Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/090214

Truncated Power Lomax Distribution with Application to Flood Data

*Amal S. Hassan, Mohamed A. H. Sabry and A. Elsehetry**

Faculty of Graduate Studies for Statistical Research, Cairo University, Cairo, Egypt

Received: 29 Aug. 2019, Revised: 16 May 2020, Accepted: 14 Jun. 2020
Published online: 1 Jul. 2020

Abstract: In this paper, we propose the truncated power Lomax distribution. Fundamental properties of the new distribution, such as moments, moment generating and characteristic functions, quantile function, incomplete moments, Lorenz and Bonferroni curves, order statistics and Rényi entropy, are investigated. Maximum likelihood estimators are derived in case of complete sample, Type I and Type II censored samples. An approximate confidence interval of the parameters is obtained for large sample sizes. Simulation issue is executed to investigate the performance of estimates. The potential utility of the truncated power Lomax model is exhibited through flood data. The application indicates that the truncated power Lomax distribution can give better fits than some other corresponding distributions.

Keywords: Power Lomax distribution, Maximum likelihood method, Type I censoring and Type II censoring.

1 Introduction

A truncated distribution is defined as a conditional distribution that results from restricting the $f(t \mid a \leq T < b)$ domain of the statistical distribution. Hence, truncated distributions are used in cases where occurrences are limited to values, which lie above or below a given threshold or within a specified range.
Let T be a random variable from a distribution with a probability density function (pdf), say $f(t)$, cumulative distribution function (cdf), say $F(t)$, and the range of the support$(-\infty,\infty)$. The density function of T defined in $a < T < b$ is given by

$$f(t|a \leq T < b) = \begin{cases} \frac{g(t)}{G(b)-G(a)} & a \leq t < b \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

(see [1]). Because it is scaled up to account for the probability of being in the restricted support, this function is a density one. The restriction can occur, either on a single side of the range which is called singly truncated or on both sides of the range which is called doubly truncated. If occurrences are limited to values which lie below a given threshold, the lower (left) truncated distribution is obtained. Similarly, if occurrences are limited to values which lie above a given threshold, the upper (right) truncated distribution arises.

Several truncated distributions have been provided by many authors. Some of the recent truncated distributions are: truncated Weibull distribution [2], doubly truncated Fréchet distribution [3], generalized exponential truncated negative binomial distribution [4], truncated inverted generalized exponential distribution [5], right truncated normal distribution [6], and truncated Weibull Frèchet distribution [7].

[8] provided a heavy-tail probability distribution. The so-called Lomax (L) distribution is often used in business, economics, and actuarial modeling. It is widely applied in many areas, for instance, analysis of income and wealth data, modeling business failure data, biological sciences, model firm size and queuing problems (see [9], [10], [11] and [12]).

Truncated power Lomax (PL) distribution which is one of extended forms of L distribution. PL distribution was provided by [13] through employing power transformation to L distribution. The PL distribution accommodates both

* Corresponding author e-mail: Ah_sehetry@hotmail.com

inverted bathtub and decreasing hazard rate. The cdf and pdf of the PL distribution are defined respectively, by

$$G(t;\alpha,\beta,\lambda)=1-\lambda^{\alpha}\left(\lambda+t^{\beta}\right)^{-\alpha} \qquad t>0, \tag{2}$$

and

$$g(t;\alpha,\beta,\lambda) = \alpha\beta\lambda^{\alpha}t^{\beta-1}\left(\lambda+t^{\beta}\right)^{-(\alpha+1)}, \tag{3}$$

where, $\alpha,\beta>0$ are two shape parameters (Ps) and $\lambda>0$ is a scale parameter. Extended forms of PL distribution have been provided by several authors(see for example [14] and [15]).

In this article, we are motivated to define a new truncated distribution referred to right truncated PL (RTPL) distribution. We obtain some main properties of the new distribution and discuss maximum likelihood estimation of its Ps based on complete and censored samples. This paper is organized, as follows: The pdf, cdf, and hazard rate function (hrf) of the RTPL model are defined in Section 2. Section 3 provides some statistical properties of RTPL distribution. The maximum likelihood estimators and simulation study are presented in Section 4. The application of RTPL distribution to a real data set is presented in Section 5. Conclusion is presented in Section 6.

2 Right Truncated Power Lomax Distribution

In this section, we introduce the information of the RTPL distribution.

Definition: *A random variable X is said to have the RTPL distribution (or [0,1] RTPL distribution) with shape Ps $\alpha,\ \beta$ and scale parameter $\lambda=1$, if its pdf is constructed by employing (1) for $a=0,\ b=1$ with cdf (2) and pdf (3) as follows:*

$$f_{RTPL}(x;\alpha,\ \beta)=\frac{g(x;\alpha,\ \beta,1)}{G(1;\alpha,\ \beta,1)-G(0;\alpha,\ \beta,1)}=\frac{\alpha\beta x^{\beta-1}\left(1+x^{\beta}\right)^{-(\alpha+1)}}{1-2^{-\alpha}},\ 0<x<1. \tag{4}$$

A random variable X with density (4) is denoted by $X\sim$ RTPL(α,β). The cdf related to (4) is given by

$$F_{RTPL}(x;\alpha,\ \beta) = \frac{G(x;\alpha,\ \beta,1)-G(0;\alpha,\ \beta,1)}{G(1;\alpha,\ \beta,1)-G(0;\alpha,\ \beta,1)}=\frac{1-\left(1+x^{\beta}\right)^{-\alpha}}{1-2^{-\alpha}}. \tag{5}$$

An important new model is obtained for $\beta=1$, which is called the right truncated Lomax (TL) distribution or we call it [0,1] TL distribution.

Depending on pdf (4) and cdf (5), the survival function and hazard rate function (hrf) are given by

$$\overline{F}_{RTPL}(x;\alpha,\ \beta)=\frac{\left(1+x^{\beta}\right)^{-\alpha}-2^{-\alpha}}{1-2^{-\alpha}},$$

and,

$$h_{RTPL}(x;\alpha,\ \beta)=\frac{\alpha\beta x^{\beta-1}\left(1+x^{\beta}\right)^{-(\alpha+1)}}{\left(1+x^{\beta}\right)^{-\alpha}-2^{-\alpha}}.$$

Fig. 1 displays a variety of possible shapes of pdf and hrf of RTPL distribution for some selected values of Ps. It can be detected a right skewed, unimodal and reversed J shaped. Also, the shape of the hrf of the RTPL distribution could be increasing, J shaped and U-shaped.

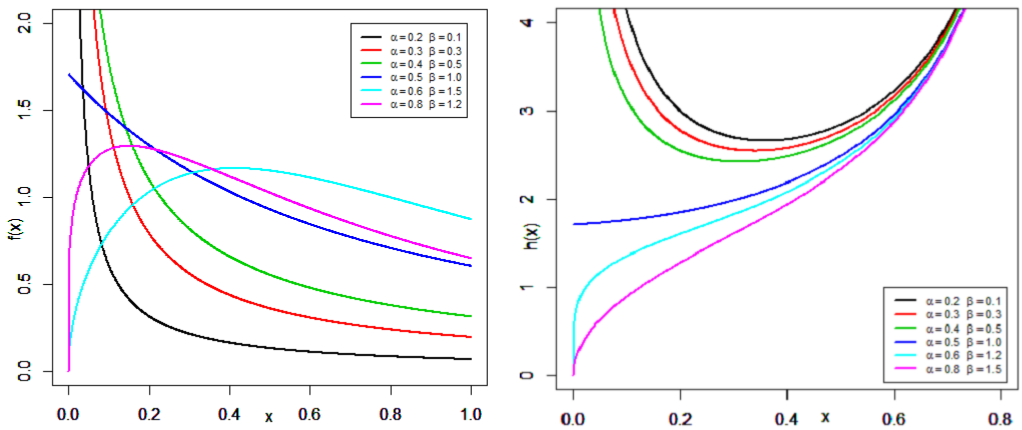

Fig. 1: The pdf and hrf of RTPL distribution for different values of parameters

In addition, the reversed hrf and cumulative hrf are obtained respectively, as follows:

$$\tau_{RTPL}(x;\alpha,\ \beta) = \frac{\alpha\beta x^{\beta-1}\left(1+x^{\beta}\right)^{-(\alpha+1)}}{1-\left(1+x^{\beta}\right)^{-\alpha}},$$

and,

$$H_{RTPL}(x;\alpha,\beta) = -ln\left(\frac{\left(1+x^{\beta}\right)^{-\alpha}-2^{-\alpha}}{1-2^{-\alpha}}\right).$$

The quantile function denoted by $Q(u)$ of random variable X which has a cdf (5) is obtained, as follows:

$$Q(u) = \left[\left[1-u(1-2^{-\alpha})\right]^{-\frac{1}{\alpha}}-1\right]^{\frac{1}{\beta}}, \tag{6}$$

where, the random variable u belongs to the uniform distribution on $[0,\ 1]$. The 1^{st}, 2^{nd}, and 3^{rd} quartiles are obtained from (6) by taking u $=$ $0.25,\ 0.5$ and 0.75 respectively.

3 Basic Properties

In this section, we present some statistical properties of RTPL distribution.

3.1 Moments

Since the moments are substantial in any statistical analysis, we derive the r^{th} moment of the RTPL distribution. If X has the pdf (4), $\grave{\mu}_r$ is obtained, as follows:

$$\grave{\mu}_r = \int_0^1 x^r f_{RTPL}(x;\alpha,\ \beta)dx = \frac{\alpha\beta}{1-2^{-\alpha}}\int_0^1 x^{r+\beta-1}\left(1+x^{\beta}\right)^{-(\alpha+1)}\ dx. \tag{7}$$

We employ the following binomial expansion

$$(1+Z)^{-\theta} = \sum_{j=0}^{\infty}(-1)^j\binom{\theta+j-1}{j}Z^j, \tag{8}$$

for $\left(1+x^{\beta}\right)^{-(\alpha+1)}$ as follows:

$$\left(1+x^{\beta}\right)^{-(\alpha+1)} = \sum_{j=0}^{\infty}(-1)^j\binom{\alpha+j}{j}x^{\beta j}. \tag{9}$$

Substituting (9) in (7), then

$$\grave{\mu}_r = A^* \int_0^1 x^{r+\beta(j+1)-1} dx = \frac{A^*}{r+\beta(j+1)}, \tag{10}$$

where, $A^* = \frac{\alpha\beta}{1-2^{-\alpha}} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha+j}{j}$.
Furthermore, the moment generating function (mgf) and characteristic function (chf) of the RTPL distribution, respectively, are derived, as follows:

$$M_X(t) = E\left(e^{tX}\right) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \grave{\mu}_r = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{A^*}{[r+\beta(j+1)]},$$

and,

$$\phi_x(t) = E\left(e^{itX}\right) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \frac{A^*}{[r+\beta(j+1)]}.$$

3.2 Incomplete moments and Inequality measures

The incomplete moments are used in several statistical areas especially in the income distribution, for measuring inequality, such as income quintiles, the Lorenz curve, Pietra and Gini measures of inequality. The s^{th} incomplete moment of the RTPL distribution is obtained, as follows:

$$\Phi_s(t) = \int_0^t x^s f_{RTPL}(x;\alpha,\beta)\, dx = A^* \int_0^t x^{s+\beta(j+1)-1} dx,$$

then,

$$\Phi_s(t) = A^* \frac{t^{s+\beta(j+1)}}{s+\beta(j+1)}.$$

Therefore, inequality measures are often calculated for distributions other than expenditure. The Lorenz $[L_F(z)]$ and Bonferroni $[B_F(z)]$ curves are calculated as below:

$$L_F(z) = \frac{\int_0^z x\, f_{RTPL}(x;\alpha,\beta)\, dx}{E(Z)} = z^{1+\beta(j+1)},$$

and

$$B_F(z) = \frac{L_F(z)}{F_{RTPL}(z;\alpha,\beta)} = \frac{z^{1+\beta(j+1)}\left(1-2^{-\alpha}\right)}{1-\left(1+z^{\beta}\right)^{-\alpha}}.$$

3.3 Rényi entropy

Entropy is a measure of the variation of the uncertainty associated with a distribution of a random variable X. The entropy of a random variable X is defined by

$$I_\delta(x) = \frac{1}{1-\delta} \log\left[\int_R f(x)^\delta\, dx\right], \tag{11}$$

where, $\delta > 0$, $\delta \neq 1$. Based on pdf (4), $f_{RTPL}(x;\alpha,\beta)^\delta$ can be formed, as follows:

$$f_{RTPL}(x;\alpha,\beta)^\delta = \left(\frac{\alpha\beta}{1-2^{-\alpha}}\right)^\delta (x)^{\delta(\beta-1)}(1+x^\beta)^{-\delta(\alpha+1)}. \tag{12}$$

Thus, the Rényi entropy of RTPL distribution is obtained by substituting (12) in (11), as follows:

$$I_\delta(x) = \frac{1}{1-\delta} \log\left[\frac{(\alpha\beta)^\delta}{(1-\delta)(1-2^{-\alpha})^\delta}\right] \int_0^1 (x)^{\delta(\beta-1)}(1+x^\beta)^{-\delta(\alpha+1)} dx. \tag{13}$$

Then, employing binomial expansion (8) in (13), we get

$$I_\delta(x) = \frac{1}{1-\delta} \log\left[\sum_{i=0}^{\infty} (-1)^i \binom{\delta(\alpha+1)+i-1}{i} \frac{(\alpha\beta)^\delta}{(1-2^{-\alpha})^\delta} \int_0^1 (x)^{\beta(i+\delta)-\delta}\, dx\right].$$

Hence, the Rényi entropy of RTPL distribution is given by:

$$I_{\delta}(x) = \frac{1}{1-\delta}\log\left[\sum_{i=0}^{\infty}\frac{w_i}{\beta(i+\delta)-\delta+1}\right],$$

where,

$$w_i = (-1)^i \binom{\delta(\alpha+1)+i-1}{i} \frac{(\alpha\beta)^{\delta}}{(1-2^{-\alpha})^{\delta}}.$$

3.4 Order statistics

Let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ be the order statistics of a random sample of size n, then the pdf of the k^{th} order statistic is given by:

$$f_{x_{(k)}}(x) = \frac{f(x)}{B(k,n-k+1)}\sum_{v=0}^{n-k}(-1)^v\binom{n-k}{v}F(x)^{v+k-1}, \tag{14}$$

where, $B(.,.)$ is the beta function.

The pdf of the k^{th} order statistic of RTPL distribution is obtained by substituting (4) and (5) in (14), as follows:

$$f_{RTPL_{x_{(k)}}}(x;\alpha,\beta) = \sum_{v=0}^{n-k}\frac{(-1)^v\,\alpha\beta\binom{n-k}{v}x^{(\beta-1)}}{(1-2^{-\alpha})^{v+k}B(k,n-k+1)}(1+x^{\beta})^{-\alpha-1}\left(1-\left(1+x^{\beta}\right)^{-\alpha}\right)^{v+k-1}.$$

Employing the following binomial expansion

$$(1-Z)^{\theta} = \sum_{u=0}^{\theta}(-1)^u\binom{\theta}{u}Z^u,$$

for $\left(1-(1+x^{\beta})^{-\alpha}\right)^{v+k-1}$ as follows:

$$\left(1-\left(1+x^{\beta}\right)^{-\alpha}\right)^{v+k-1} = \sum_{u=0}^{v+k-1}(-1)^u\binom{v+k-1}{u}\left(1+x^{\beta}\right)^{-\alpha u}.$$

$$f_{RTPL_{x_{(k)}}}(x;\alpha,\beta) = \sum_{v=0}^{n-k}\sum_{u=0}^{v+k-1}\frac{(-1)^{v+u}\,\alpha\beta\binom{n-k}{v}\binom{v+k-1}{u}}{(1-2^{-\alpha})^{v+k}B(k,n-k+1)}x^{(\beta-1)}\left(1+x^{\beta}\right)^{-\alpha(u+1)-1}.$$

Using binomial expansion in (8), the pdf $f_{RTPL_{x_{(k)}}}$ can be written as

$$f_{RTPL_{x_{(k)}}}(x;\alpha,\beta) = \sum_{v=0}^{n-k}\sum_{u=0}^{v+k-1}\sum_{m=0}^{\infty}\frac{(-1)^{v+u+m}\,\alpha\beta\binom{n-k}{v}\binom{v+k-1}{u}\binom{\alpha(u+1)+m}{m}}{(1-2^{-\alpha})^{v+k}B(k,n-k+1)}(x)^{\beta(m+1)-1}.$$

Consequently, the pdf of the k^{th} order statistic of RTPL distribution is, as follows:

$$f_{RTPL_{x_{(k)}}}(x;\alpha,\beta) = \sum_{m=0}^{\infty}w_m x^{\beta(m+1)-1},$$

where,

$$w_m = \sum_{v=0}^{n-k}\sum_{u=0}^{v+k-1}\frac{(-1)^{v+u+m}\,\alpha\beta\binom{n-k}{v}\binom{v+k-1}{u}\binom{\alpha(u+1)+m}{m}}{(1-2^{-\alpha})^{v+k}B(k,n-k+1)}.$$

Furthermore, the r^{th} moment of k^{th} order statistics for RTPL distribution is given by

$$E\left(X^r{}_{(k)}\right) = \sum_{m=0}^{\infty}w_m\int_0^1 x^{r+\beta(m+1)-1}\,dx$$

$$= \sum_{m=0}^{\infty}\frac{w_m}{r+\beta(m+1)}.$$

4 Estimation and Simulation Study

In this section, maximum likelihood (ML) estimators of the model parameters are derived in case of complete, Type I and Type II censored samples. Approximate confidence intervals are also obtained. Moreover, numerical study is provided.

4.1 Parameter Estimation Based on Censoring Samples

In reliability or lifetime testing experiments, most of the encountered data are censored due to various reasons, such as time limitation, cost, or other resources. Here, we discuss estimation of population Ps of the RTPL distribution based on two censoring schemes; namely, Type I and Type II. In Type-I censoring (TIC), we have a fixed time say; ω, but the number of items which fails during the experiment is random. Whereas, in Type-II censoring (TIIC) scheme, the experiment is continued (i.e. time varies) until the specified number of failures c occurs.

4.1.1 ML estimators in case of TIC

Suppose that $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ is a TIC sample of size c whose life time's follow RTPL distribution (4) is placed on a life test and the test is terminated at determined time ω before all n items have failed. The number of failures c and all failure times are random variables. The likelihood function, based on TIC for RTPL model, is given by

$$L_1 = \frac{n!}{(n-c)!}\left[1-\frac{1-S_{\omega}{}^{-\alpha}}{1-2^{-\alpha}}\right]^{n-c}\left\{\prod_{i=1}^{c}\left[\frac{\alpha\beta x_i^{\beta-1}S_i{}^{-(\alpha+1)}}{1-2^{-\alpha}}\right]\right\},$$

where, $S_i=\left(1+x_i^{\beta}\right)$, and $S_{\omega}=\left(1+\omega^{\beta}\right)$. For simplicity, we write S_i instead of $S_{(i)}$. Then, the log-likelihood function is obtained, as follows:

$$LnL_1 = ln\left[\frac{n!}{(n-c)!}\right] + (n-c)\ ln\left[1-\frac{1-S_{\omega}{}^{-\alpha}}{1-2^{-\alpha}}\right]$$

$$-cln\left(1-2^{-\alpha}\right) + cln\alpha + cln\beta + (\beta-1)\sum_{i=1}^{c} lnx_i - (\alpha+1)\sum_{i=1}^{c} lnS_i\ .$$

Then, the first partial derivatives of the log-likelihood with respect to the unknown Ps are given by

$$\frac{\partial lnL_1}{\partial\alpha} = \frac{c}{\alpha} - \frac{2^{-\alpha}c\ ln\ 2}{1-2^{-\alpha}} + \frac{(n-c)\left(-\frac{S_{\omega}{}^{-\alpha}ln\ S_{\omega}}{1-2^{-\alpha}} + \frac{2^{-\alpha}\ln 2\ \left(1-S_{\omega}{}^{-\alpha}\right)}{(1-2^{-\alpha})^2}\right)}{1-\frac{1-S_{\omega}{}^{-\alpha}}{1-2^{-\alpha}}} - \sum_{i=1}^{c} lnS_i\ ,$$

and,

$$\frac{\partial lnL_1}{\partial\beta} = \frac{c}{\beta} - \frac{(n-c)\,\alpha\omega^{\beta}\ln\omega\ (S_{\omega})^{-1-\alpha}}{(1-2^{-\alpha})\left(1-\frac{1-(S_{\omega})^{-\alpha}}{1-2^{-\alpha}}\right)} + \sum_{i=1}^{c}\ln x_i - (1+\alpha)\sum_{i=1}^{c}\frac{x_i^{\beta}\ln x_i}{S_i}.$$

Equating these partial derivatives with zeros and solving simultaneously yield the ML estimators of α and β based on TIC samples.

4.1.2 ML estimators in case of TIIC

If we want to ensure that the resulting data set contains a fixed number c of observed lifetimes and terminate the test as fast as possible, the design must allow for the test to terminate at the c^{th} failure such that
$X_{(1)} < X_{(2)} < \ldots < X_{(c)}$ is a TIIC sample of size n observed from lifetime testing experiment. The likelihood function of RTPL model, based on TIIC, is given by

$$L_2 = \frac{n!}{(n-c)!}\left[1-\frac{1-S_c{}^{-\alpha}}{1-2^{-\alpha}}\right]^{n-c}\left\{\prod_{i=1}^{c}\left[\frac{\alpha\beta x_i^{\beta-1}S_i{}^{-(\alpha+1)}}{1-2^{-\alpha}}\right]\right\}.$$

where, $S_i = \left(1+x_i^{\beta}\right)$, and $S_c = \left(1+c^{\beta}\right)$. Also, for simplicity, we write S_i instead of $S_{(i)}$. Then, the log-likelihood function, based on TIIC, is given by

$$LnL_2 = ln\left[\frac{n!}{(n-c)!}\right] + (n-c)\ln\left[1-\frac{1-(S_c)^{-\alpha}}{1-2^{-\alpha}}\right] - c\ln\left(1-2^{-\alpha}\right) + c\ln\alpha$$
$$+c\ln\beta + (\beta-1)\sum_{i=1}^{c}\ln x_i - (\alpha+1)\sum_{i=1}^{c}\ln S_i\,.$$

Then, the first partial derivatives of the log-likelihood are given by

$$\frac{\partial lnL_2}{\partial\alpha} = \frac{c}{\alpha} - \frac{2^{-\alpha}c\,ln\,2}{1-2^{-\alpha}} + \frac{(n-c)\left(-\frac{S_c{}^{-\alpha}ln\,S_c}{1-2^{-\alpha}} + \frac{2^{-\alpha}\ln 2\,\left(1-S_c{}^{-\alpha}\right)}{(1-2^{-\alpha})^2}\right)}{1-\frac{1-S_c{}^{-\alpha}}{1-2^{-\alpha}}} - \sum_{i=1}^{c} lnS_i,$$

and ,

$$\frac{\partial lnL_2}{\partial\beta} = \frac{c}{\beta} - \frac{(n-c)\,\alpha\,x_c{}^{\beta}\ln x_c\,(S_c)^{-1-\alpha}}{(1-2^{-\alpha})\left(1-\frac{1-(S_c)^{-\alpha}}{1-2^{-\alpha}}\right)} + \sum_{i=1}^{c}\ln x_i - (1+\alpha)\sum_{i=1}^{c}\frac{x_i^{\beta}\ln x_i}{S_i}.$$

Solving $\partial lnL_2/\partial\alpha = 0$ and $\partial lnL_2/\partial\beta = 0$ numerically using iteration technique, the ML estimators of α and β are obtained via Mathematica 7.
In addition, for $c = n$, we obtain the ML estimator under complete samples as seen in Tables 3 and 4.

For interval estimation of the Ps, it is known that under regularity condition, the asymptotic distribution of ML estimators of elements of unknown Ps for α and β is given by

$$(\widehat{\alpha}-\alpha),\left(\widehat{\beta}-\beta\right) \to N\left(0, I^{-1}(\alpha,\beta)\right),$$

where, $I^{-1}(\alpha,\beta)$ is the variance covariance matrix of unknown Ps α and β. The elements of Fisher information matrix are obtained for both censoring schemes. Therefore, the two-sided approximate $\gamma 100$ percent limits for the ML estimates of a population Ps for α and β can be obtained, respectively, as follows:

$$L_\alpha = \widehat{\alpha} - z_{\frac{\gamma}{2}}\sqrt{var(\widehat{\alpha})},\quad U_\alpha = \widehat{\alpha} + z_{\frac{\gamma}{2}}\sqrt{var(\widehat{\alpha})},$$

and,

$$L_\beta = \widehat{\beta} - z_{\frac{\gamma}{2}}\sqrt{var\left(\widehat{\beta}\right)},\quad U_\beta = \widehat{\beta} + z_{\frac{\gamma}{2}}\sqrt{var\left(\widehat{\beta}\right)},$$

where $z_{\frac{\gamma}{2}}$ is the $100(1-\gamma/2)\%$, the standard normal percentile and var(.)'s denote the diagonal elements of variance covariance matrix corresponding to the model Ps.

4.2 Simulation Studies

Here, we provide a numerical study to evaluate the behavior of the ML estimates of the RTPL based on complete sample, TIC and TIIC schemes. The algorithm used here is outlined, as follows:

- 1000 random sample of sizes n=30, 50, 100 and 150 are generated from the RTPL distribution under TIC and TIIC.
- Exact values of Ps, such as $(\alpha = 0.7,\ \beta = 1.2)$ and $(\alpha = 1.5,\ \beta = 0.5)$, are chosen.
- Three termination times, as $\omega = 0.7$, $\omega = 0.9$ and $\omega = 1$, are selected based on TIC and the number of failure items; c, based on TIIC, are selected as 70%, 90% and 100% (complete sample).
- The following measures are calculated
  1. Average ML of the simulated estimates $\widehat{\alpha}_i$ and $\widehat{\beta}_i, i = 1,2,\ldots,N$, where N=1000
  $$\frac{1}{N}\sum_{i=1}^{N}\widehat{\alpha}_i \text{ and } \frac{1}{N}\sum_{i=1}^{N}\widehat{\beta}_i.$$
  2. Average bias of the simulated estimates $\widehat{\alpha}_i$ and $\widehat{\beta}_i, i = 1,2,\ldots,N$:
  $$\frac{1}{N}\sum_{i=1}^{N}(\widehat{\alpha}_i-\alpha) \text{ and } \frac{1}{N}\sum_{i=1}^{N}(\widehat{\beta}_i-\beta).$$

3.Average mean square error (MSE) of the simulated estimates $\widehat{\alpha}_i$ and $\widehat{\beta}_i, i = 1, 2, \ldots, N$:

$$\frac{1}{N}\sum_{i=1}^{N}(\widehat{\alpha}_i-\alpha)^2 \quad \text{and} \quad \frac{1}{N}\sum_{i=1}^{N}(\widehat{\beta}_i-\beta)^2.$$

4.Average length of the N simulated confidence intervals and coverage probability with confidence level γ = 90% and 95% for $\widehat{\alpha}_i$ and $\widehat{\beta}_i, i = 1, 2, \ldots, N$ are calculated.

–Numerical results are listed in Tables 1, 2, 3 and 4.

From Tables 1- 4 we conclude that

–As the sample size n increases, the MSE of ML estimates decrease.

–As the termination time ω increases, the MSE of estimates decreases.

–Tables 1 and 2, indicate that as the sample size n increases, the MSE of estimates decreases.

–Tables 3 and 4, show that as the censoring level time ω increases, the MSE of estimates decreases

–Tables 1 to 4 reveal that the coverage probability is very close to the intended significance level for all values of n, α and β.

–The average length of confidence intervals for the unknown Ps decreases as n increases.

Table 1: ML estimates, Biases, MSE, Average length and Coverage probability of RTPL distribution under TIC for $\alpha = 0.7$, $\beta = 1.2$

n	ω	Ps	ML	Bias	MSE	Average length		Coverage probability	
						90%	95%	90%	95%
30	0.7	α	1.3049	0.60495	2.16129	5.39397	6.42686	92.70	95.70
		β	1.3294	0.12942	0.11408	1.0995	1.31004	91.40	94.40
	0.9	α	1.2314	0.53147	1.90102	5.25564	6.26204	92.80	96.20
		β	1.3243	0.12431	0.11259	1.08861	1.29707	91.70	95.30
	1	α	1.25711	0.55710	1.89013	5.27123	6.28061	92.90	96.90
		β	1.32262	0.12262	0.10315	1.0896	1.29824	92.90	95.50
50	0.7	α	1.06158	0.36158	1.17697	4.20371	5.00868	93.20	96.50
		β	1.27598	0.07598	0.06403	0.85460	1.01825	90.30	94.90
	0.9	α	1.06092	0.36091	1.0861	4.09334	4.87717	93.40	96.60
		β	1.27423	0.07422	0.06189	0.84044	1.00138	90.40	95.30
	1	α	1.08181	0.38180	1.06115	4.10358	4.88937	93.50	96.80
		β	1.27635	0.07635	0.06006	0.84154	1.00269	90.60	95.60
100	0.7	α	0.88408	0.18408	0.65140	2.99325	3.56643	92.40	96.60
		β	1.22983	0.02983	0.03175	0.60378	0.71940	90.50	95.00
	0.9	α	0.88735	0.18735	0.60037	2.92634	3.4867	92.50	96.80
		β	1.24234	0.04234	0.03144	0.60156	0.71675	90.90	95.40
	1	α	0.84577	0.14577	0.57376	2.93274	3.49433	92.70	96.80
		β	1.23644	0.03644	0.03127	0.60320	0.71870	91.00	95.60
150	0.7	α	0.80720	0.10720	0.39591	2.46112	2.93239	93.40	96.30
		β	1.2221	0.02210	0.02158	0.49953	0.59519	90.70	95.40
	0.9	α	0.80339	0.10339	0.37256	2.40556	2.8662	93.90	96.50
		β	1.22491	0.02491	0.02047	0.49399	0.58858	91.20	95.80
	1	α	0.79901	0.09900	0.35543	2.40165	2.86154	94.10	96.70
		β	1.21756	0.01755	0.02000	0.49069	0.58466	91.40	95.80

Table 2: ML estimates, Biases, MSE, Average length and Coverage probability of RTPL distribution under TIC for $\alpha = 1.5$, $\beta = 0.5$

n	ω	Ps	ML	Bias	MSE	Average length		Coverage probability	
						90%	95%	90%	95%
30	0.7	α	1.81484	0.31484	2.27123	5.27208	6.28162	92.20	95.60
		β	0.52370	0.02370	0.02095	0.39627	0.47215	81.20	87.10
	0.9	α	1.86368	0.36368	2.25639	5.27193	6.28144	92.60	95.70
		β	0.51691	0.01691	0.01975	0.38833	0.46269	82.10	87.20
	1	α	1.86767	0.36767	2.24422	5.27073	6.28002	92.80	96.20
		β	0.52159	0.02159	0.01926	0.39237	0.46750	82.70	87.80
50	0.7	α	1.63747	0.13747	1.43995	4.05412	4.83044	93.10	95.90
		β	0.49882	0.0011	0.01441	0.29895	0.35619	82.10	87.00
	0.9	α	1.61128	0.1112	1.41068	4.07381	4.8539	93.70	96.20
		β	0.49470	0.0052	0.01434	0.29840	0.35554	84.90	87.30
	1	α	1.59185	0.09184	1.4237	4.05908	4.83635	93.90	96.30
		β	0.49646	0.00353	0.01367	0.30035	0.35787	84.90	87.60
100	0.7	α	1.52734	0.02733	0.84295	2.8694	3.41886	93.70	96.00
		β	0.48805	0.01194	0.00984	0.21179	0.25234	84.30	87.50
	0.9	α	1.52061	0.02060	0.80220	2.85937	3.40691	93.70	97.60
		β	0.49002	0.00997	0.00954	0.21263	0.25335	84.70	87.70
	1	α	1.49764	0.00236	0.79306	2.86683	3.4158	94.00	97.60
		β	0.48529	0.01470	0.00937	0.21103	0.25145	85.30	87.70
150	0.7	α	1.53258	0.03258	0.59020	2.33714	2.78468	94.10	96.90
		β	0.49430	0.00569	0.00705	0.17565	0.20928	84.70	87.90
	0.9	α	1.50101	0.00101	0.58668	2.33487	2.78197	94.10	97.30
		β	0.48838	0.01161	0.00703	0.17413	0.20748	84.70	88.20
	1	α	1.46205	0.03795	0.56074	2.34002	2.78811	94.90	97.70
		β	0.48836	0.01163	0.00691	0.17497	0.20847	85.30	88.40

Table 3: ML estimates, Biases, MSE, Average length and Coverage probability of RTPL distribution under TIIC for $\alpha = 0.7$, $\beta = 1.2$

n	X_c	Ps	ML	Bias	MSE	Average length		Coverage probability	
						90%	95%	90%	95%
30	70%	α	1.4762	0.7762	3.1093	5.83458	6.95184	90.10	96.70
		β	1.3688	0.1688	0.1450	1.15777	1.37948	90.30	94.50
	90%	α	1.2831	0.5831	2.1701	5.32507	6.34477	90.80	97.40
		β	1.3317	0.1317	0.1173	1.09728	1.30739	90.40	94.70
	100%	α	1.2144	0.5144	1.7473	5.27486	6.28494	91.10	97.60
		β	1.3143	0.1143	0.1014	1.08933	1.29792	91.10	95.30
50	70%	α	1.1429	0.4429	1.4844	4.43176	5.2804	90.50	96.60
		β	1.2737	0.0737	0.0629	0.87420	1.04161	91.20	95.10
	90%	α	1.0432	0.3432	1.1968	4.11599	4.90416	90.70	96.90
		β	1.2785	0.0785	0.0617	0.84754	1.00985	92.00	95.70
	100%	α	1.0817	0.3817	1.1821	4.107	4.89345	91.20	96.90
		β	1.2819	0.0819	0.0615	0.84471	1.00647	92.50	95.70
100	70%	α	0.9270	0.2270	0.7363	3.11156	3.7074	91.10	97.20
		β	1.2469	0.0469	0.0389	0.62349	0.74288	92.00	95.20
	90%	α	0.8650	0.1650	0.5758	2.94135	3.50458	92.10	97.20
		β	1.2306	0.0306	0.0307	0.60008	0.71499	92.10	95.70
	100%	α	0.8824	0.1824	0.5666	2.92473	3.48478	92.80	97.30
		β	1.2401	0.0401	0.0304	0.60101	0.71610	92.50	96.10
150	70%	α	0.8412	0.1412	0.4779	2.53743	3.02332	92.10	97.30
		β	1.2260	0.0260	0.0228	0.50817	0.60548	92.10	96.10
	90%	α	0.7969	0.0969	0.4086	2.41101	2.8727	92.20	97.80
		β	1.2208	0.0208	0.0209	0.49322	0.58767	93.20	96.10
	100%	α	0.8320	0.1320	0.4023	2.39282	2.85102	93.80	98.10
		β	1.2240	0.0240	0.0203	0.48953	0.58327	93.30	97.30

Table 4: ML estimates, Biases, MSE, Average length and Coverage probability of RTPL distribution under TIIC for $\alpha = 1.5$,$\beta = 0.5$

n	X_c	Ps	ML	Bias	MSE	Average length		Coverage probability	
						90%	95%	90%	95%
30	70%	α	2.0398	0.5398	4.0066	6.11824	7.28982	90.70	93.60
		β	0.5286	0.0286	0.0252	0.42254	0.50345	80.60	85.60
	90%	α	1.9408	0.4408	2.7767	5.37209	6.40079	90.90	93.80
		β	0.5324	0.0324	0.0241	0.40034	0.47700	82.20	86.60
	100%	α	1.8474	0.3474	2.2934	5.24565	6.25014	92.70	94.80
		β	0.5276	0.0276	0.0209	0.39632	0.47221	82.30	86.90
50	70%	α	1.8428	0.3428	2.5558	4.58398	5.46177	93.30	93.80
		β	0.5100	0.0100	0.0173	0.32036	0.38171	85.90	86.60
	90%	α	1.6053	0.1053	1.5871	4.11046	4.89757	94.00	94.50
		β	0.4958	0.0041	0.0148	0.30164	0.35940	86.00	87.10
	100%	α	1.6965	0.1965	1.5282	4.06285	4.84084	94.70	95.60
		β	0.5035	0.0035	0.0148	0.30026	0.35776	88.00	88.40
100	70%	α	1.6164	0.1164	0.9687	3.11268	3.70873	93.30	93.80
		β	0.4997	0.0003	0.0089	0.22580	0.26904	87.30	91.10
	90%	α	1.5581	0.0581	0.8532	2.88297	3.43502	94.00	95.10
		β	0.4937	0.0062	0.0085	0.21418	0.25520	88.60	91.40
	100%	α	1.5085	0.0085	0.8379	2.86279	3.41098	95.20	97.00
		β	0.4900	0.0099	0.0081	0.21304	0.25384	90.60	92.80
150	70%	α	1.5862	0.0862	0.6713	2.52782	3.01187	93.50	95.40
		β	0.5000	.00007	0.0064	0.18555	0.22109	87.40	92.70
	90%	α	1.5176	0.0175	0.5925	2.33973	2.78777	93.60	95.40
		β	0.4916	0.0083	0.0062	0.17521	0.20876	90.10	93.40
	100%	α	1.4846	0.0153	0.5766	2.33212	2.7787	94.10	97.60
		β	0.4884	0.0115	0.0061	0.17432	0.20771	91.70	94.60

5 Application

In this section, data analysis is utilized to assess the goodness-of-fit of the RTPL distribution. The data set is obtained from [16] with respect to the flood data for 20 observations "0.265, 0.392, 0.297, 0.3235, 0.402, 0.269, 0.315, 0.654, 0.338, 0.379, 0.418, 0.423, 0.379, 0.412, 0.416, 0.449, 0.484, 0.494, 0.613, 0.74". We compare the proposed model with some existing well-known models, i.e. Kumaraswamy (Kw) distribution (see [17]) , size-biased Kumaraswamy (SBKw) distribution (see [18]), and TL distribution as a sub-model from RTPL distributions. We consider the Kolmogorov-Smirnov (K-S) test, P-value, Cramér-von Mises (CVM) and Anderson-Darling (AD) goodness-of-fit statistics. The density functions (for $0 < x < 1$) of Kw, SB-Kw and TL are presented in Table 5. Ps of

Table 5: The pdfs for some lifetime distributions

Model	The probability density function
Kumaraswamy	$f_{KW}(x;\alpha,\ \beta) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}$; $\alpha,\beta > 0.$
Size-biased Kumaraswamy	$f_{SBKW}(x;\alpha,\ \beta) = \frac{\alpha x^{\alpha}(1-x^{\alpha})^{\beta-1}}{B(1+\frac{1}{\alpha},\beta)}$; $\alpha,\beta > 0.$
Truncated Lomax	$f_{TL}(x;\alpha) = \frac{\alpha(1+x)^{-(\alpha+1)}}{1-2^{-\alpha}}$; $\alpha > 0.$

each models are estimated by ML method using Mathematica7. The goodness-of fit measures; KS, AD, CVM and, P-value are calculated in Table 6.

Table 6: MLEs and goodness-of fit measures of flood data

Model	Parameters estimate		K-S	AD	CVM	P-value
	$\hat{\alpha}$	$\hat{\beta}$				
RTPL(α,β)	16.8230	3.6560	0.1921	0.8249	0.1289	0.4515
KW(α,β)	1.5659	1.9631	0.2109	0.9723	0.1676	0.3358
SBKw(α,β)	2.7787	10.5688	0.2053	0.8972	0.1526	0.3682
TL(α)	0.00000	————	0.3391	13.6495	0.4429	0.0200

Table 6 exhibits that the RTPL distribution provides a better fit than the competitive models. It has the smallest values of K-S, AD, CVM and the largest P-value. Finally, the plots of empirical cdf of the data set and PP plots of RTPL, KW, SBKW and TL models are displayed in Figure 2, Moreover, we illustrate the usefulness of proposed model by fitted density functions over histograms of the data set in Figure 3.

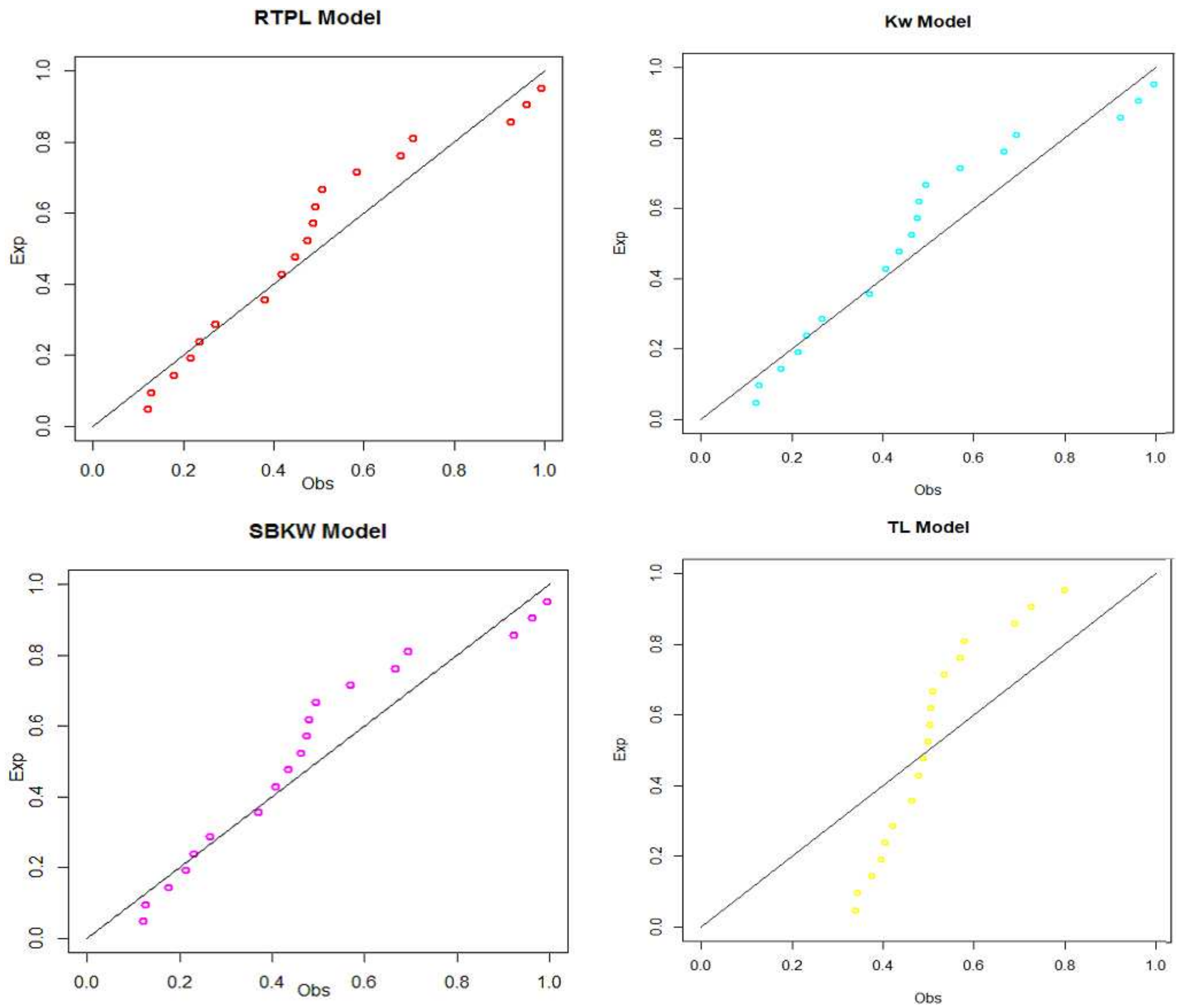

Fig. 2: PP plots of the fitted models for flood data

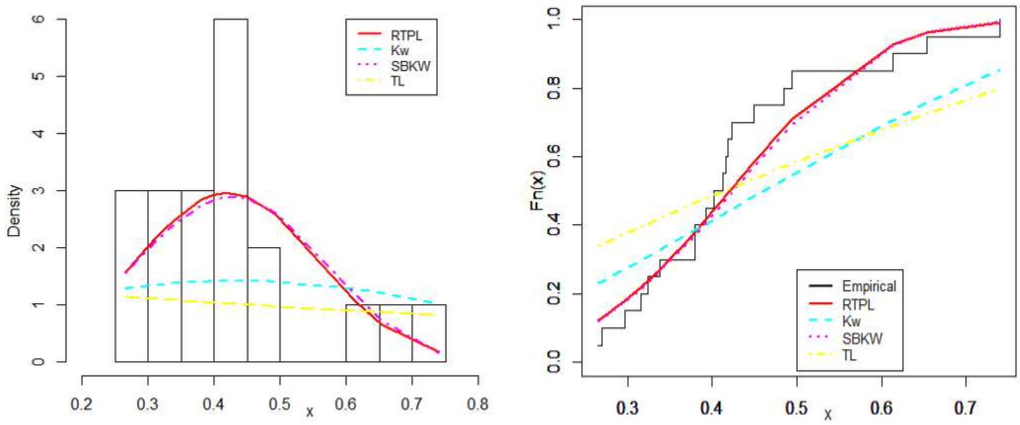

Fig. 3: Estimated pdf and cdf for the data set of models for flood data

From Fig. 2 and 3 we can see the RTPL distribution provides a better fit than the competitive models.

6 Conclusion

In this paper, we introduce a new model called the truncated power Lomax. We investigate several structural properties of the new distribution, expressions for the ordinary moments, generating function and order statistics. The model parameters are estimated by the maximum likelihood method in case of complete and censored samples. A simulation study reveals that the estimates of the model have desirable properties, for example, (i) the maximum likelihood estimates are not too far from the true parameter values; (ii) the biases and mean square errors of estimates in case of complete sample are smaller than the corresponding in censored samples; and (iii) the biases and the mean square error values decrease as the sample size increases. Application to real data empirically proves the importance and potentiality of the suggested distribution.

References

[1] J.F. Lawless. *Statistical Models and Methods for Lifetime Data, 2nd Edition*. Wiley, Hoboken, NJ.(2003).
[2] Y.M. Kantar and I. Usta. Analysis of the upper-truncated Weibull distribution for wind speed. *Energy conversion and management*, **96**, 81-88(2015).
[3] S.H. Abid. Properties of doubly-truncated Fréchet distribution. *American Journal of Applied Mathematics and Statistics*, **4(1)**, 9-15(2016).
[4] K. Jayakumar and K.K. Sankaran. Generalized exponential truncated negative binomial distribution. *American Journal of Mathematical and Management Sciences*, **36(2)**, 98-111(2017).
[5] A.I. Genç. Truncated inverted generalized exponential distribution and its properties. *Communications in Statistics-Simulation and Computation*, **46(6)**, 4654-4670(2017).
[6] N. T. Thomopoulos. *Right Truncated Normal, In Probability Distributions*, Springer, Cham, pp. 85-97(2018).
[7] A. S. Hassan, M. Elgarhy, S. G. Nassr, Z. Ahmad, and S. Alrajhi. Truncated Weibull Frèchet Distribution: Statistical Inference and Applications.*Journal of Computational and Theoretical Nanoscience*,**16(1)**, 1-9(2019).
[8] K.S. Lomax. Business failures: another example of the analysis of failure data. *Journal of the American Statistical Association*, **49**, 847-852(1954).
[9] A. Atkinson and A. Harrison. *Distribution of Personal Wealth in Britain*. Cambridge University Press, Cambridge, (1978).
[10] A. S. Hassan and A.S. Al-Ghamdi. Optimum step stress accelerated life testing for Lomax distribution. *Journal of Applied Sciences Research*, **5(12)**, 2153–2164(2009).
[11] A.S. Hassan, M.S. Assar and A. Shelbaia. Optimum step-stress accelerated life test plan for Lomax distribution with an adaptive Type-II progressive hybrid censoring. *Journal of Advances in Mathematics and Computer Science*, **13(2)**, 1-19(2016).
[12] C. M. Harris. The Pareto distribution as a queue service discipline. *Operations Research* , **16(2)**, 307–313(1968).
[13] E. H. A. Rady, W. A. Hassanein and T. A. Elhaddad. The power Lomax distribution with an application to bladder cancer data. *SpringerPlus*, **5**, 1-22(2016).
[14] A. S. Hassan and M. Abd-Allah. On the inverse power Lomax distribution. *Annals of Data Science*, **6(2)**, 259–278(2019).
[15] A. S. Hassan and S. G. Nassr. Power Lomax Poisson distribution: properties and estimation. *Journal of Data Science*, **18(1)**:105–128(2018).

[16] R.H. Dumonceaux and C.E. Antle. Discriminating between the log-normal and Weibull distribution. *Technometrics* **15(4)**,923–926(1973).
[17] P. Kumaraswamy. Generalized probability density function for double-bounded random processes. *Journal of Hydrology* **46(1–2)**, 79–88(1980).
[18] D. Sharma and T.K. Chakrabarty. On size biased Kumaraswamy distribution. *Statistics, Optimization and Information Computing*, **4(3)**, 252–64(2016).

Amal S. Hassan is Professor of Statistics at the Department of Mathematical Statistics, Faculty of Graduate Studies for Statistical Research at Cairo University, Egypt. She received the Ph.D Degree in Statistics from the Institute of Statistical Studies & Research , Cairo University, Egypt, since 1999. Now, she is Vice Dean of Community Service & Environmental Development in Faculty of Graduate Studies for Statistical Research, Cairo University. Her main research interests are: Probability distributions, Record values, Ranked Set Sampling, Stress-Strength models, Accelerated Life Tests and Goodness of Fit Tests.

Mohamed A. H. Sabry is Associate Professor of Statistics at the Department of Mathematical Statistics, Faculty of Graduate Studies for Statistical Research at Cairo University, Egypt. He received the Ph.D Degree in Statistics from the Linkoping University, Sweden, since 2005. Now, His main research interests are: Probability distributions, Record values, Ranked Set Sampling, Stress-Strength models, and Goodness of Fit Tests.

Ahmed M. Elsehetry Head of Statistical department at National Organization for Social Insurance, Egypt. His research interests include: Nonparametric hypothesis testing and its applications, Queueing Theory, Probability Distributions, Linear and Nonlinear Models.