=\varphi_4(0)(1-x)+x\varphi_3(0)+\frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int_0^x(x-\eta)\lambda_1(\eta)\eta^{\delta}D_{0\eta}^{1-\delta}\psi\left(\frac{\eta}{2}\right)d\eta$$
$$+\frac{2\Gamma(\alpha)}{k_1\Gamma(1-\delta)}\int_0^1(1-\eta)\lambda_1(\eta)\eta^{\delta}D_{0\eta}^{1-\delta}\psi\left(\frac{\eta}{2}\right)d\eta$$
$$+\Gamma(\alpha)\int_0^x(x-t)\lambda_5(t)dt-\Gamma(\alpha)x\int_0^x(1-t)\lambda_5(t)dt. \quad (50)$$

Due to class of given functions (see 41) and 31 considering $0<\alpha,\beta_k,\gamma_k,2\delta<1$ from 47-49 and 50, we will receive that $|K_1(x,t)|\le const$ and $|F(x)|\le const$. Moreover,

$$K_1(x,t)\in C([0,1]\times[0,x])\cup C_{x,t}^{2,1}((0,1)\times(0,x)),\, F(x)\in C[0,1]\cup C^2(0,1). \quad (51)$$

Because we prove that the Problem II has a unique solution, due to (51), and based on the theory Fredholm equations, Eq.(46) has a unique solution. Therefore, solving integral equation (46), we will find $\tau(x)$. Furthermore, from (13), will find $\nu^-(x)$. We will restore the solution of the Problem II in the domain Ω^+ as the first BVP for Eq. (1)(see [22,26]), and in the domain Ω^- as a solution of the Cauchy problem. Thus, Theorem 1 is proved.

6 Conclusion

BVPs for the loaded parabolic-hyperbolic type equations with the trace of solution included to Riemann-Liouville, Caputo or Atangana-Baleanu operators, will be reduced to Volterra integral equations in case if, in the parabolic domain, they were given conditions as 4 and 5 and local boundary condition on the characteristics of the hyperbolic equation. In this case, it is not required to prove uniqueness of solution separately.

BVPs for such equations will be reduced to Fredholm integral equations if, in parabolic domain, they were given conditions as 8 and 9 and local(or non-local) boundary conditions on the characteristics of the hyperbolic equation. In this case, it is required to prove uniqueness of solution separately.

Analogical problems as Problem II with non-local conditions on the characteristics of the hyperbolic equation will be investigated with the same methods.

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