

# Application of New Simulation Scheme for the Nonlinear Biological Population Model

*A. S. J. Al-Saif, and Mohammed S. Abdul-Wahab*

Department of Mathematics, College of Education for Pure Sciences, University of Basrah, Basrah, Iraq.

Received: 13 May 2019, Revised: 15 Jun. 2019, Accepted: 20 Jun. 2019.

Published online: 1 Jul. 2019.

**Abstract:** In this paper, the technique of series solution to a class of initial value problems is developed and assessed by Taylors' series for an accurate simulation of two-dimensional nonlinear biological population model. To examine the accuracy and performance of the present solution technique whose derivation is based on Taylors' series expansion, three standard test problems are simulated. To show the efficiency and accuracy of the recent solution technique, the results of an approximate analytic solutions that we obtain are compared with Adomian decomposition method (ADM), variational iteration method (VIM), improved element-free Galerkin method (IEFGM) and the modified cubic B-Spline differential quadrature method (MCB-DQM). The proposed simulation technique is precise, efficient, and has agreeable convergence for solving two-dimensional biological population model.

**Keywords:** Simulation, biological population model, Taylors' series, accuracy, convergence.

## 1 Introduction

Biological population model is a degenerate nonlinear partial differential equation which represents the distribution of population density in an area and it is written in the following form:

$$\rho_t = \nabla^2(\rho^2) + \sigma(\rho), \quad t \geq 0, x, y \in \mathbb{R}, \quad (1.1)$$

with given initial condition  $\rho(x, y, 0)$ , where  $\nabla^2$  is the Laplacian operator,  $\rho(x, y, t)$  is the population density and  $\sigma(\rho)$  is the population supply. The population density  $\rho(x, y, t)$  is the number of individuals per unit volume at locale  $(x, y)$  and time  $t$ , when we integral it over any sub region  $R$  gives the total population of  $R$  at time  $t$ . The population supply  $\sigma(\rho)$  is the rate at which individuals are supplied through births and deaths per unit volume at locale  $(x, y)$  at time  $t$ . The diffusion velocity from one locale to other in the flow of population depicted by the function  $v(x, y, t)$  which represents the average velocity of the individuals who at time  $t$  occupy locale  $(x, y)$ . The functions  $\rho, v$  and  $\sigma$  must satisfy the law of population balance for each sub region  $R$  of region  $B$  at any time  $t$ :

$$\frac{d}{dt} \int_R \rho \, dV + \int_{\partial R} \rho \cdot \mathbf{n} \, dV = \int_R \sigma \, dV, \quad (1.2)$$

where  $\mathbf{n}$  is the outward unit normal to the boundary  $\partial R$  of  $R$ . The above equation affirms that the rate of change of population of  $R$  plus the rate at which individuals leave  $R$  across its boundary must be equal to the rate at which individuals are supplied directly to  $R$  [1-3].

There are three examples for  $\sigma(\rho)$  combined in the following formula:

$$\rho_t = \nabla^2(\rho^2) + h\rho^\alpha(1 - r\rho^\beta), \quad (1.3)$$

where  $\alpha, \beta, h$  and  $r$  are real numbers. The first example is Malthusian law [1] we can obtain it by taking  $h = \mu, \alpha = 1, r = 0$ ;

$$\rho_t = \nabla^2(\rho^2) + \mu\rho, \quad (1.4)$$

where  $\mu$  is constant. The second example is Verhulst law [1] and we can obtain it by taking  $h = \mu, \alpha = \beta = 1, r = \frac{\gamma}{\mu}$ ;

$$\rho_t = \nabla^2(\rho^2) + \mu\rho - \gamma\rho^2, \quad (1.5)$$

where  $\mu, \gamma$  are positive constants and the third example is Porous Media [4,5] and we can obtain it by taking

\*Corresponding author-mail: [sattaralshaiif@yahoo.com](mailto:sattaralshaiif@yahoo.com)

$$h = -\mu, 0 < \alpha < 1, r = 0; \\ \rho_t = \nabla^2(\rho^2) - \mu\rho^\alpha, \quad \text{where } \mu \geq 0. \quad (1.6)$$

Biological population model was solved by many researchers and for different cases of  $h, \alpha, \beta, r$  and initial condition. In 2007 Shakeri [6] used the variational iteration method and Adomian decomposition method and he show that the variational iteration method is the best. In 2009 El-Sayed et al. [7] found a new solution in power series when they employ the Adomian decomposition method for finding the exact solutions of time fractional biological population model. In 2010 Roul [8] used the homotopy perturbation and found that the results were very good by comparison with Adomian decomposition method. In 2011 Arafa et al. [9] obtained exact solutions of generalized biological populations equation subject to some initial conditions by using homotopy analysis method (HAM). In 2013 Kumar et al. [10] succeeded in apply the homotopy analysis transform method (HATM) to obtain the exact solutions of the generalized biological population equations subject to some initial conditions. In 2014 Zhang et al. [11] used the improved element-free Galerkin (IEFG) method where they demonstrated numerically that the quality of the results obtained by the IEFG method is very good. In 2016 Sarwar et al. [12] used optimal homotopy asymptotic method and compared with the exact solutions and they show that numerical results were highly accurate, effective and easy in computation. In 2017 Acana et al. [13] obtained a new exact solution by used a new approximate analytic technique named three dimensional conformable reduced differential transform method (TCRDTM). In 2018 Shakeel et al. [14] succeeded in finding closed form solutions of the population mode by applied modified exp-function method. The main aim of this work is improving numerical solution for the two-dimensional biological population model and extending the application of a new simulation scheme that we proposed it in previous work [15]. The efficiency of this technique to solve linear and nonlinear equations, and the exact solutions of nonlinear biological population models are rare are stimulate us to employ a new technique to presented a comprehensive study for intricate the two-dimensional biological population model. We obtained a good numerical results with high accuracy through the very small absolute errors as well as obtaining exact solution for some problems after approximate the series solutions. The numerical results show that the new simulated scheme was best compared with ADM, VIM [6], MCB-SDQM [16], and IIEFGM [11] for different measures of errors and CPU time.

This paper is planned as follows: Generating a simulation scheme in section 2, three problems are testing and discussion of results is reported in section 3, in section 4 the convergence analysis is presented. Finally, the conclusions are recorded in section 5.

## 2 Generating a Simulation Scheme

In this section, the basic ideas for constructing new simulation scheme will be discussed.

Let's consider the initial value problem:

$$u_t(x, y, t) = F[u] + g(x, y), \quad (2.1)$$

with initial condition  $u(x, y, t_0)$ ,

where  $F[u]$  is linear and nonlinear operator and  $g(x, y)$  is the known function.

By using the integral for the two sides of equation (2.1) from  $t_0$  to  $t$ , we obtain

$$u(x, y, t) = u(x, y, t_0) + g(x, y)\Delta t + \int_{t_0}^t F[u] dt, \quad (2.2)$$

where  $\Delta t = t - t_0$ , and  $F[u]$  can be expressed by the expand Taylors' series about  $t_0$  as;

$$F[u] = [F[u]]_{t_0} + [F'[u]]_{t_0} \frac{\Delta t}{1!} + [F''[u]]_{t_0} \frac{(\Delta t)^2}{2!} + [F'''[u]]_{t_0} \frac{(\Delta t)^3}{3!} + \dots + [F^{(n)}[u]]_{t_0} \frac{(\Delta t)^n}{n!} + \dots, \quad (2.3)$$

where  $[F[u]]_{t_0} = F[u]|_{t=t_0}$ ,  $[F'[u]]_{t_0} = \frac{\partial F[u]}{\partial t}|_{t=t_0}$ ,  $[F''[u]]_{t_0} = \frac{\partial^2 F[u]}{\partial t^2}|_{t=t_0}$ ,  $\dots$ ,  $[F^{(n)}[u]]_{t_0} = \frac{\partial^n F[u]}{\partial t^n}|_{t=t_0}$

Substituting equation (2.3) into equation (2.2), and integrating resulting equation to obtain the series solution as;

$$u(x, y, t) = a_0 + a_1 \Delta t + a_2 \frac{(\Delta t)^2}{2!} + a_3 \frac{(\Delta t)^3}{3!} + \dots + a_n \frac{(\Delta t)^n}{n!} + \dots, \quad (2.4)$$

Where

$$a_0 = u(x, y, t_0), a_1 = g(x, y) + [F[u]]_{t_0}, a_2 = [F'[u]]_{t_0}, a_3 = [F''[u]]_{t_0}, \dots, a_n = [F^{(n-1)}[u]]_{t_0}. \tag{2.5}$$

Here we used the chain rule to computing the derivatives of  $F[u]$

$$F'[u] = \sum_{i=0}^n \sum_{j=0}^i F_{u_{x^{i-j}y^j}}[u] u_{x^{i-j}y^j} t, \tag{2.6}$$

$$F''[u] = \sum_{i=0}^n \sum_{j=0}^i \left( F_{u_{x^{i-j}y^j}}[u] u_{x^{i-j}y^j} t t + \sum_{k=0}^n \sum_{r=0}^k F_{u_{x^{i-j}y^j} u_{x^{k-r}y^r}}[u] u_{x^{i-j}y^j} u_{x^{k-r}y^r} t \right), \tag{2.7}$$

⋮

where  $n$  is highest derivative of  $u$ .

The series solution (2.4) at initial time ( $t_0 = 0$ ) is

$$u(x, y, t) = a_0 + a_1 t + a_2 \frac{t^2}{2!} + a_3 \frac{t^3}{3!} + \dots + a_n \frac{t^n}{n!} + \dots \tag{2.8}$$

### 3 Test Problems

**Example 1.** Consider the equation (1.3) for  $h = \frac{1}{5}$ ,  $\alpha = 1$ ,  $r = 0$ , and  $\rho(x, y, 0) = \sqrt{xy}$  becomes  $\rho_t = \nabla^2(\rho^2) + \frac{1}{5}\rho$ .

The exact solution for this problem is  $\rho(x, y, t) = \sqrt{xy}e^{\frac{1}{5}t}$ , compare with equation (2.1), we have

$$g(x, y) = 0, \tag{3.1}$$

$$F[\rho] = (\rho^2)_{xx} + (\rho^2)_{yy} + \frac{1}{5}\rho. \tag{3.2}$$

Let  $\xi = xy$ , we note that the highest derivative of  $\rho$  is  $n = 2$  and  $t_0 = 0$  then according to (2.5), we get

$$a_0 = \rho(x, y, 0) = \xi^{\frac{1}{2}}, \tag{3.3}$$

$$a_1 = [F[\rho]]_0 = ((a_0)^2)_{xx} + ((a_0)^2)_{yy} + \frac{1}{5}a_0 = \frac{1}{5}\xi^{\frac{1}{2}}, \tag{3.4}$$

$$a_2 = [F'[\rho]]_0 = \sum_{i=0}^2 \sum_{j=0}^i F_{\rho_{x^{i-j}y^j}}[a_0] (a_1)_{x^{i-j}y^j} = \frac{1}{25}\xi^{\frac{1}{2}}, \tag{3.5}$$

$$a_3 = [F''[\rho]]_0 = \sum_{i=0}^2 \sum_{j=0}^i \left( F_{\rho_{x^{i-j}y^j}}[a_0] (a_2)_{x^{i-j}y^j} + \sum_{k=0}^2 \sum_{r=0}^k F_{\rho_{x^{i-j}y^j} \rho_{x^{k-r}y^r}}[a_0] (a_1)_{x^{i-j}y^j} (a_1)_{x^{k-r}y^r} \right) = \frac{1}{125}\xi^{\frac{1}{2}}, \tag{3.6}$$

from equation (2.8), we get the analytical approximate solution

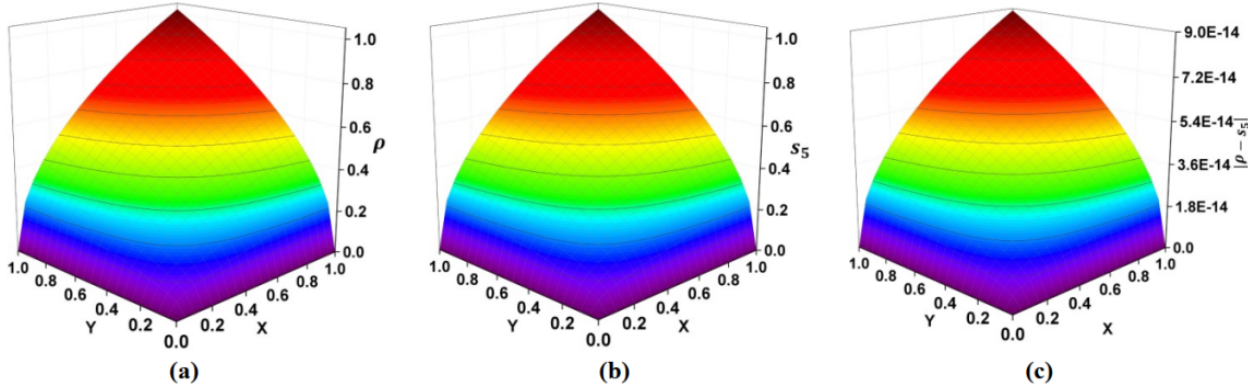
$$\begin{aligned} \rho(x, y, t) &= \xi^{\frac{1}{2}} + \frac{1}{5}\xi^{\frac{1}{2}}t + \frac{1}{25}\xi^{\frac{1}{2}}\frac{t^2}{2!} + \frac{1}{125}\xi^{\frac{1}{2}}\frac{t^3}{3!} + \dots \\ &= \xi^{\frac{1}{2}}\left(1 + \frac{t}{5} + \frac{\left(\frac{t}{5}\right)^2}{2!} + \frac{\left(\frac{t}{5}\right)^3}{3!} + \dots\right) = \xi^{\frac{1}{2}}e^{\frac{t}{5}} = \sqrt{xy}e^{\frac{1}{5}t}. \end{aligned} \tag{3.7}$$

**Table 1:** Comparison of solutions and CPU time between MCBS-DQM, IEFGM and present study for **Example 1** at  $x= y=0.5$

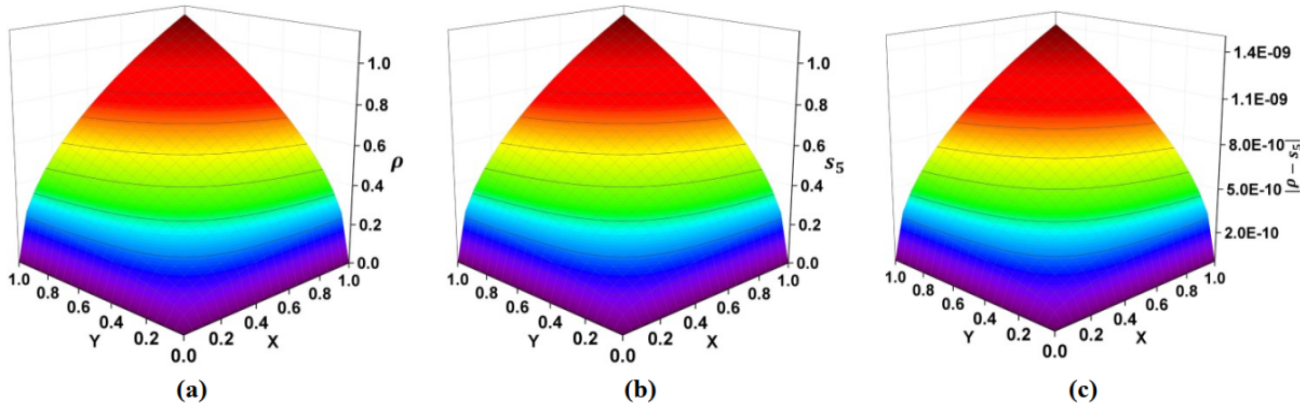
Method	$t= 0.1$	$t= 0.2$	$t= 0.3$	$t= 0.4$	$t= 0.5$
<b>Exact</b>	0.5101	0.5204	0.5309	0.5416	0.5526
<b>Present study</b>	0.5101	05204	0.5309	0.5416	0.5526
<b>CPU(s)</b>	0	0	0	0	0
<b>MCBS-DQM[16]</b> ( $\Delta t = 0.0001$ )	0.5078	0.5170	0.5273	0.5379	0.5488
<b>CPU(s)</b>	0.31	0.61	0.94	1.20	1.48
<b>IEFGM [11]</b> ( $\Delta t = 0.001$ )	0.5069	0.5165	0.5268	0.5373	0.5481
<b>CPU(s)</b>	-	-	-	-	-

**Table 2:** Comparison of errors and CPU time between MCBS-DQM and present study for **Example 1** with  $h_x = h_y = 1/12$ .

Method	Errors	$t=2$	$t=5$	$t=8$	$t=10$
MCBS-DQM[16]	$L_2$	7.94E-03	1.50E-02	3.06E-02	2.07E-02
	$L_\infty$	2.31E-03	4.10E-03	1.13E-02	9.34E-02
	CPU(s)	5.87	14.84	23.79	29.60
Present study	$L_2$	1.85E-07	1.23E-04	3.58E-03	1.81E-02
	$L_\infty$	3.42E-07	2.26E-04	6.62E-03	3.35E-02
	CPU(s)	0.094	0.094	0.094	0.094



**Fig. 1:** (a) Exact solution, (b) Approximate solution ( $S_5$ ), (c) Absolute errors for **Example 1** where  $t=0.1$  and  $h=0.05$ .



**Fig. 2:** (a) Exact solution, (b) Approximate solution ( $S_5$ ), (c) Absolute errors for **Example 1** where  $t=0.5$  and  $h=0.05$ .

Figs.(1-2) display the exact solution, approximate solution and absolute errors at time 0.1 and 0.5 with  $N=20$  and  $h=0.05$ . Comparison of numerical solutions at  $x=0.5, y=0.5$  for present study, MCBS-DQM and IEFGM are demonstrated in Table 1. Table 2 shows the comparison between the present study and MCBS-DQM at time 2, 5, 8 and 10 with  $N=12$  and  $h=1/12$ .

**Example 2.** Consider the equation (1.3) with  $h = -1, \alpha = 1, r = -\frac{8}{9}, \beta = 1$ , and  $\rho(x, y, 0) = e^{\frac{1}{3}(x+y)}$ . That is the equation becomes  $\rho_t = \nabla^2(\rho^2) - \rho \left(1 + \frac{8}{9}\rho\right)$  and has the exact solution  $\rho(x, y, t) = e^{\frac{1}{3}(x+y)-t}$ . Compare with equation (2.1), we have

$$g(x, y) = 0, \tag{3.8}$$

$$F[\rho] = (\rho^2)_{xx} + (\rho^2)_{yy} - \rho - \frac{8}{9}\rho^2. \tag{3.9}$$

Let  $\xi = e^{\frac{1}{3}(x+y)}$ , we note that the highest derivative of  $\rho$  is  $n = 2$  and  $t_0 = 0$ , then according to (2.5), we get

$$a_0 = \rho(x, y, 0) = \xi, \tag{3.10}$$

$$a_1 = [F[\rho]]_0 = ((a_0)^2)_{xx} + ((a_0)^2)_{yy} - a_0 - \frac{8}{9}(a_0)^2 = -\xi, \tag{3.11}$$

$$a_2 = [F'[\rho]]_0 = \sum_{i=0}^2 \sum_{j=0}^i F_{\rho_{x^i-jy^j}} [a_0] (a_1)_{x^i-jy^j} = \xi, \tag{3.12}$$

$$a_3 = [F''[\rho]]_0 = \sum_{i=0}^2 \sum_{j=0}^i (F_{\rho_{x^i-jy^j}} [a_0] (a_2)_{x^i-jy^j} + \sum_{k=0}^2 \sum_{r=0}^k F_{\rho_{x^i-jy^j} \rho_{x^k-ry^r}} [a_0] (a_1)_{x^i-jy^j} (a_1)_{x^k-ry^r}) = -\xi, \tag{3.13}$$

from equation (2.8), we get the analytical approximate solution

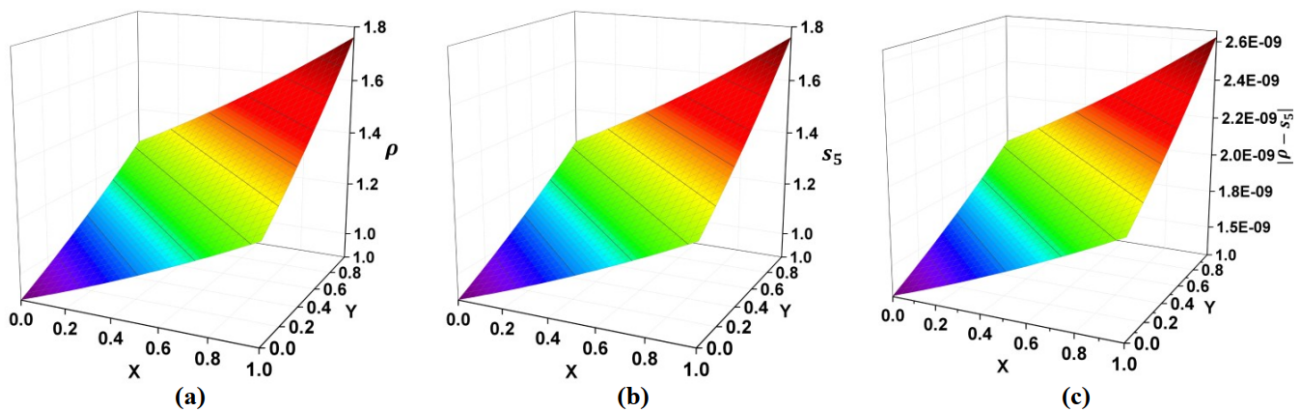
$$\begin{aligned} \rho(x, y, t) &= \xi - \xi t + \xi \frac{t^2}{2!} - \xi \frac{t^3}{3!} + \dots \\ &= \xi (1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots) = \xi e^{-t} = e^{\frac{1}{3}(x+y)-t}. \end{aligned} \tag{3.14}$$

**Table 3:** Comparison between solutions of MCBS-DQM, IIEFGM and present study for **Example 2** at  $x=0.5, y=0.5$ .

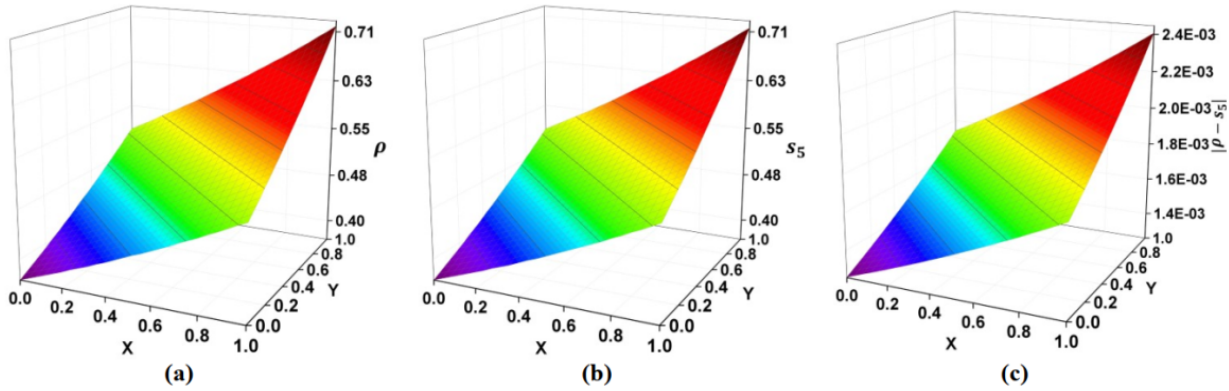
Method	t= 0.1	t= 0.2	t= 0.3	t= 0.4	t= 0.5
Exact	1.26280	1.14263	1.03389	0.93550	0.84645
Present study	1.26280	1.14263	1.03389	0.93550	0.84645
CPU(s)	0	0	0	0	0
MCBS-DQM[16] ( $\Delta t = 0.0001$ )	1.26282	1.14254	1.03381	0.93543	0.84641
CPU(s)	0.31	0.59	0.90	1.22	1.51
IIEFGM[11] ( $\Delta t = 0.0001$ )	1.2574	1.1382	1.0302	0.9325	0.8440
CPU(s)	-	-	-	-	-

**Table 4:** Comparison of errors and CPU time between MCBS-DQM and present study for **Example 2** with  $h_x = h_y = 1/12$ .

Method	Errors	t= 0.1	t= 0.2	t= 0.3	t= 0.4	t= 0.5	t= 5
MCBS-DQM [16]	$L_2$	2.689E-05	2.391E-05	2.141E-05	1.929E-05	1.747E-05	2.33E-07
	$L_\infty$	1.052E-04	8.604E-05	7.037E-05	5.756E-05	4.709E-05	3.40E-07
	CPU(s)	0.33	0.64	0.96	1.29	15.17	30.31
Present study	$L_2$	8.253E-10	8.328E-10	7.741E-10	6.967E-10	3.731E-10	1.24E-07
	$L_\infty$	2.000E-09	2.000E-09	2.000E-09	2.000E-09	2.000E-09	1.59E-07
	CPU(s)	0.109	0.109	0.109	0.109	0.109	0.109



**Fig. 3:** (a) Exact solution, (b) Approximate solution ( $S_5$ ), (c) Absolute errors for **Example 2** where  $t=0.1$  and  $h=0.04$ .



**Fig. 4:** (a) Exact solution, (b) Approximate solution ( $S_5$ ), (c) Absolute errors for **Example 2** where  $t=1$  and  $h=0.04$ .

Figs. (3-4) display the exact solution, approximate solution and absolute errors at time 0.1 and 1 with  $N=25$  and  $h=0.04$ . Comparison of approximate solution at  $x=0.5, y=0.5$  for present study, MCBS-DQM and IEFGM are demonstrated in Table 3. Table 4 shows the comparison between the present study and MCBS-DQM at time 0.1, 0.2, 0.3, 0.4, 0.5 and 5 with  $N=12$  and  $h=1/12$ .

**Example 3.** Consider the equation (1.3) with  $h = \frac{1}{96}, \alpha = -1, r = 48, \beta = 1$ , and  $\rho(x, y, 0) = \frac{1}{4}\sqrt{2x^2 + y(2y + 1)} + 5$ . That is the equation becomes  $\rho_t = \nabla^2(\rho^2) + \frac{1}{96}\rho^{-1}(1 - 48\rho)$ , and has exact solution  $\rho(x, y, t) =$

$\frac{1}{4}\sqrt{2x^2 + y(2y + 1) + \frac{t}{3}} + 5$ . Compare with equation (2.1), we have

$$g(x, y) = -\frac{1}{2}, \tag{3.15}$$

$$F[\rho] = (\rho^2)_{xx} + (\rho^2)_{yy} + \frac{1}{96}\rho^{-1}. \tag{3.16}$$

Let  $\xi = 2x^2 + y(2y + 1) + 5$ , we note that the highest derivative of  $\rho$  is  $n = 2$  and  $t_0 = 0$ , then according to (2.5), we get

$$a_0 = \rho(x, y, 0) = \frac{1}{4}\xi^{\frac{1}{2}}, \tag{3.17}$$

$$a_1 = g(x, y) + [F[\rho]]_0 = ((a_0)^2)_{xx} + ((a_0)^2)_{yy} + \frac{1}{96}(a_0)^{-1} = \frac{1}{24}\xi^{-\frac{1}{2}}, \tag{3.18}$$

$$a_2 = [F'[\rho]]_0 = \sum_{i=0}^2 \sum_{j=0}^i F_{\rho_{x^i-j,y^j}}[a_0] (a_1)_{x^i-j,y^j} = -\frac{1}{144}\xi^{\frac{3}{2}}, \tag{3.19}$$

$$a_3 = [F''[\rho]]_0 = \sum_{i=0}^2 \sum_{j=0}^i (F_{\rho_{x^i-j,y^j}}[a_0] (a_2)_{x^i-j,y^j} + \sum_{k=0}^2 \sum_{r=0}^k F_{\rho_{x^i-j,y^j} \rho_{x^k-r,y^r}}[a_0] (a_1)_{x^i-j,y^j} (a_1)_{x^k-r,y^r}) = \frac{1}{288}\xi^{-\frac{5}{2}}, \tag{3.20}$$

from equation (2.8), we get the analytical approximate solution

$$\rho(x, y, t) = \frac{1}{4}\xi^{\frac{1}{2}} + \frac{1}{24}\xi^{-\frac{1}{2}}t - \frac{1}{144}\xi^{-\frac{3}{2}}\frac{t^2}{2!} + \frac{1}{288}\xi^{-\frac{5}{2}}\frac{t^3}{3!} + \dots, \tag{3.21}$$

**Table 5:** Comparison of absolute errors obtained by ADM, VIM and present study for **Example 3** at  $t=10$ .

$(x, y)$	ADM[6]	VIM[6]	Present study
(-450,-450)	2.573384E-6	7.272155E-12	2.525523E-21
(-400,-400)	3.257144E-6	1.106995E-11	5.761320E-21
(-300,-300)	5.791548E-6	3.578650E-11	4.319163E-20
(-250,-250)	8.341020E-6	7.431441E-11	1.548500E-19

(0,0)	8.094717E-1	1.777776E-1	2.963822E-3
(50,50)	2.068923E-4	4.572776E-8	1.182434E-14
(100,100)	5.192803E-5	2.877083E-9	9.332434E-17
(200,200)	1.300298E-5	1.803953E-10	7.325588E-19
(350,350)	4.248496E-6	1.906919E-11	1.460245E-20
(500,500)	2.082251E-6	4.562573E-12	1.203501E-21

**Table 6:** Comparison of absolute errors obtained by ADM, VIM and present study for **Example 3** at  $t= 20$ .

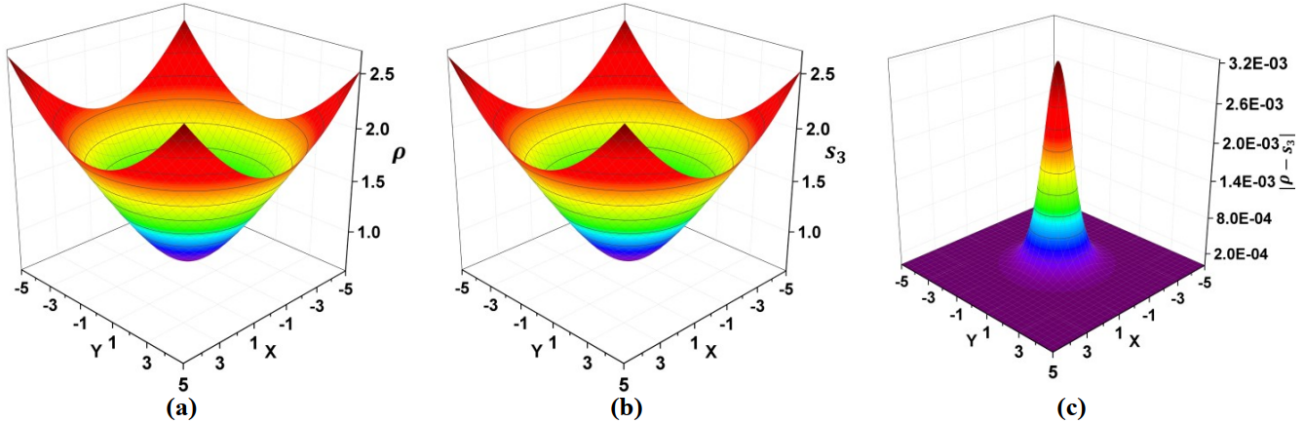
$(x, y)$	ADM[6]	VIM[6]	Present study
(-450,-450)	1.029354E-5	7.272155E-12	4.040827E-20
(-400,-400)	2.316619E-5	1.106995E-11	9.218078E-20
(-300,-300)	2.316619E-5	3.578650E-11	6.910615E-19
(-250,-250)	3.336408E-5	7.431441E-11	2.477577E-18
(0,0)	3.254777	1.777776E-1	3.637376E-2
(50,50)	8.275692E-4	3.658223E-7	1.891451E-13
(100,100)	2.077121E-4	2.301665E-8	1.493103E-15
(200,200)	5.201193E-5	1.442949E-9	1.172077E-17
(350,350)	1.699398E-5	1.538449E-10	2.336381E-19
(500,500)	8.329005E-6	3.683809E-11	1.925597E-20

**Table 7:** Comparison of errors and CPU time between MCBS-DQM and present study for **Example 3**.

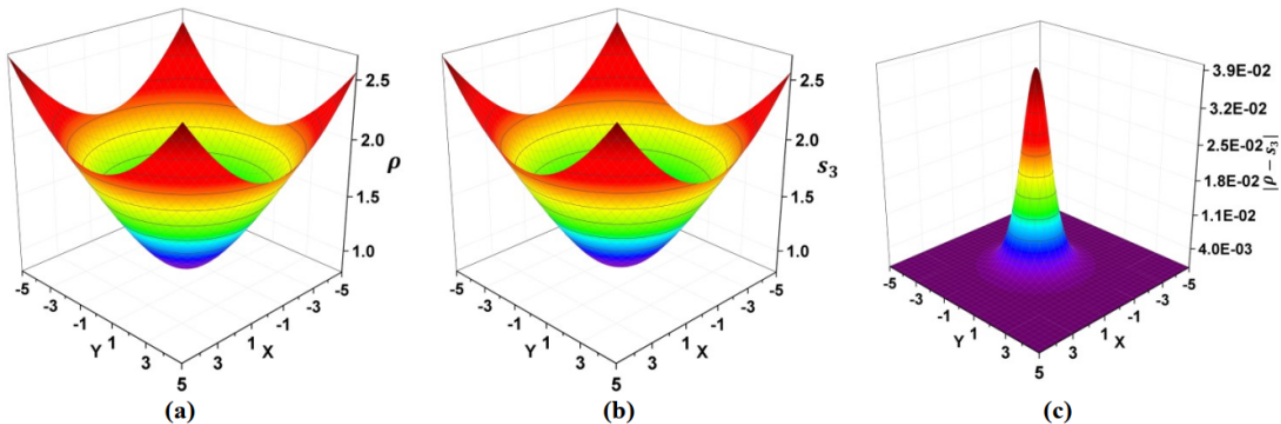
Method	$h_x = h_y$	Errors	$t = 0.1$	$t = 0.2$	$t = 0.5$
MCBS-DQM[16]	0.03	$L_2$	6.43E-06	6.39E-06	6.36E-06
		$L_\infty$	3.70E-05	3.69E-05	3.69E-05
		CPU(s)	4.84	9.76	24.4
	0.04	$L_2$	9.46E-06	9.53E-06	9.56E-06
		$L_\infty$	3.65E-05	3.65E-05	3.65E-05
		CPU(s)	2.23	4.48	11.24
Present study	0.03	$L_2$	5.70E-09	4.60E-08	7.11E-07
		$L_\infty$	1.04E-08	8.24E-08	1.27E-06
		CPU(s)	0.391	0.484	0.547
	0.04	$L_2$	5.82E-09	4.66E-08	7.20E-07
		$L_\infty$	1.04E-08	8.24E-08	1.27E-06
		CPU(s)	0.251	0.282	0.250

**Table 8:** Comparison of errors and CPU time between MCBS-DQM and present study for **Example 3** with  $h_x = h_y = 1/12$ .

Method	Errors	$t=5$	$t=10$	$t=15$
MCBS-DQM[16]	$L_2$	2.71E-05	2.50E-05	2.33E-05
	$L_\infty$	3.64E-05	3.65E-05	3.65E-05
	CPU(s)	12.14	30.22	45.67
Present study	$L_2$	1.78E-10	2.42E-10	3.62E-06
	$L_\infty$	4.00E-10	6.00 E-10	3.64E-05
	CPU(s)	1.749	1.641	1.532



**Fig. 5:** (a) Exact solution, (b) Approximate solution ( $S_3$ ), (c) Absolute errors for **Example 3** where  $t = 10$ .



**Fig. 6:** (a) Exact solution, (b) Approximate solution ( $S_3$ ), (c) Absolute errors for **Example 3** where  $t = 20$ .

Figs. (5-6) display the exact solution, approximate solution and absolute errors at time 10 and 20. Comparison of numerical solution for the present study, ADM and VIM at time 10 and 20 respectively are demonstrated in Tables (5-6). The comparison between the present study and MCBS-DQM are demonstrated in Table 7 at time 0.1, 0.2 and 0.5 with  $N=33$ ,  $N=25$ ,  $h=0.03$  and  $h=0.04$ , and in Table 8 at time 5, 10 and 15 with  $N=12$  and  $h=1/12$ . The present method is perfect and gives a good approximate solutions. The above figures and tables show that this method is efficient and has high accuracy. We get exact solution for examples 1 and 2 and approximate solutions for example 3.  $L_2, L_\infty$  for the series approximation solutions  $S_n$  of the three test examples computed by

$$\|E\|_{L_2} = \sqrt{h^2 \sum_{i=0}^N \sum_{j=0}^N |\rho_{exact}(x, y) - \rho_{approx}(x, y)|^2},$$

$$\|E\|_{L_\infty} = \max_{i,j=0,1,\dots,N} (|\rho_{exact}(x, y) - \rho_{approx}(x, y)|),$$

where  $t$  is fixed and  $\rho_{approx}(x, y, t) = S_n = \sum_{i=0}^n a_i \frac{t^i}{(i)!}$ . Tables (1, 3) show that the approximation solution  $S_5$  is closer to



the exact solution with very small CPU time at point (0.5, 0.5) and times 0.1, 0.2, ..., 0.5 than IEFGM and MCBS-DQM. Tables (2, 4, 8) illustrate the approximation solutions for the three tests and show that the  $L_2$ ,  $L_\infty$  and CPU time for new technique are smaller than MCBS-DQM with  $\Delta t = 0.0001$  and  $h_x = h_y = 1/12$ . Also, for example 3 at time 10 and 20 Tables (5-6) documenting that the new method have absolute errors smaller than ADM [4] and VIM [4]. Table 7 show that  $L_2$ ,  $L_\infty$  and CPU time for approximation solution  $S_2$  is better than MCBS-DQM with  $\Delta t = 0.0001$  and  $h_x = h_y = 0.03, 0.04$  at time 0.1, 0.2 and 0.5. The series of approximation solutions in Tables (1-4, 7, 8) was taken with least  $n$  make the results best than MCBS-DQM. Figures (1-2), (3-4) and (5-6) display that the approximate solutions  $S_5, S_5$  and  $S_3$  are identical with exact solution for examples 1, 2 and 3 at times  $\{0.1, 0.5\}$ ,  $\{0.1, 1\}$  and  $\{10, 20\}$  respectively. The numerical results show that the present method compared with other methods is the most efficient and the best, because it gives good results with a few iteration of solutions and small errors.

### 4 Convergence Analyses

In this section, we study the analysis of convergence depending on some definitions and theorems. From equation (1.3) let

$$F[\rho] = F(\rho, \rho_x, \rho_y, \rho_{xx}, \rho_{yy}) = \frac{\partial^2(\rho^2)}{\partial x^2} + \frac{\partial^2(\rho^2)}{\partial y^2} + h\rho^\alpha(1 - r\rho^\beta), \tag{4.1}$$

and by applying the current method, we get

$$\rho(x, y, t) = a_0 + a_1t + a_2 \frac{t^2}{2!} + a_3 \frac{t^3}{3!} + \dots + a_n \frac{t^n}{n!} + \dots, \tag{4.2}$$

where

$$a_0 = \rho_0(x, y, 0), a_1 = g(x, y) + [F[\rho]]_0, a_2 = [F'[\rho]]_0, a_3 = [F''[\rho]]_0, a_n = [F^{(n-1)}[\rho]]_0. \tag{4.3}$$

#### Definition 1 (Taylor’s Formula)

Let  $f$  be a function with derivatives of all orders throughout some interval containing  $a$  as an interior point. Then the Taylor series generated by  $f$  at  $x = a$  is

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n(x), \tag{4.4}$$

where  $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1}$  for some  $c$  between  $a$  and  $x$ .

#### Definition 2 (Radius of Convergence)

Suppose that  $a_n \neq 0$  for all sufficiently large  $n$  and the  $R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$  exists or diverges to infinity. Then the power series  $\sum_{n=0}^{\infty} a_n(x - c)^n$  has radius of convergence  $R$ .

Now, we will state theorem for convergence biological population model.

#### Theorem 4.1 (Convergence of Biological Population Model)

If  $\rho(x, y, t)$  have  $(n + 1)$  times differentiable with respect to  $t$  and  $\lim_{n \rightarrow \infty} \left| F^{(n)}[\rho_c] \frac{t^{n+1}}{(n+1)!} \right| = 0$  for some  $c$  between 0 and  $t$ , then the approximate solution  $\sum_{i=0}^{\infty} a_i \frac{t^i}{i!}$  of the biological population model in equation (1.3) is convergent to exact solution  $\rho(x, y, t)$ .

**Proof:** Since  $\rho(x, y, t)$  have  $(n + 1)$  times differentiable with respect to  $t$  and  $\lim_{n \rightarrow \infty} \left| F^{(n)}[\rho_c] \frac{t^{n+1}}{(n+1)!} \right| = 0$  then

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| F^{(n)}[\rho_c] \frac{t^{n+1}}{(n+1)!} \right| &= \lim_{n \rightarrow \infty} \left( \rho(x, y, t) - \sum_{i=0}^n a_i \frac{t^i}{i!} \right) = 0 \\ \rho(x, y, t) - \lim_{n \rightarrow \infty} \sum_{i=0}^n a_i \frac{t^i}{i!} &= 0 \\ \rho(x, y, t) &= \sum_{i=0}^{\infty} a_i \frac{t^i}{i!}. \end{aligned}$$

□

#### Theorem 4.3 (The Ratio Test)

Let  $\sum_{n=0}^{\infty} b_n$  be a series with positive terms and suppose that  $\lim_{n \rightarrow \infty} \frac{b_{n+1}}{b_n} = p$ , then

- (a) The series converges if  $p < 1$ ,
- (b) The series diverges if  $p > 1$  or  $p$  is infinite,
- (c) The test is inconclusive if  $p = 1$ .

We will determine the radius of convergence to the series approximation solutions by the following theorem.

#### Theorem 4.4 (Radius of Convergence to Biological Population Model)

The series approximation solutions of the biological population model  $\sum_{i=0}^{\infty} a_i \frac{t^i}{i!}$  have radius of convergence

$$R = \lim_{i \rightarrow \infty} \frac{\left| \frac{a_i}{(i)!} \right|}{\left| \frac{a_{i+1}}{(i+1)!} \right|} = \lim_{i \rightarrow \infty} \left| \frac{a_i}{a_{i+1}} \right| (i+1) \text{ hold one of the following cases ;}$$

- (a) If  $R = \infty$ , then for all  $t$ ,
- (b) If  $0 < R < \infty$ , then for all  $|t| < R$ ,
- (c) If  $R = 0$ , then only for  $t = 0$ .

**Proof:** Let  $p = \lim_{i \rightarrow \infty} \frac{\left| \frac{a_{i+1} \frac{t^{i+1}}{(i+1)!} \right|}{\left| \frac{a_i \frac{t^i}{i!} \right|} = \lim_{i \rightarrow \infty} \left| \frac{a_{i+1}}{a_i} \right| \frac{|t|}{i+1} = \frac{|t|}{R}$ ,

then by theorem 4.3 the series converges if  $p < 1 \Rightarrow \frac{|t|}{R} < 1 \Rightarrow |t| < R$ , so, the radius of convergence is  $R$ , therefore, if  $R = \infty$ , then the series approximation solutions converges for all  $t$ , if  $0 < R < \infty$ , then the series converges for all  $|t| < R$  and if  $R = 0$ , then the series approximation solutions converges only for  $t = 0$ .

We have to know if the series approximation solutions converges at the end points  $t = \pm R$ , to do this we must check it directly.

Now Let's find the convergence analysis for the above test problems. Clearly that the exact solutions  $\rho$  for the three tests have  $(n+1)$  times differentiable with respect to  $t$  and

$$\lim_{n \rightarrow \infty} \left| F^{(n)}[\rho_c] \frac{t^{n+1}}{(n+1)!} \right| \leq m \lim_{n \rightarrow \infty} \frac{t^{n+1}}{(n+1)!} = 0, \text{ where } m \geq 0,$$

hence, the series approximation solution converges to the exact solution function for the three tests.

To find the radius of convergence

#### In the first example

$$\rho(x, y, t) = \xi^2 + \frac{1}{5} \xi^2 t + \frac{1}{25} \xi^2 \frac{t^2}{2!} + \frac{1}{125} \xi^2 \frac{t^3}{3!} + \dots = \sum_{i=0}^{\infty} \xi^2 \frac{\left(\frac{t}{5}\right)^i}{i!} \quad (4.6)$$

$$\lim_{i \rightarrow \infty} \left| \frac{a_i}{a_{i+1}} \right| (i+1) = \lim_{i \rightarrow \infty} \left| \frac{\xi^2 \left(\frac{1}{5}\right)^i}{\xi^2 \left(\frac{1}{5}\right)^{i+1}} \right| (i+1) = \lim_{i \rightarrow \infty} 5(i+1) = \infty, \quad (4.7)$$

hence  $|t| < \infty$ .

#### The second example

$$\rho(x, y, t) = \xi - \xi t + \xi \frac{t^2}{2!} - \xi \frac{t^3}{3!} + \dots = \sum_{i=0}^{\infty} \xi \frac{(-t)^i}{i!} \quad (4.7)$$

$$\lim_{i \rightarrow \infty} \left| \frac{a_i}{a_{i+1}} \right| (i+1) = \lim_{i \rightarrow \infty} \left| \frac{\xi (-1)^i}{\xi (-1)^{i+1}} \right| (i+1) = \lim_{i \rightarrow \infty} (i+1) = \infty \quad (4.8)$$

hence  $|t| < \infty$ .

#### The third example

$$\begin{aligned} \rho(x, y, t) &= \frac{1}{4} \xi^2 + \frac{1}{24} \xi^{-\frac{1}{2}} t - \frac{1}{144} \xi^{-\frac{3}{2}} \frac{t^2}{2!} + \frac{1}{288} \xi^{-\frac{5}{2}} \frac{t^3}{3!} + \dots \\ &= \frac{1}{4} \xi^2 + \frac{1}{24} \xi^{-\frac{1}{2}} t + \sum_{i=2}^{\infty} \frac{(-1)^{i-1}}{24} \left( \prod_{j=1}^{i-1} \frac{(2j-1)}{6} \right) \xi^{\frac{(1-2i)}{2}} \frac{t^i}{i!} \end{aligned} \quad (4.9)$$

$$\text{a. } \left| \frac{a_0}{a_1} \right| = \left| \frac{\frac{1}{4} \xi^2}{\frac{1}{24} \xi^{-\frac{1}{2}}} \right| = 6 \xi \quad (4.10)$$

$$\text{b. } \left| \frac{a_1}{a_2} \right| = \left| \frac{\frac{1}{24} \xi^{-\frac{1}{2}}}{-\frac{1}{144} \xi^{-\frac{3}{2}}} \right| = 12 \xi \quad (4.11)$$

$$\begin{aligned} \text{c. } \lim_{i \rightarrow \infty} \left| \frac{a_i}{a_{i+1}} \right| (i+1) &= \lim_{i \rightarrow \infty} \left| \frac{\frac{(-1)^{i-1}}{24} \left( \prod_{j=1}^{i-1} \frac{(2j-1)}{6} \right) \xi^{\frac{(1-2i)}{2}}}{\frac{(-1)^i}{24} \left( \prod_{j=1}^i \frac{(2j-1)}{6} \right) \xi^{\frac{(-1-2i)}{2}}} \right| (i+1) = \lim_{i \rightarrow \infty} \left| \frac{1}{-\frac{(2i-1)}{6}} \right| \xi (i+1) \\ &= \lim_{i \rightarrow \infty} \frac{6(i+1)}{(2i-1)} \xi = 3 \xi \end{aligned} \quad (4.12)$$

$$R = \min(a, b, c) = 3 \xi \quad (4.13)$$

hence  $|t| < 3\xi = 3(2x^2 + y(2y + 1) + 5)$ .

Let  $x = y = 0 \Rightarrow |t| < 15$ ,

$$\begin{aligned} \text{if } t = 15 \Rightarrow \sum_{i=0}^{\infty} a_i \frac{t^i}{i!} &= \frac{1}{4}5^{\frac{1}{2}} + \frac{1}{24}5^{-\frac{1}{2}}15 + \sum_{i=2}^{\infty} \frac{(-1)^{i-1}}{24} \prod_{j=1}^{i-1} \frac{(2j-1)}{6} 5^{\frac{(1-2i)}{2}} \frac{15^i}{i!} \\ &= 0.84 - 0.07 + 0.03 - 0.02 + \dots, \end{aligned} \tag{4.14}$$

$$\begin{aligned} \text{if } t = 20 \Rightarrow \sum_{i=0}^{\infty} a_i \frac{t^i}{i!} &= \frac{1}{4}5^{\frac{1}{2}} + \frac{1}{24}5^{-\frac{1}{2}}20 + \sum_{i=2}^{\infty} \frac{(-1)^{i-1}}{24} \prod_{j=1}^{i-1} \frac{(2j-1)}{6} 5^{\frac{(1-2i)}{2}} \frac{20^i}{i!} \\ &= 0.93 - 0.12 + 0.08 - 0.07 + \dots, \end{aligned} \tag{4.15}$$

the equations (4.14) and (4.15) is convergent alternating series since each term is smaller in absolute value than the one preceding it and as  $i$  goes to  $\infty$ , the  $i$ th term converges to 0. Therefore, the series approximation solution converges at  $t = 15$  and  $20$ , as well as between them. From the above proofs and their application, we conclude that the convergent of analytical approximate solutions obtained by the present method is valid.

## 5 Conclusions

In this paper, we used a new simulation scheme for solving two-dimensional biological population model. The numerical results of three test problems confirm that the present method is an efficient method with good convergence, high accuracy with small absolute errors,  $L_2$ ,  $L_\infty$ , and CPU time. This method can be used to solve a nonlinear partial differential equations because it has the best formula needed to derivatives only; so that, its application is easy compared with difficult computation of Adomian decomposition method which needs to apply Adomian polynomial. The results of the present method are the best in measurements of errors and CPU time with good convergent of solutions. The computations are done using Maple18.

## References

- [1] M.E. Gurtin and R.C. MacCamy, On the diffusion of biological population, *Math. Biosci.*, **33**, 35–49, 1977.
- [2] W.S.C. Gurney and R.M. Nisbet, The regulation of inhomogeneous populations, *J. Theor. Biol.*, **52**, 441–457, 1975.
- [3] Y.G. Lu, Hölder estimates of solutions of biological population equations, *Appl. Math. Lett.*, **13**, 123–126, 2000.
- [4] J. Bear, Dynamics of fluids in porous media, American Elsevier, New York, 1972.
- [5] A. Okubo, Diffusion and ecological problem, *Mathematical Models*, Biomathematics 10, Sp, Berlin (1980).
- [6] F. Shakeri and M. Dehghan, Numerical solution of a biological population model using He’s variational iteration method, *Comput. Math. Appl.*, **54**, 1197–1209, 2007.
- [7] A.M.A. El-Sayed, S.Z. Rida and A.A.M. Arafa, Exact solutions of fractional-order biological population model, *Commun. Theor. Phys.*, **52**, 992–996, 2009.
- [8] P. Roul, Application of homotopy perturbation method to biological population model, *Appl. Appl. Math.*, **10**, 1369–1378, 2010.
- [9] A. A. M. Arafa, S. Z. Rida and H. Mohamed, Homotopy analysis method for solving biological population model, *Commun. Theor. Phys.*, **56**, 797–800, 2011.
- [10] D. Kumar, J. Singh and S. Rathore, Application of homotopy analysis transform method to fractional biological population model, *Romanian Reports in Physics.*, **65(1)**, 63–75, 2013.
- [11] L. W. Zhang, Y. J. Deng, and K. M. Liew, An improved element-free Galerkin method for numerical modelling of biological population problems, *Engineering Analysis with Boundary Elements.*, **40**, 181–188, 2014.
- [12] S. Sarwar, M. A. Zahid and S. Iqbal, Mathematical study of fractional-order biological population model using optimal homotopy asymptotic method, *Int. J. Biomath.*, **9**, 1650081, 2016.
- [13] O. Acana, M. M. Al Qurashi and D. Baleanu, New exact solution of generalized biological population model, *J. Nonlinear Sci. Appl.*, **10**, 3916–3929, 2017.
- [14] M. Shakeel, M. A. Iqbal and S. T. Mohyud-Din, Closed form solutions for nonlinear biological population model., **26(1)**, 207–223, 2018.
- [15] A. S. J. Al-Saif and M.S. Abdul-Wahab, A new technique for simulation the Zakharov-Kuznetsov equation, *Journal of Advances in Mathematics.*, **14(2)**, 7912-7920, 2018.
- [16] B.K. Singh, A novel approach for numeric study of 2D biological population model, *Cogent Math.*, **3**, 1261527, 2016.