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Multiple Positive Solutions for a Continuous Fractional Boundary Value Problem with Fractional $q-$Differences

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Abstract: In this paper, a continuous fractional boundary value problem invovling fractional q-derivative of the Riemann-Liouville type is considered. Using Krasnoselskii's fixed point theorem, the existence of positive solutions for the problem is obtained. By applying Leggett-Williams fixed-point theorem, the multiplicity of positive solutions is also achieved. Moreover, two examples are presented to illustrate the existence and multiplicity of positive solutions for the problem.

Keywords: Continuous fractional calculus; Fractional $q-$differences; Positive solutions; Existence and multiplicity; Fixed point theorem.

1 Introduction

q-difference calculus (i.e. quantum calculus) was first introduced by Jackson[1,2]. Then, it was extended to fractional q-difference calculus by Agarwal[3] and Al-Salam[4]. Due to its various applications in many subjects, including quantum mechanics, particle physics and hypergeometric series, many researchers devoted their efforts to develop the theory in this field and many results were made, such as the q-Taylor's formula, q-Laplace transform[5], ...etc.

During the last few years, the study of positive solutions for fractional boundary value problems as well as their its various applications in physics and engineering flourished. Many results were obtained by applying Caputo derivative and standard Riemann-Liouville fractional derivative (see [6,7,8,9,10] and references therein). But for fractional q-difference boundary value problems, there were few. In 2011, R. Ferreira[11] investigated existence of positive solutions for a class of fractional q-difference boundary value problems

$$(A)\begin{cases} D_q^{\alpha}u(t)=-f(t,u(t)), & 0<t<1,\\ u(0)=D_qu(0)=D_qu(1)=0. \end{cases}$$

where $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is nonnegative continuous and $2<\alpha\leq 3$. Recently, Li et al.[12,13] have addressed eigenvalue problems of problem (A), some existence and nonexistence theorems for solutions with different eigenvalues were obtained.

In this paper, we consider a continuous fractional boundary value problem

$$(P)\begin{cases} D_q^{\alpha}u(t)=-f(t,u(t)), & 0<t<1,\\ u(0)=D_q^{k}u(0)=0, & 1\leq k\leq n-2, k\in\mathbb{N},\\ D_q^{\beta}u(1)=0, & 1\leq\beta\leq n-2. \end{cases}$$

where $n-1<\alpha\leq n$, $n\geq 3$, $n\in\mathbb{N}$, $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is nonnegative continuous and D_q^{α} is the fractional q-derivative of Riemann-Liouville type with order α. Let $n=3$, $\beta=1$ in the problem (P), it becomes problem (A). Thus, the problem (P) we considered is a more general case and the results we obtained will extend the work of R. Ferreira and Li et al.

This paper is designed as follows: In Section 2, we recall some notations of q-integral and q-derivative. Then, the corresponding Green's function is obtained and some fixed points theorems are given. In Section 3, the existence and multiplicity of positive solutions are obtained using the fixed point theorems on cone. In Section 4, two examples are presented to show the main results.

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2 Preliminaries

We use the notations indicated in Jackson's work [1,2]. q-derivative of function $u(t)$ for $0<q<1$ is defined by

$$(D_q u)(t)=\frac{u(t)-u(qt)}{(1-q)t},\quad (D_q u)(0)=\lim_{t\to 0}(D_q u)(t),$$

and for higher order q-derivative,

$$(D_q^0 u)(t)=u(t),\quad (D_q^n u)(t)=D_q((D_q^{n-1}u)(t)),\ n\in\mathbb{N}.$$

For q-integral of function u defined on $[0,b]$, we have

$$(I_q u)(t)=\int_0^t u(s)d_q s=t(1-q)\sum_{n=0}^{\infty}q^n u(tq^n),\ t\in[0,b],$$

The fundamental formula of calculus can also be applied to q-derivative and q-integral, i.e.

$$(D_q I_q u)(t)=u(t),$$

and if $\lim_{t\to 0}u(t)=u(0)$, then

$$(I_q D_q u)(t)=u(t)-u(0).$$

Let $[\alpha]_q=\frac{1-q^\alpha}{1-q}$, $\alpha\in\mathbb{R}$. The power function $(a-b)^\alpha$ can be expressed as

$$(a-b)^{(\alpha)}=a^\alpha\prod_{n=0}^{\infty}\frac{a-bq^n}{a-bq^{\alpha+n}}.$$

From the definition, we can see $(a-b)^{(\alpha)}=a^\alpha(1-\frac{b}{a})^{(\alpha)}$ and $a^{(\alpha)}=a^\alpha$.

Let

$$\Gamma_q(\alpha)=\frac{(1-q)^{(\alpha-1)}}{(1-q)^{\alpha-1}},\ \alpha\in\mathbb{R}\setminus\{0,-1,-2,\cdots\},$$

and we have $\Gamma_q(\alpha+1)=[\alpha]_q\Gamma_q(\alpha)$.

Fractional q-difference calculus was established by Agarwal [3] and Al-Salam [4]. Here we use Agarwal's notations.

Definition 1.[3] *Let $\alpha\geq 0$ and u be a function defined on interval $[0,1]$. The fractional q-integral of Riemann-Liouville type is $(I_q^0 u)(t)=u(t)$ and*

$$(I_q^\alpha u)(t)=\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-qs)^{(\alpha-1)}u(s)d_q s,\ \alpha>0,\ t\in[0,1].$$

Definition 2.[3] *The definition of fractional q-derivative of Riemann-Liouville type on interval $[0,1]$ is $(D_q^0 u)(t)=u(t)$ and*

$$(D_q^\alpha u)(t)=(D_q^m I_q^{m-\alpha}u)(t),\ \alpha>0,$$

where m is the smallest integer satisfying $m\geq\alpha$.

Lemma 1.[3] *The following formulas hold for $t\in[0,1]$ and $\alpha,\beta>0$*

(1) $(I_q^\beta I_q^\alpha u)(t)=(I_q^\alpha I_q^\beta u)(t)=(I_q^{\beta+\alpha}u)(t)$;

(2) $(D_q^\alpha I_q^\alpha u)(t)=u(t)$;

(3) $(I_q^\alpha D_q^p u)(t)=(D_q^p I_q^\alpha u)(t)-\sum_{k=0}^{p-1}\frac{t^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)}(D_q^k u)(0)$, $p\in\mathbb{N},p\geq\alpha$.

Next, we list some equalities that will be used later. They have been proved by R. Ferreira[11].

Lemma 2.[11] *Let ${}_tD_q$ be the derivative with respect to t. Then the following formulas hold:*

(1) ${}_tD_q(t-s)^{(\alpha)}=[\alpha]_q(t-s)^{(\alpha-1)}$;

(2) $D_q^\beta t^\alpha=\frac{\Gamma_q(\alpha+1)}{\Gamma_q(\alpha-\beta+1)}t^{\alpha-\beta}$;

(3) $(t-a)^{(\alpha)}\geq(t-b)^{(\alpha)},\ a\leq b\leq t,\ \alpha>0.$

Lemma 3. *Problem (P) has a unique solution with*

$$u(t)=\int_0^1 G(t,s)f(s,u(s))d_q s$$

where $G(t,s)$ is the Green's function of problem (P) with

$$G(t,s)=\frac{1}{\Gamma_q(\alpha)}\begin{cases}(1-qs)^{(\alpha-\beta-1)}t^{\alpha-1}-(t-qs)^{(\alpha-1)}, & 0\leq s\leq t\leq 1,\\ (1-qs)^{(\alpha-\beta-1)}t^{\alpha-1}, & 0\leq t\leq s\leq 1.\end{cases}$$

Proof. From $(D_q^\alpha u)(t)=-f(t,u(t))$ and Lemma 2.1(3) with $p=n$, we have $(I_q^\alpha D_q^n I_q^{n-\alpha}u)(t)=-I_q^\alpha f(t,u(t))$. Then the general solution to the problem is

$$u(t)=-\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-qs)^{(\alpha-1)}f(s,u(s))d_q s+ c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n}.$$

From condition $u(0)=0$, we can get $c_n=0$. Then, differentiating both sides with respect of t, we can get

$$(D_q u)(t)=-\frac{1}{\Gamma_q(\alpha)}\int_0^t[\alpha-1]_q(t-qs)^{(\alpha-2)}f(s,u(s))d_q s +[\alpha-1]_q c_1t^{\alpha-2}+[\alpha-2]_q c_2t^{\alpha-3}+\cdots+[\alpha-n+1]_q c_{n-1}t^{\alpha-n}.$$

Boundary condition $(D_q u)(0)=0$ means $c_{n-1}=0$. Differentiating both sides with respect of t successively and using boundary condition $(D_q^k u)(0)=0$, $k=2,3,\ldots,n-2$ similarly, we obtain $c_{n-2}=c_{n-3}=\cdots=c_2=0$. Now, the equation becomes $u(t)=-I_q^\alpha f(t,u(t))+c_1t^{\alpha-1}$. Differentiating both sides with order β, where $1\leq\beta\leq n-2$ and with Lemma 2.2(2), we obtain

$$\begin{aligned}(D_q^\beta u)(t)&=-D_q^\beta I_q^\beta I_q^{\alpha-\beta}f(t,u(t))+c_1D_q^\beta t^{\alpha-1}\\&=-I_q^{\alpha-\beta}f(t,u(t))+c_1\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\beta)}t^{\alpha-\beta-1}\\&=-\frac{1}{\Gamma_q(\alpha-\beta)}\int_0^t(t-qs)^{(\alpha-\beta-1)}f(s,u(s))d_q s\\&+c_1\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\beta)}t^{\alpha-\beta-1}.\end{aligned}$$

By boundary condition $(D_q^{\beta}u)(1) = 0$, we have $c_1 = \frac{1}{\Gamma_q(\alpha)}\int_0^1(1-qs)^{(\alpha-\beta-1)}f(s,u(s))d_qs$. Then, we can get

$$\begin{aligned}u(t) &= \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^1(1-qs)^{(\alpha-\beta-1)}f(s,u(s))d_qs\\&-\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-qs)^{(\alpha-1)}f(s,u(s))d_qs\\&=\frac{1}{\Gamma_q(\alpha)}\Big\{\int_0^t((1-qs)^{(\alpha-\beta-1)}t^{\alpha-1}s\\&-(t-qs)^{(\alpha-1)})f(s,u(s))d_q\\&+\int_t^1(1-qs)^{(\alpha-\beta-1)}t^{\alpha-1}f(s,u(s))d_qs\Big\}\\&=\int_0^1 G(t,s)f(s,u(s))d_qs.\end{aligned}$$

The proof is complete.

Next, we will give some properties of Green's function.

Lemma 4.*The Green's function of problem (P) satisfies following conditions:*

$$\begin{aligned}&(1)\quad for\ 0\le t,s\le 1, G(t,s)\ge 0,\\&(2)\quad for\ 0\le s\le 1, \max_{0\le t\le 1}G(t,s)=G(1,s),\\&(3)\quad for\ any\ 0<\tau<1\ and\ 0\le s\le 1,\\&\quad \min_{\tau\le t\le 1}G(t,s)\ge \tau^{\alpha-1}G(1,s).\end{aligned}$$

Proof.(1) For $0\le s\le t\le 1$, we write

$$G_1(t,s)=\frac{1}{\Gamma_q(\alpha)}\{(1-qs)^{(\alpha-\beta-1)}t^{\alpha-1}-(t-qs)^{(\alpha-1)}\},$$

and for $0\le t\le s\le 1$,

$$G_2(t,s)=\frac{1}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-\beta-1)}t^{\alpha-1}.$$

Obviously, $G_1(0,s)=0$ and $G_2(t,s)\ge 0$. For $t\neq 0$, using Lemma 2.2 (3), we can obtain

$$\begin{aligned}G_1(t,s)=&\frac{1}{\Gamma_q(\alpha)}\{(1-qs)^{(\alpha-\beta-1)}t^{\alpha-1}-(t-qs)^{(\alpha-1)}\}\\=&\frac{1}{\Gamma_q(\alpha)}\{(1-qs)^{(\alpha-\beta-1)}t^{\alpha-1}-(1-q\frac{s}{t})^{(\alpha-1)}t^{\alpha-1}\}\\\ge&\frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\{(1-qs)^{(\alpha-\beta-1)}-(1-qs)^{(\alpha-1)}\}\ge 0.\end{aligned}$$

Therefore, for $0\le t,s\le 1$, we obtain $G(t,s)\ge 0$.
(2) Obviously, for variable t, $G_2(t,s)$ increases monotonically. And for fixed $0\le t\le 1$, we get

$$\begin{aligned}{}_tD_qG_1(t,s)=&\frac{1}{\Gamma_q(\alpha)}\{(1-qs)^{(\alpha-\beta-1)}[\alpha-1]_qt^{\alpha-2}\\&-[\alpha-1]_q(t-qs)^{(\alpha-2)}\}\\=&\frac{t^{\alpha-2}}{\Gamma_q(\alpha)}[\alpha-1]_q\{(1-qs)^{(\alpha-\beta-1)}-(1-q\frac{s}{t})^{(\alpha-2)}\}\\\ge&\frac{t^{\alpha-2}}{\Gamma_q(\alpha)}[\alpha-1]_q\{(1-qs)^{(\alpha-\beta-1)}-(1-qs)^{(\alpha-2)}\}.\end{aligned}$$

Here, $\alpha-\beta-1\le\alpha-2$ since $1\le\beta\le n-2$. So ${}_tD_qG_1(t,s)\ge 0$. Then, for variable t, $G_1(t,s)$ also increases monotonically. Hence, $\max_{0\le t\le 1}G(t,s)=G(1,s)$.
(3) If $0\le s\le t\le 1$,

$$\begin{aligned}\frac{G_1(t,s)}{G(1,s)}=&\frac{(1-qs)^{(\alpha-\beta-1)}t^{\alpha-1}-(t-qs)^{(\alpha-1)}}{(1-qs)^{(\alpha-\beta-1)}-(1-qs)^{(\alpha-1)}}\\=&\frac{t^{\alpha-1}\{(1-qs)^{(\alpha-\beta-1)}-(1-q\frac{s}{t})^{(\alpha-1)}\}}{(1-qs)^{(\alpha-\beta-1)}-(1-qs)^{(\alpha-1)}}\\\ge&\frac{t^{\alpha-1}\{(1-qs)^{(\alpha-\beta-1)}-(1-qs)^{(\alpha-1)}\}}{(1-qs)^{(\alpha-\beta-1)}-(1-qs)^{(\alpha-1)}}=t^{\alpha-1}.\end{aligned}$$

If $0\le t\le s\le 1$,

$$\frac{G_2(t,s)}{G(1,s)}=t^{\alpha-1}.$$

Hence $G(t,s)\ge t^{\alpha-1}G(1,s)$.
The proof is complete.

Lemma 5.[14](*Krasnoselskii's fixed point theorem) Let subset K be a cone in Banach space E. There exist two bounded open subsets Ω_1, Ω_2 in K satisfying $0\in\Omega_1\subset\overline{\Omega_1}\subset\Omega_2$. If operator $T:\overline{\Omega_2}\setminus\Omega_1\to K$ is completely continuous and satisfies one of following conditions:*
(1) $||Tu||\le||u||$, $u\in\partial\Omega_1$ and $||Tu||\ge||u||$, $u\in\partial\Omega_2$;
(2) $||Tu||\ge||u||$, $u\in\partial\Omega_1$ and $||Tu||\le||u||$, $u\in\partial\Omega_2$.
Then, T has a fixed point in $\overline{\Omega_2}\setminus\Omega_1$.

Definition 3.[15] *Let K be a cone of Banach space E. If $\theta:K\to[0,+\infty)$ is a continuous map and for $u,v\in K$, $0<t<1$, the following inequality holds*

$$t\theta(u)+(1-t)\theta(v)\le\theta(tu+(1-t)v).$$

Then, function θ is concave, nonnegative and continuous on cone K.

Lemma 6.[15](*Leggett-Williams fixed-point theorem) Let K be a cone of Banach space E. θ defined above is concave, nonnegative and continuous on K and satisfies $\theta(u)\le||u||$. Denote $K_{r_1}=\{u\in K:||u||\le r_1\}$. For all $u\in K_{r_1}$, let $K(\theta,r_1,r_2)=\{u\in K:||u||\le r_1,\theta(u)\ge r_2\}$. If $T:K_{r_1}\to K_{r_1}$ is completely continuous and the following conditions hold for $0<r_3<r_2<r_1$:*
(C1) $||Tu||<r_3$ for $u\le r_3$;

(C2) Set $\{u \in K(\theta, r_1, r_2) : \theta(u) > r_2\}$ is non-empty, and $\theta(Tu) > r_2$ for $u \in K(\theta, r_1, r_2)$.
Then, there exist three different fixed points u_1, u_2, u_3 of T with

$$\theta(u_1) < r_2,\ ||u_1|| > r_3,\ \theta(u_2) > r_2,\ ||u_3|| < r_3.$$

3 Main Results

We consider problem (P) in Banach space $C[0,1]$. The norm $||u||$ is defined by $||u|| = \max\limits_{0\le t\le 1} |u(t)|$.

Let $K = \{u \in C[0,1] : \min\limits_{\tau \le t \le 1} u(t) \ge \tau^{\alpha-1}||u||, u(t) \ge 0\}$, where $\tau = q^n$ for a given $n \in \mathbb{N}$. Then $K \in C[0,1]$ is a cone containing nonnegative functions.

For $0 \le t \le 1$, $u \in K$, denote T as

$$(Tu)(t) = \int_0^1 G(t,s) f(s,u(s)) d_q s.$$

Then $Tu \in C[0,1]$ is well defined.

Lemma 7. *Operator $T : K \to K$ is completely continuous.*

Proof. Given $u \in K$, from nonnegativity of G, f and Lemma 2.4, one has $Tu \ge 0$ and

$$\begin{aligned}\min_{\tau\le t\le 1} Tu(t) &= \min_{\tau\le t\le 1} \int_0^1 G(t,s) f(s,u(s)) d_q s\\ &\ge \int_0^1 \tau^{\alpha-1} \max_{0\le t\le 1} G(t,s) f(s,u(s)) d_q s\\ &= \tau^{\alpha-1}||Tu||.\end{aligned}$$

Thus, $Tu \in K$. T is continuous by continuity of f, G. For any bounded set $\Omega \subset K$ and $u \in \Omega$, we have $||u|| < M$ where M is a positive constant. Let $L = \max\limits_{0\le t\le 1, 0\le u\le M} f(t,u) + 1$. We can get

$$\begin{aligned}|Tu(t)| &\le \int_0^1 |G(t,s) f(s,u(s))| d_q s \le L \int_0^1 G(1,s) d_q s\\ &\le \frac{L}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} d_q s\\ &\le \frac{L}{\Gamma_q(\alpha)} < +\infty.\end{aligned}$$

Therefore, $T(\Omega)$ is bounded. Next, we consider the equicontinuity of $T(\Omega)$. That is, given $\varepsilon > 0$, let $\delta = \frac{\Gamma_q(\alpha)}{L}\varepsilon$, for each $u \in \Omega$, $0 \le t_1 < t_2 \le 1$, and $t_2 - t_1 < \delta$, then $|(Tu)(t_2) - (Tu)(t_1)| < \varepsilon$. In fact,

$$\begin{aligned}&|(Tu)(t_2) - (Tu)(t_1)| = \left| \int_0^1 (G(t_2,s) - G(t_1,s)) f(s,u(s)) d_q s \right|\\ &\le L \int_0^1 (G(t_2,s) - G(t_1,s)) d_q s\\ &\le \frac{L}{\Gamma_q(\alpha)} \Big(\int_0^{t_1} ((1-qs)^{(\alpha-\beta-1)}(t_2{}^{\alpha-1} - t_1{}^{\alpha-1}) - ((t_2-qs)^{(\alpha-1)}\\ &\quad - (t_1-qs)^{(\alpha-1)})) d_q s\\ &\quad + \int_{t_1}^{t_2} ((1-qs)^{(\alpha-\beta-1)}(t_2{}^{\alpha-1} - t_1{}^{\alpha-1}) - (t_2-qs)^{(\alpha-1)}) d_q s\\ &\quad + \int_{t_2}^{t_1} (1-qs)^{(\alpha-\beta-1)}(t_2{}^{\alpha-1} - t_1{}^{\alpha-1}) d_q s \Big)\\ &\le \frac{L}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-\beta-1)}(t_2{}^{\alpha-1} - t_1{}^{\alpha-1}) d_q s \le \frac{L}{\Gamma_q(\alpha)}\delta = \varepsilon.\end{aligned}$$

Then $T : K \to K$ is completely continuousby by Arzela-Ascoli theorem. The proof is complete.

Denote

$$A = \int_0^1 G(1,s) d_q s, \quad B = \min_{\tau\le t\le 1} \int_\tau^1 G(t,s) d_q s.$$

Theorem 1. *Suppose $f(t,u)$ defined on $[0,1] \times [0,+\infty)$ is nonnegative continuous. Let $\tau = q^n, n \in \mathbb{N}$ and $0 < r_1 < r_2$ be two positive constants. If the following assumptions are satisfied*

(H1) for $(t,u) \in [0,1] \times [0,r_1]$, $f(t,u) \le \frac{r_1}{A}$;

(H2) for $(t,u) \in [\tau,1] \times [\tau^{\alpha-1} r_2, r_2]$, $f(t,u) \ge \frac{r_2}{\tau^{\alpha-1}B}$.

Then, there exists one positive solution $u^* \in K$ to problem (P).

Proof. Denote $\Omega_1 = \{u \in K : ||u|| < r_1\}$, then for $u \in \partial\Omega_1$, we get $0 \le u \le r_1$ on $[0,1]$. By assumption $(H1)$, we can obtain

$$||Tu|| = \max_{0\le t\le 1} \left| \int_0^1 G(t,s) f(s,u(s)) d_q s \right| \le \int_0^1 G(1,s) \frac{r_1}{A} d_q s = r_1 = ||u||.$$

Hence, for $u \in \partial\Omega_1$, we have $||Tu|| \le ||u||$.
Denote $\Omega_2 = \{u \in K : ||u|| < r_2\}$, then for $u \in \partial\Omega_2$, we have $\tau^{\alpha-1} r_2 = \tau^{\alpha-1}||u|| \le u \le r_2$ on $[\tau,1]$. By assumption $(H2)$ for $t \in [\tau,1]$, we can obtain

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s) f(s,u(s)) d_q s \ge \tau^{\alpha-1} \int_\tau^1 G(1,s) \frac{r_2}{\tau^{\alpha-1}B} d_q s\\ &\ge \tau^{\alpha-1} \int_\tau^1 \min_{\tau\le t\le 1} G(t,s) \frac{r_2}{\tau^{\alpha-1}B} d_q s = r_2.\end{aligned}$$

Hence, for $u \in \partial\Omega_2$, we have $||Tu|| \ge ||u||$. Hence, by completely continuity of T on K and Lemma 2.5, there exists one fixed point u^* of T in $\overline{\Omega_2} \setminus \Omega_1$ with $r_1 \le ||u^*|| \le r_2$. Therefore, the proof is complete.

The following theorem concerns the multiplicity of positive solutions.

Theorem 2. *Suppose $f(t,u)$ defined on $[0,1]\times[0,+\infty)$ is nonnegative continuous. Let $\tau=q^n, n\in\mathbb{N}$ and $0<a<\tau^{\alpha-1}b$. If the following assumptions are satisfied:*

(A1) for $(t,u)\in[0,1]\times[0,b]$, $f(t,u)\le\frac{b}{A}$;

(A2) for $(t,u)\in[0,1]\times[0,a]$, $f(t,u)\le\frac{a}{A}$;

(A3) for $(t,u)\in[\tau,1]\times[\tau^{\alpha-1}b,b]$, $f(t,u)\ge\frac{\tau^{\alpha-1}b}{B}$.

Then there exist three positive solutions u_1, u_2, u_3 of problem (P) satisfying

$$\max_{0\le t\le1}|u_1|>a>\max_{0\le t\le1}|u_3|,\ \min_{\tau\le t\le1}|u_2|>\tau^{\alpha-1}b>\min_{\tau\le t\le1}|u_1|$$

Proof. From Lemma 2.6, we denote $\theta(u)=\min_{\tau\le t\le1}|u|$, $r_1=b$, $r_2=\tau^{\alpha-1}b$ and $r_3=a$.

If $u\in K_b$, then $0\le u\le b$. By assumption $(A1)$, we obtain

$$||Tu||=\max_{0\le t\le1}\left|\int_0^1 G(t,s)f(s,u(s))d_qs\right|\le\int_0^1 G(1,s)\frac{b}{A}d_qs=b.$$

Hence $Tu\in K_b$, $T:K_b\to K_b$ is completely continuous. From assumption $(A2)$, for $u\le a$, we can similarly get $||Tu||<a$. This is condition $(C1)$.

Let $u(t)=\frac{\tau^{\alpha-1}+1}{2}b$. Since $0<\tau=q^n<1$, then $\tau^{\alpha-1}b<\theta(u)=\frac{\tau^{\alpha-1}+1}{2}b<b$. This means $u(t)=\frac{\tau^{\alpha-1}+1}{2}b\in K(\theta,b,\tau^{\alpha-1}b)$ and $\theta(u)=\frac{\tau^{\alpha-1}+1}{2}b>\tau^{\alpha-1}b$.

So set $\{u\in K(\theta,b,\tau^{\alpha-1}b):\theta(u)>\tau^{\alpha-1}b\}$ is non-empty. For $u\in K(\theta,b,\tau^{\alpha-1}b)$, then for $t\in[\tau,1]$, we have $\tau^{\alpha-1}b\le u(t)\le b$. From assumption $(A3)$, we have

$$\theta(Tu)=\min_{\tau\le t\le1}|(Tu)(t)|=\min_{\tau\le t\le1}\left|\int_0^1 G(t,s)f(s,u(s))d_qs\right|$$
$$>\int_\tau^1\min_{\tau\le t\le1}G(t,s)\frac{\tau^{\alpha-1}b}{B}d_qs=\tau^{\alpha-1}b.$$

Hence, condition $(C2)$ holds. Therefore, by Lemma 2.6, the result is achieved.

4 Examples

To illustrate existence of positive solution, we present the following example:

Example 1. Consider following fractional boundary value problem

$$\begin{cases}D_{0.5}^{3.5}u(t)=-u^2e^u,\ 0<t<1,\\ u(0)=D_{0.5}u(0)=D_{0.5}^2u(0)=0,\ D_{0.5}^{1.5}u(1)=0.\end{cases}$$

In this problem, $\alpha=3.5, q=0.5, \beta=1.5$, $f(t,u)=u^2e^u$.

$$A=\int_0^1 G(1,s)d_qs\le\frac{1-(1-q)^{\alpha-1}}{\Gamma_q(\alpha)}\approx0.3762,$$

and

$$B=\min_{0.5\le t\le1}\int_{0.5}^1 G(t,s)d_qs\ge0.5^{\alpha-1}\int_{0.5}^1 G(1,s)d_qs$$
$$\ge0.5^{\alpha-1}\frac{(1-\frac{q}{2})^{\alpha-\beta-1}-(1-q)^{\alpha-1}}{2\Gamma_q(\alpha)}\approx0.1301.$$

Let $r_1=\frac{3}{4}$, $r_2=25$, $\tau^{\alpha-1}\approx0.1767$,

$f(t,u)=u^2e^u\le r_1^2e^{r_1}\approx1.5878r_1<2.6582r_1=\frac{r_1}{0.3762}\le\frac{r_1}{A}$, for $u\in[0,r_1]$;

$f(t,u)=u^2e^u\ge(\tau^{\alpha-1}r_2)^2e^{\tau^{\alpha-1}r_2}\approx64.70r_2>43.497r_2=\frac{1}{0.1767\times0.1301}r_2\ge\frac{r_2}{\tau^{\alpha-1}B}$, for $u\in[\tau^{\alpha-1}r_2,r_2]$.

Then by Theorem 3.1, there exists one solution $u(t)$ of this example with $\frac{3}{4}\le||u||\le25$.

To illustrate multiplicity of positive solutions, we present another example as follows:

Example 2. Consider following fractional boundary value problem

$$\begin{cases}D_{0.5}^{3.5}u(t)=-f(u),\ 0<t<1,\\ u(0)=D_{0.5}u(0)=D_{0.5}^2u(0)=0,\ D_{0.5}^{1.5}u(1)=0.\end{cases}$$

where

$$f(u)=\begin{cases}10u^2,\ u\le1,\\ 9+u,\ u\ge1.\end{cases}$$

From Example 4.1, we obtain $A\le0.3762$, $B\ge0.1301$. Choose $a=0.1$, and b such that $\tau^{\alpha-1}b=1$, then $b=\frac{1}{\tau^{\alpha-1}}\approx5.6593$,

$f(u)\le9+u\approx14.6593<15.04\approx\frac{5.6593}{0.3762}\le\frac{b}{A}$, for $u\in[0,b]$;

$f(u)=10u^2=0.1<0.2658\approx\frac{0.1}{0.3762}\le\frac{a}{A}$, for $u\in[0,a]$;

$f(u)=9+u>9>7.686\approx\frac{1}{0.1301}\ge\frac{\tau^{\alpha-1}b}{B}$, for $u\in[1,b]$.

Then by Theorem 3.2, there exist three positive solutions u_1, u_2, u_3 of this example with

$$\max_{0\le t\le1}|u_1|>0.1>\max_{0\le t\le1}|u_3|,\ \min_{0.5\le t\le1}|u_2|>1>\min_{0.5\le t\le1}|u_1|$$

That is

$$0.1<\max_{0\le t\le1}|u_1|\le5.65935,\quad 0<\min_{0.5\le t\le1}|u_1|<1,$$

$$1<\min_{0.5\le t\le1}|u_2|<\max_{0\le t\le1}|u_2|\le5.65935,\quad 0<\max_{0\le t\le1}|u_3|<0.1.$$

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