Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/090308

Accelerated Life Test Plans and Age-Replacement Policy under Warranty on Burr Type-X distribution with Type-II Censoring

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Received: 16 Aug. 2019, Revised: 2 Jul. 2020, Accepted: 5 Aug. 2020
Published online: 1 Nov. 2020

Abstract: In this paper, we describe how to design and analyze the accelerated life testing (ALTg) plans for the improvement of the quality and the reliability of the product. We also focus on finding the expected cost rate and the expected total cost for age-replacement under warranty policy. The problem is studied using constant stress, under the assumption that the lifetimes of the units follow Burr Type-X distribution for predicting the cost of age replacement under warranty policy. Asymptotic variance and covariance matrix of the estimators are obtained by using the Fisher Information Matrix. Confidence intervals for parameters and respective errors are also obtained. A simulation study is performed to illustrate the statistical properties of the parameters and confidence bound. In the last, numerical examples are also carried out to illustrate the theoretical results.

Keywords: Accelerated Life Testing, Type-II Censoring, Age-Replacement Policy, Burr Type-X distribution, Expected Cost Rate, Expected Total Cost, a Simulation study.

1 Introduction

Accelerated life test (ALT) is the most commonly used method of finding the reliability of the products/items. This method is used for the prediction of products/items reliabilities at normal operating conditions and using data obtained at accelerated conditions. In accelerated life testing, the product/item is put at higher stress to quickly get information about the life distribution of the material, component, or product. This testing involves the acceleration of failures with the only purpose of quantifying the life characteristics of the product/item at normal use conditions. There are two types of accelerated life testing; (i) Qualitative accelerated life testing and (ii) Quantitative accelerated life testing. Qualitative accelerated life testing is used to identify the failures and failures modes. Qualitative testing is used without attempting to make any predictions about the item's life with normal use conditions. While quantitative accelerated life testing uses to predict the life of any item at normal use conditions.

Now, we present brief literature on accelerated life testing [ALTg] and Warranty models that are related to our study in this paper. Eman A. [1] described accelerated life testing and age-replacement policy under warranty with the Exponentiated Pareto distribution. Eman A. [2] also described maintenance service policy using step stress partially accelerated life testing for the extension of the Exponential distribution using Type-II censored samples. Xiujie et al. [3] provided a framework to predict the warranty cost and risk under one-dimensional. In the present time, the two-dimensional [4–6] and extended warranty [7, 8] have taken an important place in warranty policy analysis. However, it is very tough and challenging to design warranty policies and predict warranty cost for new products that have not been in the market, because the failure rate of such types of products is not available. A literature review is presented by Murthy and Djamaludin [9] on new product warranty by considering marketing, logistics, etc. and a joint optimization model that involved reliability, warranty, and price for new products, is presented by Huang et al. [10]. Xie, and Ye. [11] proposed an aggregate discounted warranty cost forecast under the new item.

Accelerated life tests could be conducted to adumbrate the reliability of new products. In an ALT, the ecology stresses animated to advance the occurrences of failures. By celebratory abortion times (either exact or censored) from an ALT,

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the accelerated life (AL) model ambit estimated and analysis planners can use these estimators to accomplish statistical inferences of the lifetime administration of analysis products, which helps to adumbrate the number of warranty claims. In Yang [12], optimal 3-level accommodation ALT affairs were discussed to minimize the asymptotic about-face of best likelihood appraisal of the assurance cost. For an overview of accelerated believability tests, one can refer to as Meeker and Escobar [13]. Orlando et al. [14] presented a case study for the household's gadgets; he proposed accelerated life tests for new product qualification. Guangbin Yan [10] proposed a technique for item population for predicting the warranty cost and its confidence interval. It will experience varying stress levels using the accelerated life tests conducted in the early stages of the item life cycle.

The main purpose of this paper is finding the expected total cost and expected cost rate under warranty policy for age replacement. We assume that constant stress is independently distributed, and lifetimes are also independently distributed. This problem is studied when the constant stress of the components is independent identicaly distributed. Maximum Likelihood Estimator (MLE) obtained for the Burr Type-X failure model. While, also, estimating the expected total cost and expected cost rate for age replacement under the pro-rate rebate warranty policy.

The rest of the paper is organized as follows. In section 2, the ALT plan on Burr Type-X distribution is presented. In section 3, the MLE and Fisher Information matrix is derived. Simulations results are presented in section 4. The age-replacement warranty under pro-rate rebate warranty policy is presented in section 5. At last, conclusions are made in section 6.

2 Model description and test procedure

ALT is usually conducted by one of the two approaches, (a) accelerated failure time, which means ALT is conducted for the item or component by experiencing at normal conditions but more intensively than normal. This is good for items or components that are used on a continuous time basis. (b) Accelerated stress means ALT is conducted by using item or component at higher stress than normal. For designing of ALT plans, the following are needed to determine

(i) The stress application testing method.
(ii) The stress levels for each stress type selected.
(iii) The stress type to be used in the testing.
(iv) Relationship between life and stress (life-stress relationship) described by the mathematical model.
(v) At last, the proportion of test units to be allocated to each level of stress.

Several authors dealt with constant stress, such as Abdel-Ghaly [15], presented a study on Pareto distribution and estimate reliability function and parameters of the distribution with the use of accelerated life testing. Attia et al. [16] presented a study on Accelerated life testing for Birnbaum-saunders distribution using censored data with constant stress. Attia et al. [17] also presented a study on Accelerated life testing for Generalized Logistic distribution using Type-I censored data with constant stress. In the following section:

(i) Stress V_j has k-levels.
(ii) Assuming that is normal use condition and satisfying $V_u < V_1 < V_2 \ldots V_k$.
(iii) There are n_j units puts on testing at each stress level.
(iv) The experiment ended when r_j units reached among these n_j units.
(v) This current study is dealt with Type-II censoring (item censoring) and constant stress with the assumption that the lifetime of the units follows the Burr Type-X failure model.
The Burr Type-X distribution is proposed by Burr [18] in 1942. He proposed twelve different forms of cumulative distribution functions for modeling data. But in these distribution functions, Burr Type- X and Burr Type-XII received the most important attention.
Here, we consider the two-parameter Burr Type-X distribution. The two-parameter Burr Type-X distribution has the following probability density function *(pdf)*

$$f(t_{ij},\theta,\lambda) = 2\theta\lambda t_{ij}e^{-\lambda t_{ij}^2}\left[1-e^{-\lambda t_{ij}^2}\right]^{\theta-1} \qquad t_{ij},\theta,\lambda > 0 \tag{1}$$

where θ, λ are shape and scale parameters, respectively.

The cumulative distribution function *(cdf)* of two-parameter Burr Type-X is given as

$$F(t_{ij},\theta,\lambda) = \left[1-e^{-\lambda t_{ij}^2}\right]^{\theta} \qquad t_{ij},\theta,\lambda > 0 \tag{2}$$

The reliability function of Burr Type-X distribution is given as

$$S(t_{ij},\theta,\lambda)=1-\left[1-e^{-\lambda t_{ij}^2}\right]^{\theta} \tag{3}$$

The failure rate or hazard rate of Burr Type-X distribution is given as

$$H(t_{ij},\theta,\lambda)=\frac{2\theta\lambda t_{ij}e^{-\lambda t_{ij}^2}\left[1-e^{-\lambda t_{ij}^2}\right]^{\theta-1}}{1-\left[1-e^{-\lambda t_{ij}^2}\right]^{\theta}} \tag{4}$$

The hazard rate of a Burr Type-X distribution can be either bathtub type or an increasing function, depending on the shape parameter θ.

For $\theta\leq 1/2$, the hazard rate is bathtub type, and for $\theta>1/2$, it has an increasing hazard function. The two-parameter Burr Type-X distribution can be used quite effectively in general lifetime data and also modeling strength data.

The two-parameter Burr Type-X distribution has several types of distributions like Rayleigh distribution when shape parameter, $\theta=1$ and Burr Type-X distribution when one parameter, and when the value of scale parameter, $\lambda=1$.

Some authors have studied Burr-Type X distribution with one parameter, for example, Ahmad Sartawi and Abu-Salih [19,20], Jaheen [21], Ahmad et al. [22], Raqab [23] and Surles and Padgett [24].

(vi) The stress only affects the shape parameter of the Burr Type-X model, θ through a certain acceleration function considered to be power rule model and takes the following form

$$\theta_j=CV_j^{-p} \qquad C>0,p>0,j=1,2,\dots,k \tag{5}$$

Where C is proportionality constant and p is the power of the applied stress. Both are the parameters of the model.

3 Estimation procedure

Many methods are available for the estimation of parameters, but we use the maximum likelihood (ML) estimation method. We use this method because it is very powerful and gives estimates of parameters with very good statistical properties. However, the ML estimation method is very easy for distributions that have one parameter, but its application in ALT is mathematically more powerful, and mainly, estimates of parameters do not exist in closed form. So, numerical methods like the Newton-Raphson method and many other computer programs are used to find them.

There are n_j units put on the test at each stress level V_j. So, the total number of units in the test is $N=\sum_{i=1}^{k}n_j$. If Type-II censoring is used at each stress level, the test ends once the number of failures r_j from the total number of units n_j is reached. So the likelihood function under Type-II censoring takes the following form

$$L(\theta,C,p)=\prod_{j=1}^{k}\frac{n_j}{(n_j-r_j)!}\left[\prod_{i=1}^{r_j}f(t_{ij},\theta,C,p)\right]\left[1-F(t_{r_jj})\right]^{n_j-r_j} \tag{6}$$

The logarithm of the likelihood function is given as

$$lnL=K+\sum_{j=1}^{k}r_j(ln\theta+lnC)-p\sum_{j=1}^{k}r_jlnV_j-\sum_{j=1}^{k}\sum_{i=1}^{r_j}CV_j^{-p}t_{ij}^2+(\theta-1)\sum_{j=1}^{k}\sum_{i=1}^{r_j}ln\left(1-e^{-CV_j^{-p}t_{ij}^2}\right)$$
$$+\sum_{j=1}^{k}(n_j-r_j)\left[1-(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta}\right] \tag{7}$$

Where K is constant and $lnL=lnL(\theta,C,p)$. The Maximum likelihood estimators (MLEs) of parameters θ, C and p are obtained by getting the partial derivatives of equation (7) with respect to θ, C and p respectively and equating to zero. The first derivatives of the likelihood function are

$$\frac{\partial lnL}{\partial\theta}=\sum_{j=1}^{k}\frac{r_i}{\theta}+\sum_{j=1}^{k}\sum_{i=1}^{r_j}\left(1-e^{-CV_j^{-p}t_{ij}^2}\right)+\sum_{j=1}^{k}\frac{\theta(n_j-r_j)(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta-1}}{1-(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta}}=0$$

$$\frac{\partial lnL}{\partial C}=\sum_{j=1}^{k}\frac{r_i}{C}+(\theta-1)\sum_{j=1}^{k}\sum_{i=1}^{r_j}\frac{e^{-CV_j^{-p}t_{ij}^2}V_j^{-p}t_{ij}^2}{(1-e^{-CV_j^{-p}t_{ij}^2})}+\sum_{j=1}^{k}\frac{\theta(n_j-r_j)V_j^{-p}t_{r_jj}^2e^{-CV_j^{-p}t_{r_jj}^2}(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta-1}}{1-(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta}}$$
$$-\sum_{j=1}^{k}\sum_{i=1}^{r_j}V_j^{-p}t_{ij}^2=0$$

$$\frac{\partial lnL}{\partial p}=-\sum_{j=1}^{k}r_jlnV_j+\sum_{j=1}^{k}\sum_{i=1}^{r_j}pV_j^{-p-1}t_{ij}^2+(\theta-1)\sum_{j=1}^{k}\sum_{i=1}^{r_j}e^{-CV_jt_{ij}^2}V_j^{-p}lnV_j$$
$$+\sum_{j=1}^{k}\frac{\theta(n_j-r_j)lnV_jCV_j^{-p}t_{r_jj}^2e^{-CV_j^{-p}t_{r_jj}^2}(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta-1}}{1-(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta}}=0$$

The closed-form solution of the above partial equations does not exist, so the Newton-Raphson method will be used to solve these equations. We use the logarithm of equation (6) defined in equation (7) to estimate the variance-covariance matrix of parameters. The Fisher Information matrix is given as

$$F=\begin{bmatrix}-\frac{\partial^2 lnL}{\partial\theta^2} & -\frac{\partial^2 lnL}{\partial\theta\partial C} & -\frac{\partial^2 lnL}{\partial\theta\partial p}\\ -\frac{\partial^2 lnL}{\partial C\partial\theta} & -\frac{\partial^2 lnL}{\partial C^2} & -\frac{\partial^2 lnL}{\partial C\partial p}\\ -\frac{\partial^2 lnL}{\partial p\partial\theta} & -\frac{\partial^2 lnL}{\partial p\partial C} & -\frac{\partial^2 lnL}{\partial p^2}\end{bmatrix} \tag{8}$$

The elements of the Fisher Information matrix are

$$-\frac{\partial^2 lnL}{\partial\theta^2}=\sum_{j=1}^{k}r_j\theta^{-2}+\sum_{j=1}^{k}\frac{\theta(n_j-r_j)(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta-1}}{1-(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta}}\left[\frac{1}{\theta}+ln(1-e^{-CV_j^{-p}t_{r_jj}^2})+\frac{\theta(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta-1}}{1-(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta}}\right]$$

$$-\frac{\partial^2 lnL}{\partial\theta\partial C}=-\sum_{j=1}^{k}\sum_{i=1}^{r_j}\frac{e^{-CV_j^{-p}t_{ij}^2}V_j^{-p}t_{ij}^2}{1-e^{-CV_j^{-p}t_{ij}^2}}$$
$$-\sum_{j=1}^{k}\frac{\theta(n_j-r_j)(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta-1}}{1-(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta}}\left[\frac{(\theta-1)e^{-CV_j^{-p}t_{r_jj}^2}V_j^{-p}t_{r_jj}^2}{1-e^{-CV_j^{-p}t_{r_jj}^2}}+\frac{\theta e^{-CV_j^{-p}t_{r_jj}^2}V_j^{-p}t_{r_jj}^2(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta-1}}{1-(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta}}\right]$$

$$-\frac{\partial^2 lnL}{\partial C^2}=\sum_{j=1}^{k}r_jC^{-2}+\left[(\theta-1)\sum_{j=1}^{k}\sum_{i=1}^{r_j}\frac{(1-e^{-CV_j^{-p}t_{ij}^2})e^{-CV_j^{-p}t_{ij}^2}V_j^{-p}t_{ij}^2+e^{-2CV_j^{-p}t_{ij}^2}V_j^{-p}t_{ij}^2}{(1-e^{-CV_j^{-p}t_{ij}^2})}\right]V_j^{-p}t_{ij}^2$$
$$-\sum_{j=1}^{k}\theta(n_j-r_j)V_j^{-p}t_{r_jj}^2\left[\frac{(1-(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta})(e^{-CV_j^{-p}t_{r_jj}^2}+2e^{-2CV_j^{-p}t_{r_jj}^2})}{(1-(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta})^2}\right]$$
$$+\sum_{j=1}^{k}\theta(n_j-r_j)V_j^{-p}t_{r_jj}^2\left[\frac{(e^{-CV_j^{-p}t_{r_jj}^2}-2e^{-2CV_j^{-p}t_{r_jj}^2}V_j^{-p}t_{r_jj}^2)((e^{-CV_j^{-p}t_{r_jj}^2}V_j^{-p}t_{r_jj}^2)^{\theta-1}V_j^{-p}t_{r_jj}^2)}{(1-(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta})^2}\right]$$

$$-\frac{\partial^2 lnL}{\partial C\partial p}=(\theta-1)\sum_{j=1}^{k}\sum_{i=1}^{r_j}\frac{e^{-CV_j^{-p}t_{ij}^2}CpV_j^{-p-1}t_{ij}^2}{1-e^{-CV_j^{-p}t_{ij}^2}}\left[CpV_j^{-p}t_{ij}^2-lnV_j+\frac{e^{-CV_j^{-p}t_{ij}^2}CpV_j^{-p-1}t_{ij}^2}{1-e^{-CV_j^{-p}t_{ij}^2}}\right]$$
$$-\sum_{j=1}^{k}\theta(n_j-r_j)t_{r_jj}^2\left[\frac{e^{-CV_j^{-p}t_{r_jj}^2}CpV_j^{-p-1}t_{r_jj}^2}{1-(1-e^{-CV_j^{-p}t_{r_jj}^2})}+CpV_j^{-p}t_{r_jj}^2-lnV_j-\frac{\theta(1-e^{-CV_j^{-p}t_{r_jj}^2})e^{-CV_j^{-p}t_{r_jj}^2}CpV_j^{-p-1}t_{r_jj}^2}{1-(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta}}\right]$$

$$-\frac{\partial^2 lnL}{\partial p\partial\theta}=\sum_{j=1}^{k}\sum_{i=1}^{r_j}\frac{e^{-CV_j^{-p}t_{ij}^2}}{1-e^{-CV_j^{-p}t_{ij}^2}}-\sum_{j=1}^{k}C(nj-r_j)t_{r_jj}^2\left[\frac{\theta^2(1-e^{-CV_j^{-p}t_{r_jj}^2})^{2(\theta-1)}}{(1-(1-e^{-CV_j^{-p}t_{r_jj}^2}))^2}\right]$$
$$-\sum_{j=1}^{k}C(n_j-r_j)t_{r_jj}^2\left[\frac{(1-(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta})(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta-1}+\theta(\theta-1)(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta-2}}{(1-(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta})^2}\right]$$

$$-\frac{\partial^2 lnL}{\partial p^2}=\sum_{j=1}^{k}\sum_{i=1}^{r_j}t_{ij}^2(p(p+1)V_j^{-pt_{ij}^2})+(\theta-1)\sum_{j=1}^{k}\sum_{i=1}^{r_j}\frac{e^{-CV_j^{-p}t_{ij}^2}V_j^{-p-1}}{1-e^{-CV_j^{-p}t_{ij}^2}}\times$$
$$\left[-CV_j^{-p-1}t_{ij}^2-(p+1)V_j^{-1}+\frac{e^{-CV_j^{-p}t_{ij}^2}}{1-e^{-CV_j^{-p}t_{ij}^2}}\right]$$
$$-\sum_{j=1}^{k}(n_j-r_j)\frac{\theta(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta-1}CV_j^{-p}t_{r_jj}^2}{1-(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta}}\times$$
$$\left[\frac{e^{-CV_j^{-p}t_{r_jj}^2}CpV_j^{-p-1}}{1-e^{-CV_j^{-p}t_{r_jj}^2}}-\frac{t_{r_jj}^2CV_j^{-2p-3}}{V_j^{-p}}-\frac{\theta(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta-1}e^{-CV_j^{-p}t_{r_jj}^2}V_j^{-p-1}pCt_{r_jj}^2}{1-(1-e^{-CV_j^{-p}t_{r_jj}^2})^{\theta}}\right]$$

So the MLE of $\hat{\theta}$, $\hat{C}$ and $\hat{p}$ have an asymptotic variance-covariance matrix, which is obtained by inverting the Fisher Information matrix given in equation (8).

$$v=F^{-1} \tag{9}$$

The approximate $100(1-\lambda)\%$ confidence intervals for $\hat{\theta}$, $\hat{C}$ and $\hat{p}$ are given as

$$\hat{\theta}+Z_{\lambda/2}\sqrt{(var\hat{\theta})},\hat{C}+Z_{\lambda/2}\sqrt{(var\hat{C})}\ \ and\ \ \hat{p}+Z_{\lambda/2}\sqrt{(var\hat{p})} \tag{10}$$

$Z_{\lambda/2}$ is the $100(1-\lambda/2)\%$ percentile of a standard normal variate.

4 Simulation results

In this section, we use the Monte-Carlo simulation technique for simulation studies. Numerical studies are performed to examine the performances of the MLEs through their absolute relative bias (RABs) and mean square error (MSE). Using the invariance property of MLEs, we can estimate the MLEs of shape Parameter θ_j through the following equation.

$$\theta_j=CV_j^{-p}\qquad C,p>0\quad j=1,2,\dots,k$$

The detailed steps are given below:

1. First, 1000 random samples of sizes 40, 80, 120, and 160 are generated from Burr Type-X distribution. Different initial values are selected for all sets of parameters.
2. There are only three different levels of stress; the stress values are selected $(V_1=1,V_2=1.5,V_3=2),n_j=\frac{n}{3}\ and\ r_j=60\%n_j$
3. For all sample sizes, the parameters of the model are estimated under Type-II censoring.
4. The Newton-Raphson method is used for solving all equations.
5. The estimates of the shape parameter θ_j are calculated from equation (5).
6. The RABs and MSEs are tabulated for all sets of (θ_0,C_0,p_0).
7. Using the invariance property of MLEs, we calculate the MLEs of the scale parameter θ_u at the usual stress level $V_u=0.5$.
8. The reliability function at the same normal stress for different values θ,C,p and t_0 is calculated. $\hat{R}_u(t)=1-[1-e^{-\lambda_0t^2}]^{\theta_0}$
9. At mission time $(t_0=1.5,1.8,2.2)$, the MLEs of reliability function are predicted under the same usual conditions for all sets of parameters.

Table 1: The Estimates, Relative Bias and MSE of the parameters $(\lambda, C, p, \theta_1, \theta_2, \theta_3)$ under Type-II censoring

n	**Parameters**	$(\lambda_0 = 0.25, C_0 = 1.5, p_0 = 1)$			$(\lambda_0 = 1, C_0 = 1.5, p_0 = 1)$		
		Estimator	RABs	MSEs	Estimator	RABs	MSEs
40	λ	1.229	0.095	0.088	1.930	0.080	0.086
	C	2.211	0.070	0.069	2.443	0.062	0.073
	p	1.430	0.080	0.072	1.725	0.095	0.081
	θ_1	2.456	0.065	0.057	2.821	0.061	0.077
	θ_2	1.943	0.062	0.051	1.765	0.053	0.050
	θ_3	1.878	0.055	0.062	2.101	0.061	0.069
80	λ	1.534	0.086	0.076	1.876	0.073	0.076
	C	2.543	0.067	0.055	2.346	0.052	0.061
	p	1.893	0.077	0.061	2.098	0.088	0.097
	θ_1	2.762	0.054	0.040	2.989	0.064	0.051
	θ_2	2.688	0.054	0.031	1.760	0.050	0.042
	θ_3	2.097	0.049	0.055	1.886	0.036	0.057
120	λ	3.239	0.044	0.042	2.945	0.069	0.061
	C	2.649	0.054	0.044	2.791	0.050	0.051
	p	2.846	0.065	0.054	2.152	0.072	0.086
	θ_1	1.956	0.046	0.035	2.328	0.042	0.047
	θ_2	2.090	0.048	0.040	1.904	0.039	0.037
	θ_3	2.196	0.041	0.022	1.867	0.011	0.050
160	λ	3.252	0.018	0.029	2.976	0.064	0.050
	C	2.112	0.023	0.013	2.460	0.042	0.023
	p	1.813	0.113	0.010	1.905	0.045	0.031
	θ_1	1.750	0.027	0.014	1.843	0.062	0.069
	θ_2	1.922	0.031	0.023	2.011	0.018	0.016
	θ_3	2.089	0.032	0.023	1.825	0.033	0.025

Table 2: The Estimates, Relative Bias and MSE of the parameters $(\lambda, C, p, \theta_1, \theta_2, \theta_3)$ under Type-II censoring

n	**Parameters**	$(\lambda_0 = 0.25, C_0 = 1, p_0 = 1)$			$(\lambda_0 = 1, C_0 = 1, p_0 = 1.5)$		
		Estimator	RABs	MSEs	Estimator	RABs	MSEs
40	λ	1.342	0.076	0.089	1.987	0.099	0.087
	C	2.003	0.223	0.087	2.171	0.210	0.087
	p	1.929	0.090	0.088	1.621	0.082	0.071
	θ_1	2.101	0.093	0.062	2.523	0.081	0.074
	θ_2	0.970	0.212	0.081	1.210	0.077	0.068
	θ_3	0.788	0.099	0.077	0.803	0.093	0.096
80	λ	0.837	0.062	0.077	0.663	0.071	0.075
	C	2.018	0.092	0.072	2.156	0.144	0.070
	p	0.737	0.073	0.070	2.136	0.052	0.095
	θ_1	2.064	0.081	0.058	2.289	0.074	0.061
	θ_2	0.980	0.188	0.065	2.170	0.069	0.061
	θ_3	0.787	0.068	0.065	0.986	0.088	0.079
120	λ	0.739	0.054	0.045	0.998	0.063	0.062
	C	1.949	0.089	0.063	2.091	0.081	0.056
	p	0.678	0.052	0.050	3.998	0.050	0.066
	θ_1	2.061	0.086	0.061	1.930	0.066	0.055
	θ_2	0.998	0.070	0.047	3.132	0.054	0.016
	θ_3	0.994	0.059	0.076	0.998	0.077	0.098
160	λ	0.739	0.054	0.045	0.998	0.063	0.062
	C	1.949	0.089	0.063	2.091	0.081	0.056
	p	0.678	0.052	0.050	3.998	0.050	0.066
	θ_1	2.061	0.086	0.061	1.930	0.066	0.055
	θ_2	0.998	0.070	0.047	3.132	0.054	0.016
	θ_3	0.994	0.059	0.076	0.998	0.077	0.098

4.1 Estimation of the reliability function and shape parameter at normal stress

In the following table, we estimate the reliability function at the normal stress level $V_u = 0.5$ for different values of parameters λ, C, p and t_0, also find the shape parameter for the same stress level.

Table 3: Estimated reliability function and shape parameter at normal stress

λ_0	C_0	p_0	θ_0	t_0	$R_u(t_0)$
0.25	1.5	1	3.1246	(0.3,0.6,0.9)	(0.58268,0.52515,0.44820)
1	1.5	1	3.2376	(0.3,0.6,0.9)	(0.53164,0.53428,0.49725)
0.25	1	1	2.0332	(0.3,0.6,0.9)	(0.36510,0.31170,0.30849)
1	1	1.5	2.8216	(0.3,0.6,0.9)	(0.39504,0.28456,0.20757)

5 The Age-Replacement policy under pro-rate rebate warranty for Burr Type-X distribution

Under this warranty policy,
(i) A non-repairable item is replaced at a certain time age τ or upon failure, which occurs first.
(ii) When the item fails at $t \leq \tau$ a failure, replacement is performed with a purchasing cost C_p and downtime cost C_d, where $C_p, C_d > 0$.
(iii) The customer is refunded a proportion of sales price C_p if the item fails over the warranty period (w),

So, the rebate function under the pro-rata warranty is

$$R(t) = \begin{cases} C_p\left(1 - \frac{t}{w}\right) & 0 \leq t \leq w \\ 0 & t > w \end{cases} \tag{11}$$

John Mamer [25] presented a study on cost analysis of pro-rata with a free replacement warranty approaches; he examines the long-run average and total costs of items with warranty. Timothy et al. [26] proposed a pro-rata study for combined warranty approaches; he used different repair options in his study. Huang et al. [27] studied the problem of estimating the expected warranty cost for the case where the product usage is intermittent and of heterogeneous usage intensity by the product life cycle when sales occur regularly.

Major Assumptions
1. Item is replaced at the time of failure (corrective replacement), or age τ (preventive replacement), which comes first.
2. Items are sold with a pro-rata rebate warranty approach.
3. There is no salvage value for the preventive replaced product.
4. The warranty period (w) is less than age replacement τ, i.e. $(w) < \tau$.

When the item's life reaches τ, then the preventive replacement carried out with cost C_p only because it is a planned preventive maintenance action.
The total cost incurred in a renewal cycle for this policy is

$$C(d) = \begin{cases} C_p + C_d - R(t) & 0 \leq t \leq w \\ 0 & w > t > \tau \\ C_p & t \geq \tau \end{cases} \tag{12}$$

The expected total cost [28, 29] under this policy is given by

$$E(C(t)) = C_d + C_p \frac{\int_0^w \bar{F}(u)}{w} \tag{13}$$

The expected cost rate is

$$E(CR(t)) = \frac{E(C(t))}{\int_0^\tau \bar{F}(u)} \tag{14}$$

Where $\int_0^\tau \bar{F}(u)$ is the expected cycle time, which is denoted by $E(T(\tau))$.

Under the Burr Type-X distribution

The *cdf* is

$$F(u) = [1 - e^{-\lambda u^2}]^\theta \qquad u, \lambda, \theta > 0$$

So,

$$\int_0^\tau \bar{F}(u)du = \tau - \int_0^\tau [1 - e^{-\lambda u^2}]^\theta du$$

And,

$$\int_0^w \bar{F}(u)du = w - \int_0^w [1 - e^{-\lambda u^2}]^\theta du$$

By putting the above values in equation (13) and equation (14), we get the expected total cost and expected cost rate for the non-repairable product.

6 The expected total cost, the expected cycle time and the expected cost rate for age-replacement under warranty policy on Burr Type-X distribution

In this section, we estimate the expected total cost, expected cycle time, and expected cost rate with a downtime cost and purchasing cost.

For example, if the failure replacement is performed with a downtime cost $C_d = 50$ and purchasing cost $C_p = 1000$, then we estimate the expected total cost, expected cycle time, and expected cost rate for age-replacement under warranty policy for different values of warranty periods and the parameters of Burr Type-X and at normal use.

Table 4: The expected total cost, the expected cycle time and the expected cost rate

θ	λ	w	τ	**Expected Total Cost** $E(C(\tau))$	**Expected Cycle Time** $E(T(\tau))$	**Expected Cost Rate** $CR(\tau)$
0.3	3	6	8	730.543	5.2341	296.8237
0.3	3	6	6	735.564	3.9876	386.4821
0.6	3	6	8	718.987	5.0987	310.8761
0.6	3	6	6	683.835	3.5673	390.4712
0.9	3	6	8	593.871	4.9453	320.1246
0.9	4	6	8	725.837	5.3451	312.9812
0.9	4	6	6	743.113	3.7532	388.9817
0.9	5	6	8	789.586	5.7987	305.8895
0.9	5	7	7	769.980	5.0981	343.0981
0.9	5	7	8	770.500	5.8712	310.1272
0.9	5	7	9	771.997	6.8712	237.6667

7 Conclusions

From the Table (1) and (2), it is concluded that the absolute value of the difference between the true value of the parameter and its estimate is very small positive value converges to zero. So, these estimates are said to be consistent estimators.

From the Table (3), we estimate the values of the shape parameter. We can see that if the value of mission time (t_0) increases, then the value of reliability function decreases. So, we can conclude that there is an inverse relationship between mission time and the reliability function $R_u(t_0)$.

The following results are concluded from the Table (4):

(i) There is an inverse relationship between the shape parameter and expected total cost, with a direct relationship between the shape parameter and expected cost rate.

(ii) There is a direct relationship between scale parameter and expected total cost and expected cycle time, with an inverse relationship between shape parameter and expected cost rate.

(iii) There is a direct relationship between the warranty period, expected total cost and expected cost rate. Increasing the warranty period does not affect on expected cycle time.

(iv) There is a direct relationship between age replacement of time, expected total cost and expected cycle time, while an inverse relationship between age replacement time and expected cost rate.

Acknowledgement

The authors are grateful to the anonymous referees for a careful checking of the details and for helpful comments that improved this paper.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article

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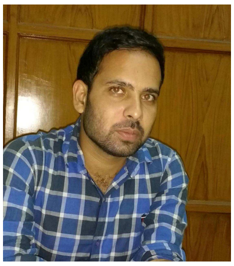

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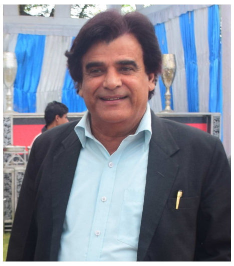

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