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A New Sumudu Type Integral Transform and Its Applications

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Abstract: In this paper, we introduce generalize Sumudu type integral transform to investigate conformable derivative, convolution as well as commutative semigroup property and to obtain solution of conformable heat transfer problems.

Keywords: Sumudu type integral transform, conformable derivative, convolution

1 Introduction, motivation and preliminaries

Mainly the concept of IT has originated from Fourier integral formula. The importance of ITs is to provide an effective method for solving many mathematical models, IVPs and BVPs that appear in DEs. The IT is a prominent tool for solving the DEs, FDEs and IEs arising in various sciences [1,2,3,4]. The DEs and FDEs have become important tools in mathematical model [1,4]. There are several types of fractional derivatives Riemann-Liouvelli, Caputo, ABR and ABC [5]. The fractional diffusion equation plays an important role in dynamical system, bio-information and finance [6]. In recent years, considerable interest in ST [7,8,9,10] has been stimulated by the numerous applications in various fields, such as applied mathematics, physics, and engineering. Also in [11] Belgacem and Karaballi investigated ST fundamental properties and applications. Recently, Khalil et al. have defined the conformable derivative in [12]. In [13], Mohamed et al. studied nonlocal telegraph model equation in frame omit using the conformable time-derivative. In [14], Zhao and Li investigated the concepts of conformable delta derivative and conformable delta integral on time scales. Recently, Ghadle and Magar [10] applied ST to the fractional advection-diffusion equation to determine the pollution and dissolved oxygen in river water. In [15] it was derived the formulae for the ST and double ST of ordinary and partial fractional derivatives and applied them to solve Caputo type FDEs. Also they showed the applicability of this new transform and its efficiency in solving such problems. Moreover, Panchal et al.[8] defined k-Prabhakar fractional derivative as well as k-Hilfer-Prabhakar fractional derivative and its regularized version and found LT and ST of k-Prabhakar fractional derivative, k-Hilfer-Prabhakar fractional derivative. The LT of k-Prabhakar fractional derivative defined in [9] are obtained. Yang [16] introduced new IT method which is different from the LT, ST, and ET operators to solve the DEs in the steady heat-transfer problem.

The classical ITs are not reduced conformable derivative. In the present paper, we introduce efficacious generalize Sumudu type IT to reduce conformable derivative and omit obtain basic properties. We find a solution to unveil effective applications to Sumudu type IT.

In section 2, some useful definitions are presented. In section 3, we prove some basic properties, convolution theorem by new Sumudu type IT and apply it to conformable derivative, convolution operation and semigroup properties. Also, we study few applications in section 4.

Abbreviations
FDEs - Fractional Differential Equations
IT - Integral Transform

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ST - Sumudu Transform
LT - Laplace Transform
FT - Fourier Transform
ET - Elzaki Transform
DEs - Differential Equations
BVPs - Boundary Value Problems
IVPs - Initial Value Problems
IE - Integral Equation

2 Preliminary

We begin by introducing some necessary definitions and basic results required for further developments in this paper.

Definition 2.1. [7] ST is defined for function of exponential order and we consider functions in the set A omit defined by,

$$A = \{f(t)/\exists M, \tau_1, \tau_2 |f(t)| \leq Me^{|t|/T_j}, if \quad t\varepsilon(-1^j) \times [0,\infty)\}$$

the ST is defined by,

$$G(u) = S[f(t)] = \int_0^\infty f(ut)e^{-t}dt, \qquad u\varepsilon(-\tau_1, \tau_2)$$

for a given function in the set A. The constant M must be finite number, τ_1, τ_2 may be finite or infinite.

Definition 2.2. [12] (Conformable integral)
Let $\alpha \in (0,1]$ and $F : [0,\infty) \to \mathbb{R}$.
The conformable integral of f of order α from zero to t is defined by

$$\begin{aligned} I_\alpha f(t) &= \int_0^t F(s)d_\alpha s = \int_0^t f(s)s^{\alpha-1}ds \\ &= I_1(t^{\alpha-1})(t), \quad t \geq 0, \end{aligned}$$

where the above integral is the usual improper Riemann integral.

Definition 2.3. [12] Let $f \in C^n(0,\infty)$, $\alpha \in (p-1,p]$ for all $p \in \mathbb{N}$ and $\lceil \alpha \rceil$ denote the smallest integer greater than or equal to α. The conformable derivative of order α denoted by $T_\alpha[f(t)]$ is defined as,

$$\begin{aligned} T_\alpha[f(t)] &= \lim_{\varepsilon \to 0} \frac{f^{(\lceil \alpha \rceil - 1)}(t + \varepsilon t^{(\lceil \alpha \rceil - \alpha)}) - f^{(\lceil \alpha \rceil - 1)}(t)}{\varepsilon}, \\ &= t^{(\lceil \alpha \rceil - \alpha)} f^{(\lceil \alpha \rceil)}(t). \end{aligned}$$

The relationship between the conformable derivative and first derivative can be represented,

$$T_\alpha f(t) = t^{1-\alpha} f'(t), \quad f \in C^1.$$

Definition 2.4. ([17]) The conformable exponential function is defined for every $t \geq 0$ by

$$E_\alpha(c,t) = exp(c\frac{t^\alpha}{\alpha}),$$

where $c \in \mathbb{R}$ and $0 < \alpha \leq 1$.

Definition 2.5. Let

$$A = \{f \in \mathscr{L}^p(\mathbb{R}^+)/\|f(t)\|_{p,\alpha} = \int_0^\infty |t^{\alpha-p} f(t)|dt \leq M; \tag{1}$$

for some, $M > 0, \quad \forall \quad p \in \mathbb{N} \quad$ and $\quad \alpha \in (p-1, p]\}$.
The Sumudu type IT of a function $f \in A$ of order $\alpha \in (0,1]$ is defined as,

$$\mathscr{S}_{\alpha}[f(t)](u) = F_{\alpha}(u) = \int_0^{\infty} \frac{1}{u} e^{\frac{-t^{\alpha}}{u\alpha}} t^{\alpha-1} f(t)dt,$$

where $\mathscr{S}_{\alpha}$ is Sumudu type IT operator and $u \in \mathbb{R}$.

In perspective of these outcomes, we have omit characterized the accompanying Sumudu type IT and studied its different applications.

3 Main results

In this section, we study Sumudu type IT and basic results.

Theorem 1. *Let $f \in A$ then Sumudu type IT $S_{\alpha}[f(t)](u) = F_{\alpha}(u)$ is bounded for all $u \in \mathbb{N}$.*

Proof: By Definition 2.5, we have

$$|F_{\alpha}(u)| = \int_0^{\infty} |\frac{1}{u} e^{\frac{-t^{\alpha}}{u\alpha}} t^{\alpha-1} f(t)dt| \leq \frac{1}{u} \int_0^{\infty} |e^{\frac{-t^{\alpha}}{u\alpha}}||t^{\alpha-1} f(t)|dt \leq \qquad \frac{1}{u} M_1 M_2 < \infty,$$

where

$$\begin{aligned} M_1 &= Sup|e^{-\frac{t^{\alpha}}{u\alpha}}| \quad for \quad t \in [0,\infty) \\ &= 1 \\ &\leq \frac{1}{u} \quad M_2, \end{aligned}$$

for some, $\quad M_3 \geq 0, \quad \frac{1}{u} \leq M_3, \quad$ for $\quad u \neq 0,$

where

$$M_2 = \frac{1}{u} \int_0^{\infty} |t^{\alpha-1} f(t)|dt \quad < M,$$

from the Definition 2.5.

Theorem 2. *Let $f, g \in A$, $\mathscr{S}_{\alpha}[f(t)](u) = F_{\alpha}(u)$, $\quad \mathscr{S}_{\alpha}[g(t)](u) = G_{\alpha}(u)$ and a, b are any scalars then*

$$\begin{aligned}
&(a) \qquad \mathscr{S}_{\alpha}[af(t) + bg(t)](u) = aF_{\alpha}(u) + bG_{\alpha}(u). \\
&(b) \qquad \mathscr{S}_{\alpha}[f(at)](u) = \frac{1}{a^{\alpha}} F_{\alpha}(\frac{1}{ua^{\alpha}}). \\
&(c) \qquad \mathscr{S}_{\alpha}[e^{\frac{iat^{\alpha}}{\alpha}} f(t)](u) = F_{\alpha}(\frac{1}{u} - ia). \\
&(d) \qquad \mathscr{S}_{\alpha}\left[f(t)cosh\frac{ax^{\alpha}}{\alpha}\right](u) = \frac{1}{2}[F_{\alpha}(\frac{1}{u} - a) + F_{\alpha}(\frac{1}{u} + a)]. \\
&(e) \qquad \mathscr{S}_{\alpha}\left[f(t)sinh\frac{ax^{\alpha}}{\alpha}\right](u) = \frac{1}{2}[F_{\alpha}(\frac{1}{u} - a) - F_{\alpha}(\frac{1}{u} + a)].
\end{aligned}$$

Theorem 3. *The Sumudu type IT of conformable derivative of $f \in \mathscr{L}^1(\mathbb{R}^+)$ of order $\alpha \in (0,1]$ is given by*

$$\mathscr{S}_{\alpha}[T_{\alpha}(f(t))](u) = \frac{1}{u} F_{\alpha}(u) - \frac{1}{u} f(0),$$

where $T_{\alpha}(f(t)) \in A$ and $\mathscr{S}_{\alpha}[f(t)](u) = F_{\alpha}(u)$.

Proof: Form the Definition 2.5, we have

$$\begin{aligned}\mathscr{S}_\alpha[T_\alpha f(t)](u) &= \int_0^\infty \frac{1}{u}e^{\frac{-t^\alpha}{u\alpha}}t^{\alpha-1}t^{1-\alpha}f^{'}(t)dt\\ &= -\frac{1}{u}f(0)+\frac{1}{u}\int_0^\infty \frac{1}{u}e^{\frac{-t^\alpha}{u\alpha}}t^{\alpha-1}f(t)dt\\ &= \frac{1}{u}F_\alpha(u)-\frac{1}{u}f(0).\end{aligned}$$

Note that,
(i) The Sumudu type IT of $e^{\frac{-at^\alpha}{\alpha}}$ for $a\in\mathbb{R}$ is given by

$$\mathscr{S}_\alpha[e^{\frac{-at^\alpha}{\alpha}}](u) = \int_0^\infty \frac{1}{u}e^{\frac{-(\frac{1}{u}+a)t^\alpha}{\alpha}}t^{\alpha-1}dt = \int_0^\infty \frac{1}{u}e^{\frac{-(\frac{1}{u}+a)x}{\alpha}}\frac{dx}{\alpha} = \frac{1}{(1+au)}. \tag{2}$$

(ii)The Sumudu type IT of $cos(\frac{at^\alpha}{\alpha})$ for $a\in\mathbb{R}$ is given by

$$\begin{aligned}\mathscr{S}_\alpha\left[f(t)cos\frac{at^\alpha}{\alpha}\right](u) &= \int_0^\infty \frac{1}{u}e^{\frac{-t^\alpha}{u\alpha}}\left(\frac{e^{\frac{iat^\alpha}{\alpha}}+e^{\frac{-iat^\alpha}{\alpha}}}{2}\right)t^{\alpha-1}dt\\ &= \frac{1}{2}\int_0^\infty \frac{1}{u}\left[e^{\frac{-(\frac{1}{u}-ia)t^\alpha}{\alpha}}+e^{\frac{-(\frac{1}{u}+ia)t^\alpha}{\alpha}}\right]t^{\alpha-1}dt\\ &= \frac{1}{1+a^2u^2}.\end{aligned} \tag{3}$$

(iii)The Sumudu type IT of $\frac{sin(\frac{at^\alpha}{\alpha})}{a}$ for $a\in\mathbb{R}$ is given by

$$\begin{aligned}\mathscr{S}_\alpha\left[f(t)\frac{sin(\frac{at^\alpha}{\alpha})}{a}\right](u) &= \int_0^\infty \frac{1}{u}e^{\frac{-t^\alpha}{u\alpha}}t^{\alpha-1}\frac{sin(\frac{at^\alpha}{\alpha})}{a}dt\\ &= \frac{1}{a}\left\{\int_0^\infty \frac{1}{u}e^{\frac{-t^\alpha}{u\alpha}}\left(\frac{e^{\frac{iat^\alpha}{\alpha}}-e^{\frac{-iat^\alpha}{\alpha}}}{2i}\right)t^{\alpha-1}dt\right\}\\ &= \frac{u}{1+a^2u^2}.\end{aligned}$$

(iv)The Sumudu type IT of

$$\delta(t)=\begin{cases}\infty, & t=0\\ 0; & t\neq 0\end{cases}$$

$$\mathscr{S}_\alpha[\delta(t)]=\tfrac{1}{u}.$$

Now for $f,g\in A$, we define the convolution $(f\circledast g)$ as

$$\begin{aligned}(f\circledast g)(t)&=u\int_0^t e^{\frac{-1}{u\alpha}(x^\alpha-t^\alpha)}\frac{1}{u}e^{\frac{-1}{u\alpha}(t-x)^\alpha}(t-x)^{\alpha-1}x^{\alpha-1}t^{1-\alpha}f(x)g(t-x)dx\\ &=ue^{\frac{1}{u\alpha}t^\alpha}t^{1-\alpha}\int_0^t\widehat{f}(x)\widehat{g}(t-x)dx\\ &=ue^{\frac{1}{u\alpha}t^\alpha}t^{1-\alpha}(\widehat{f}*\widehat{g})(t)\end{aligned}$$

where

$$(\widehat{f}*\widehat{g})(t)=\int_0^t\widehat{f}(x)\widehat{g}(t-x)dx$$

for $\widehat{f}(x)=ue^{\frac{-1}{u\alpha}x^\alpha}x^{\alpha-1}f(x)$ and $\widehat{g}(t-x)=\frac{1}{u}e^{\frac{-1}{u\alpha}(t-x)^\alpha}(t-x)^{\alpha-1}g(t-x)$.

Theorem 4.*(Convolution Theorem)*
Let $f,g\in A$ and $\mathscr{S}_\alpha[f(t)](u)=F_\alpha(u)$, $\mathscr{S}_\alpha[g(t)](u)=G_\alpha(u)$ then the Sumudu type IT of convolution $(f\circledast g)$ is given by

$$\mathscr{S}_\alpha[(f\circledast g)(t)](u)=uF_\alpha(u)G_\alpha(u).$$

Proof: Let $f,g\in A$ then from the Definition 2.5 and convolution $(f\circledast g)$, we have

$$\begin{aligned}\mathscr{S}_\alpha[(f\circledast g)(t)](u)&=\int_0^\infty\frac{1}{u}e^{\frac{-t^\alpha}{u\alpha}}t^{\alpha-1}\\ \times&\left(u\int_0^t e^{\frac{-1}{u\alpha}(x^\alpha-t^\alpha)}\frac{1}{u}e^{\frac{-1}{u\alpha}(t-x)^\alpha}(t-x)^{\alpha-1}x^{\alpha-1}t^{1-\alpha}f(x)g(t-x)dx\right)dt\\ &=u\int_0^\infty\frac{1}{u}e^{\frac{-x^\alpha}{u\alpha}}x^{\alpha-1}f(x)\int_x^\infty\frac{1}{u}e^{\frac{-1}{u\alpha}(t-x)^\alpha}(t-x)^{\alpha-1}g(t-x)dtdx\\ &=u\int_0^\infty\frac{1}{u}e^{\frac{-x^\alpha}{u\alpha}}x^{\alpha-1}f(x)\int_0^\infty\frac{1}{u}e^{\frac{-1}{u\alpha}(y)^\alpha}(y)^{\alpha-1}g(y)dydx=uF_\alpha(u)G_\alpha(u).\end{aligned}$$

Theorem 5.*The space $(A,\circledast)$ is commutative semigroup.*

Proof: Let $f,g,h\in A$, $(f\circledast g)=s$ and $(g\circledast h)=v$. We have to show that operation $\circledast$ is commutative and associative in A.

(i) Commutative:

$$\begin{aligned}(f\circledast g)(t)&=u\int_0^t e^{\frac{-1}{u\alpha}(x^\alpha-t^\alpha)}\frac{1}{u}e^{\frac{-1}{u\alpha}(t-x)^\alpha}(t-x)^{\alpha-1}x^{\alpha-1}t^{1-\alpha}f(x)g(t-x)dx\\ &=u\int_0^t e^{\frac{-1}{u\alpha}(y^\alpha-t^\alpha)}\frac{1}{u}e^{\frac{-1}{u\alpha}(t-y)^\alpha}(t-y)^{\alpha-1}y^{\alpha-1}t^{1-\alpha}f(t-y)g(y)dy\\ &=(g\circledast f)(t).\end{aligned}$$

(ii) Associative:

$$
\begin{aligned}
((f \circledast g) \circledast h)(t) &= (s \circledast h)(t) \\
&= ue^{\frac{1}{u\alpha}t^{\alpha}} t^{1-\alpha} \int_0^t \frac{1}{u} e^{\frac{-1}{\alpha}x^{\alpha}} e^{\frac{-1}{u\alpha}(t-x)^{\alpha}} (t-x)^{\alpha-1} x^{\alpha-1} s(x) h(t-x) dx \\
&= ue^{\frac{1}{u\alpha}t^{\alpha}} t^{1-\alpha} (\widehat{s} * \widehat{h})(t) = ue^{\frac{1}{u\alpha}t^{\alpha}} t^{1-\alpha} ((\widehat{f} * \widehat{g}) * \widehat{h})(t).
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
(f \circledast (g \circledast h))(t) &= ue^{\frac{1}{u\alpha}t^{\alpha}} t^{1-\alpha} (\widehat{f} * \widehat{v})(t) \\
&= ue^{\frac{1}{u\alpha}t^{\alpha}} t^{1-\alpha} (\widehat{f} * (\widehat{g} * \widehat{h}))(t) = ue^{\frac{1}{u\alpha}t^{\alpha}} t^{1-\alpha} ((\widehat{f} * \widehat{g}) * \widehat{h})(t).
\end{aligned}
$$

This implies $((f \circledast g) \circledast h)(t) = (f \circledast (g \circledast h))(t).$

4 Applications

In this section, we obtain the solution of IE, diffusion equation and heat-transfer problem involving conformable derivative using Sumudu type IT and the FT.

*Example 1.*For $s \in \mathscr{L}^1(\mathbb{R}^+)$ and $t > 0$, the solution of the conformable diffusion equation

$$
\begin{aligned}
T_{\alpha} s(x,t) &= \lambda s_{xx}(x,t), \qquad (4) \\
s(x,0) &= f(x), \qquad (5)
\end{aligned}
$$

is given by

$$
s(x,t) = \sqrt{\frac{\alpha}{4\pi\lambda t^{\alpha}}} \int_{-\infty}^{\infty} f(\xi) e^{\frac{-\alpha(x-\xi)^2}{4\lambda t^{\alpha}}} d\xi. \qquad (6)
$$

Solution: Let $\overline{s}(x,s)$ and $\widehat{s}(l,t)$ denote the Sumudu type IT and the FT of $s(x,t)$ respectively. $\widehat{f}(k)$ is the FT of $f(x)$. Now, taking Sumudu type IT and the FT of (6) and using (2) and (7) we get

$$
\begin{aligned}
\frac{1}{u}\overline{\widehat{s}}(l,u) - \frac{\widehat{f}(l)}{u} &= -\lambda l^2 \overline{\widehat{s}}, \\
\overline{\widehat{s}}(l,s) &= \frac{\widehat{f}(l)}{1+\lambda u l^2}.
\end{aligned}
$$

Taking inverse Sumudu type IT and using (3)

$$
\widehat{s}(l,t) = \widehat{f}(l) e^{\frac{-\lambda l^2 t^{\alpha}}{\alpha}} \qquad for \quad t > 0.
$$

Now using inverse FT, we get

$$
s(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(l) e^{\frac{-\lambda l^2 t^{\alpha}}{\alpha}} dl.
$$

From the convolution property of FT, we have

$$
s(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) g(x-\xi) d\xi,
$$

where $g(x) = \mathscr{F}^{-1}[e^{\frac{-\lambda l^2 t^\alpha}{\alpha}}] = \sqrt{\frac{\alpha}{2\lambda t^\alpha}} e^{\frac{-\alpha x^2}{4\lambda t^\alpha}}$ for $t > 0$.
The operator $\mathscr{F}^{-1}$ is inverse FT operator.
Finally, we see that the special case involving impulsive initial condition $u(x,0) = \delta(x)$ and the property of delta function given by,
$\int_{-\infty}^{\infty} \delta(\xi) y(\xi) d\xi = y(0)$, the solution reduces to (8)

$$s(x,t) = \sqrt{\frac{\alpha}{4\pi\lambda t^\alpha}} e^{\frac{-\alpha x^2}{4\lambda t^\alpha}}, \qquad t > 0. \tag{7}$$

*Example 2.*The solution of the conformable heat-transfer equation

$$-pN\theta(t) = \rho U a_h T_\alpha \theta(t) \tag{8}$$
$$\theta(0) = \eta, \qquad for \quad \eta \geq 0, \tag{9}$$

where ρ- density, U- volume, a_h- specific heat of material, p- convection heat-transfer coefficient, N- surface area of the body and $\theta \in \mathscr{L}^1[0,\infty)$, $0 < t < \infty; \alpha \in (0,1]$ is given by,

$$\theta(t) = \eta e^{\left(\frac{-pN}{\rho U a_h}\right)\frac{t^\alpha}{\alpha}}. \tag{10}$$

Solution: Let $\Theta_\alpha(u)$ denote the Sumudu type IT of $\theta(t)$. Taking Sumudu type IT of (10) and using (2) and (11), we get,

$$-pN\Theta_\alpha(u) = \rho U a_h \left[\frac{1}{u}\Theta_\alpha(u) - \frac{\eta}{u}\right]$$
$$\Theta_\alpha(u) = \frac{\eta}{u\left(\frac{1}{u} + \frac{pN}{\rho U a_h}\right)},$$
$$\Theta_\alpha(u) = \frac{\eta}{\left(1 + \frac{pNu}{\rho U a_h}\right)},$$

Taking the inverse Sumudu type IT of the above equation and using (2), we get the required solution.

$$\theta(t) = \eta e^{-\left(\frac{pN}{\rho U a_h}\right)\frac{t^\alpha}{\alpha}}. \tag{11}$$

5 Conclusion

The solutions of DE and FDE play a crucial role in mathematical physics. The classical ITs are not reduced conformable derivative. In the present paper, we introduced efficacious generalized Sumudu-type IT and obtained basic properties. This new IT reduced conformable derivative omit. We found a solution to conformable heat transfer equation omit using generalized Sumudu-type IT. This generalized Sumudu-type IT is useful to find the solutions to DEs, such as diffusion equations and wave equations involving conformable derivative.

Conflict of Interest

The authors declare that they have no conflict of interest.

References

[1] L. Debnath, Recent applications of fractional calculus to science and engineering, *Int. J. Math. Sci.* **54**, 3413-3442 (2003).
[2] B. Goodwine, *Engineering differential equations, theory and applications*, Springer, New York, USA, 2010.
[3] I. N. Sneddon, *The use of integral transforms*, McGraw-Hill International Editions, New York, 1972.
[4] I. Podlubny, *Fractional differential equations*, Academy Press, San-Diego,1999.
[5] A. Atangana and D. Baleanu, New fractional derivatives with non singular kernel, *Therm. Sci.* **20**, 757-763 (2016).
[6] A. A. Kilbas, H. M. Srivastva and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.
[7] G. K. Watugala, Sumudu transform an integral transform to solve differential equations and control engineering problems, *Int. J. Math. Educ. Sci. Tech.* **24**, 35-43 (1993).
[8] S. K. Panchal, P. V. Dole and A. D. Khandagale, k-Hilfer-Prbhakar fractional derivatives and its applications, *Ind. J. Math.* **59**, 367-383 (2017).
[9] G. A. Dorrego, Generalized Riemann-Liouville fractional operators associated with a generalization of the Prabhakar integral operator, *Progr. Fract. Differ. Appl.* **2**, 131-140 (2016).
[10] K. P. Ghadle and S. K. Magar, To determine oncentration of pollution and dissolved oxygen in river water using fractional advection-diffusion equation through Sumudu transform, *Ind. J. Envirom. Protect.* **39**, 271-276 (2019).
[11] F. B. M. Belgacem and A. A. Karaballi, Sumudu transform fundamental properties investigation and applications, *J. Appl. Nath. Stoch. Anal.* Article ID 91083, 1-23 (2006).
[12] R. Khalil, M. Al Horani, A. Yousef and M. Sababhehb, A new definition of fractional derivative, *J. Comput. Appl. Math.* **264**, 65-70 (2014).
[13] M. Bouaouid, K. Hilal and S. Melliani, Nonlocal telegraph equation in frame of the conformable time-fractional derivative, *Adv. Math. Phys.* **7** (2019).
[14] D. Zhaoa and T. Li, On conformable delta fractional calculus on time scales, *J. Math. Comput. Sci.* **16**, 324-335 (2016).
[15] F. Jarad and K. Tas, Application of Sumudu and double Sumudu transforms to Caputo-fractional differential equations, *J. Comput. Anal. Appl.* **14**, 475-483 (2012).
[16] X. J. Yang, A new integral transform with an application in heat-transfer, *Therm. Sci.* **20**, S677-S681 (2016).
[17] T. Abdeljawad, On conformable fractional calculus, *J. Comput. Appl. Math.* **279**, 57-66 (2015).