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New Travelling Wave Solutions for the Two-Dimensional Navier-Stokes and Heat Equations

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Abstract: In this work, the Kudryashov method is used to find the exact travelling wave solutions of the two-dimensional Navier-Stokes and Heat equations. some new forms of exact travelling wave solutions are found.

Keywords: The Kudryashov Method, Travelling Wave Solutions, Navier-Stokes and Heat equation.
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Introduction

Analytical solutions have offered important advantages in the development of various fluid flow systems.These solutions serve as fundamental basis for the comparison between fluid dynamic nature and system evolution trends. The analytical solutions of incompressible flow and constant coefficient of heat conduction have been derived from the bases of fluid dynamics and heat transfer[1,2]. The Navier-Stokes and heat equations are one of the most useful sets of equations that is used to describe the flow field with functions of space and the time in Eulerian representation. These equations can be used to describe many different physical and engineering problems[3-5]. These equations are nonlinear in nature and they have solutions for description the velocity of the fluid and temperature distributions at a any point in space and time. Nonlinear partial differential equations (NPDEs) are difficult to be solved therefore this motives the researchers to rise their interesting with some phenomena such as chaos, Navier-Stokes and heat equations. The exact solutions of these NPDEs are an important scope to study of the nonlinear phenomena. In past decades, many methods have been developed to find exact solutions for NPDEs such as Kudryashos method(KM) [9-11], new similarity transformation method [21], homogeneous balance method [22] and tanh function method [23-25]. The Kudryashov Method is considered as transformation methods: due to it has the application of the transform $\xi = \sum_{i=0}^{n} a_i x_i$, the partial differential equations (PDEs) which can be reduced to tractable ordinary differential equation, where ξ is an independent variable of ordinary differential equations and it is also called a phase of the wave. $x_1 \dots x_n$ are independent variables of the partial differential equations and $a_1 \dots a_n$ are arbitrary constants. Some of the problems in this field are solved numerically a little of them are solved analytically . So that solving this equations analytically be difficult. To our under standing, Navier â€“stokes equations and heat equation are unresolved by the Kudryashov method that is represented as a good way to find exact wave traveling solutions. These information with regard to finding exact traveling wave solutions encourage us to solve these equations by kudryashov method which deals with the complicated problems by combination of from heat transfer equation and fluid flow equation(Naviers-Stokes equations). The scope of this paper is to determined the efficiency of the Kudryashov method for finding exact solutions of the fluid flow systems that are specified by Navier -Stokes equations and heat equation. In this work, we use the Kudryashov method to find the exact traveling wave solutions for two-dimensional Navier-Stokes and heat equations. Two cases were discussed to find exact solutions of the Navier -Stokes equations and each case contains three types of exact wave travelling solution, which can be written in terms of the exponential, sech and csch functions. The exact travelling wave solutions for Navier-Stokes with heat equations divided into sets, each set contains two cases, which can be written in terms of the exponential, sech and csch functions. The layout of this paper is systematized as follows: The description of the Kudryashov method in section (2). In section (3) the two-dimensional Navier -Stokes

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equations, and the two-dimensional Navier -Stokes and heat equations are described. Applications of KM to solve of these equations are given insection (4). Finally, conclusions are reported.

1.1 The Kudryashov Method

The method has been extensively used for solving many nonlinear partial differential equations. Consider the nonlinear equation in three independent variables x , y and t:

$$F(u,\frac{\partial u}{\partial t},\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x^2},\dots)=0,\quad x,y\varepsilon\Omega,t>0, \tag{1}$$

where Ω is a smooth bounded domain with boundary $\partial\Omega$, $u=u(x,y,t)$ is an unknown function, x and y are the spatial variables, t is the time variable, F is a polynomial in u and its various partial derivatives, which are involved the highest order derivatives and nonlinear terms.

The main steps of the Kudryashov method are written as follows:

Step 1: The traveling wave variable $\xi=ex+ky-ct$ transform Equation(1) into an ordinary differential equation of the form :

$$G(u,-c\frac{du}{d\xi},e\frac{du}{d\xi},k\frac{du}{d\xi},e^2\frac{d^2u}{d\xi^2},\dots)=0, \tag{2}$$

where e and k are the wave numbers and c is a wave speed, Equation(2) may be integrated as many times as required. For simplicity, the constants of integration, should be set to zero.

Step 2: Suppose that the solution of Equation (2) has the following form:

$$u(\xi)=\sum_{i=0}^{N} a_iQ^i, \tag{3}$$

where $a_i(i=0,1,2,\dots,N)$ are constants to be determined afterward such that $a_N\neq 0, Q=Q(\xi)$, the following ordinary differential equation becomes :

$$\frac{d^2Q(\xi)}{d\xi^2}=Q^2(\xi)-Q(\xi), \tag{4}$$

The solution of Equation (4) becomes as follows:

$$Q(\xi)=\frac{1}{1+AExp(\xi)}, \tag{5}$$

where A is a constant of integration.

Step 3: The positive integers N appearing in Equation (3) can be found by consideration of the homogenous balance between the highest order derivatives and the term $u^p(\frac{d^qu}{d\xi^q})^s$ which obtain in Equation (2), The degree of $u(\xi)$ as $D(u(\xi))=N$ which gives rise to the degree of other expression can be defined as follows:

$$D[\frac{d^ku)}{d\xi^k}]=N+k,\quad k=1,2,\dots \tag{6}$$

$$D[u^p(\frac{d^qu}{d\xi^q})^s]=N(p+1)+s(N+q),\quad p=0,1,\dots\quad q=0,1,\dots\quad s=1,2,\dots \tag{7}$$

where k,p,q,s are integer numbers. Therefore, the value of N can be found in Equation (3).

1.2 Description of the Equations

- **The two-dimensional Navier-Stokes equations**

In this subsection, the Navier-Stokes equations are described. Let us consider the unsteady state two-dimensional incompressible Navier-Stokes equations [12] as follows:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{8}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\mu\nabla^2u,\quad t>0,(x,y)\varepsilon\Omega \tag{9}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\mu\nabla^2v, \tag{10}$$

where u and v are velocity components in $x-$ direction and $y-$ direction, respectively. p is the pressure, t is the time,x and y are the space coordination, μ is the kinematic viscosity, ρ is the fluid density and ∇^2 is the Laplacian operator. In order to facilitate analysis, the following dimensionless variables become:-

$$u'=\frac{u}{U_0},v'=\frac{v}{U_0},x'=\frac{x}{L},y'=\frac{y}{L},p'=\frac{p}{U_0^2} \tag{11}$$

where U_0 is a reference velocity, and L is a reference length. After drop the primes for the new variables definitions, the equations becomes:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{12}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{Re}\nabla^2 u, \tag{13}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{1}{Re}\nabla^2 v, \tag{14}$$

where $Re=\frac{U_0 L}{\mu}$ is Reynolds number. In Cartesian coordinate system, we can defined the vorticity function ω and the velocity components as follows:

$$\omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}. \tag{15}$$

$$u=\frac{\partial \psi}{\partial y}, v=-\frac{\partial \psi}{\partial x}, \tag{16}$$

where ψ is the stream function. Differentiating Equations (13) and (14) with respect to x and y respectively also subtraction the results equations from Equations (16) and (12) we obtain :

$$-\frac{\partial^3 \psi}{\partial t\,\partial x^3}-\frac{\partial^3 \psi}{\partial t\,\partial y^3}-\frac{\partial \psi}{\partial y}\frac{\partial^3 \psi}{\partial x^3}-\frac{\partial \psi}{\partial y}\frac{\partial^3 \psi}{\partial x\,\partial y^2}+\frac{\partial \psi}{\partial x}\frac{\partial^3 \psi}{\partial y\,\partial x^2}+\frac{\partial \psi}{\partial x}\frac{\partial^3 \psi}{\partial y^3}+$$

$$\frac{1}{Re}\left[\frac{\partial^4 \psi}{\partial x^4}+\frac{\partial^4 \psi}{\partial y^4}+\frac{\partial^4 \psi}{\partial x^2\,\partial y^2}+\frac{\partial^4 \psi}{\partial y^2\,\partial x^2}\right]=0 \tag{17}$$

This partial differential equation is called the stream equation. Now, a Poisson equation for ψ can be obtained by substitution of the velocity components Equation (16) in the Equation (15) as,

$$\omega=-\frac{\partial^2 \psi}{\partial x^2}-\frac{\partial^2 \psi}{\partial y^2}. \tag{18}$$

- **The two-dimensional Navier-Stokes and heat equations**

In this subsection, the Banard problem can be considered. The unsteady state two-dimensional Heat Benard [6] as follows:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{19}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+pr\nabla^2 u, \tag{20}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+pr\nabla^2 v+prRaT, \tag{21}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}, \tag{22}$$

The governing equations are the continuity equation, the $x-$ is momentum equation, the $y-$ is momentum equation, and the energy equation can be drawn in accordance to Figure(1).

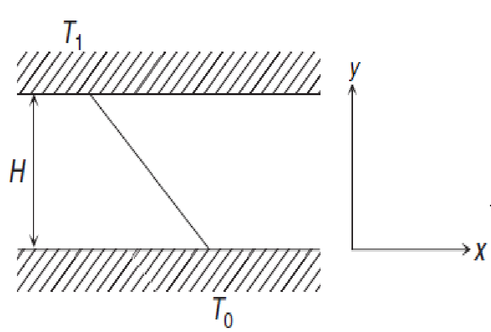

Fig.1: Schematic of the Benard problem.

In these equations, pr is the prandtl number which expresses the ratio (viscous diffusion)/(thermal diffusion)and given as $pr = \frac{\mu}{k}$. The characteristic non-dimensional number for this problem is the Rayleigh number as defined below:

$$Ra = \frac{g\alpha\Delta T H^3}{k\mu}, \tag{23}$$

here, g is the gravitational acceleration, α is the thermal expansion coefficient, ΔT is the temperature difference between the lower(hot) surface and the upper(cold) surface, H is the height of the fluid layer, k is the thermal diffusivity and μ is the kinematic viscosity of the fluid. Differential Equation (20) and Equation (21) with respect to x and y respectively, and subtracting the results equations with use Equations (16) and (19) give the following system of equations

$$-\frac{\partial^3\psi}{\partial t\,\partial x^3} - \frac{\partial^3\psi}{\partial t\,\partial y^3} - \frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial x^3} - \frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial x\,\partial y^2} + \frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial y\,\partial x^2} + \frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial y^3} +$$

$$pr[\frac{\partial^4\psi}{\partial x^4} + \frac{\partial^4\psi}{\partial y^4} + \frac{\partial^4\psi}{\partial x^2\,\partial y^2} + \frac{\partial^4\psi}{\partial y^2\,\partial x^2}] - prRa\frac{\partial T}{\partial x} = 0, \tag{24}$$

$$\frac{\partial T}{\partial t} + \frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y} - \frac{\partial^2 T}{\partial x^2} - \frac{\partial^2 T}{\partial y^2} = 0, \tag{25}$$

2 Applications

In this section, we apply the Kudryashov method for the two-dimensional Navier -stokes equations and the two -dimensional Navier -stokes equations with heat equation as the following:

- **Two-dimensional Navier -Stokes equations**

we will apply the Kudryashov method for the two-dimensional Navier -Stokes equations by use the traveling wave transformation as:

$$\psi(x,y,t) = F(\xi), \xi = ex + ky - ct, \tag{26}$$

substituting this traveling wave into Equation (17), $F(\xi)$ satisfies

$$c(e^2+k^2)\frac{d^3F(\xi)}{d\xi^3} + \frac{1}{Re}(e^2+k^2)^2\frac{d^4F(\xi)}{d\xi^4} = 0, \tag{27}$$

Integrating Equation (27) three times with respect to ξ and choosing the constant of integration as zero to obtain the following ordinary differential equation:

$$cF(\xi) + \frac{1}{Re}(e^2+k^2)\frac{dF(\xi)}{d\xi} = 0, \tag{28}$$

Now, homogeneous balancing the highest order derivative $\frac{dF(\xi)}{d\xi}$ and the term $F^0(\xi)\frac{d^0F(\xi)}{d\xi^0}$ with apply Equation (6) and (7) satisfies;

$$D(\frac{dF(\xi)}{d\xi}) = N+1, \tag{29}$$

$$D(F^0(\xi)\frac{d^0F(\xi)}{d\xi^0}) = 2N, \tag{30}$$

from Equations (29) and(30), we get $N = 1$, therefore Equation (3) reduces to

$$F(\xi) = a_0 + a_1 Q, \tag{31}$$

$$\frac{dF(\xi)}{d\xi} = a_1Q^2 - a_1Q, \tag{32}$$

where Q is given in Equation (5), substituting Equations (31) and (32) in Equation (28) implies that :

$$ca_0 + ca_1Q - \frac{1}{Re}(e^2+k^2)a_1Q + \frac{1}{Re}(e^2+k^2)a_1Q^2 = 0. \tag{33}$$

Equating the coefficients of this polynomial of the same powers of $Q^k (k = 0,1,2)$ to zero, we obtain a system of algebraic equations:

$$Q^0: ca_0 = 0, \tag{34}$$

$$Q^1: ca_1 - \frac{1}{Re}(e^2+k^2)a_1 = 0, \tag{35}$$

$$Q^2: \frac{1}{Re}(e^2+k^2)a_1 = 0, \tag{36}$$

from Equations (34) and (35), we obtain $a_1 \neq 0, c \neq 0, a_0 = 0, c = \frac{1}{Re}(e^2 + k^2)$. Then we get:

$$F(\xi) = a_1 Q = \frac{a_1}{1+AExp(\xi)}, \quad (37)$$

since $\psi(x,y,t) = F(\xi)$, then

$$\psi(x,y,t) = \frac{a_1}{1+AExp(ex+ky-ct)}. \quad (38)$$

There are several cases for exact traveling wave solutions can be summarized as:

case 1 :$k = e, c = \frac{2e^2}{Re}, \frac{\partial p}{\partial y} = -\frac{\partial p}{\partial x}$,

$$u(x,y,t) = -v(x,y,t) = \frac{-Aea_1 Exp(e(x+y)-\frac{2e^2}{Re}t)}{(1+AExp(e(x+y)-\frac{2e^2}{Re}t))^2},$$

$$p(x,y,t) = -\frac{2a_1 cAExp(e(x+y)-\frac{2e^2}{Re}t)}{(1+AExp(e(x+y)-\frac{2e^2}{Re}t))3},$$

$$\omega(x,y,t) = \frac{2a_1 e^2[AExp(e(x+y)-\frac{2e^2}{Re}t)-A^2 Exp(2e(x+y)-\frac{4e^2}{Re}t)]}{(1+AExp(e(x+y)-\frac{2e^2}{Re}t))^3},$$

if $A = 1$, the results become:

$$u(x,y,t) = -v(x,y,t) = \frac{-a_1 e}{4} sech^2(\frac{e(x+y)-\frac{2e^2}{Re}t}{2}),$$

$$p(x,y,t) = -\frac{ca_1 sech^2(\frac{e(x+y)-\frac{2e^2 t}{Re}}{2})}{2(1+Exp(e(x+y)-\frac{2e^2}{Re}t))},$$

$$\omega(x,y,t) = \frac{a_1 e^2}{2} sech^2(\frac{e(x+y)-\frac{2e^2 t}{Re}}{2})[\frac{1-Exp(e(x+y)-\frac{2e^2 t}{Re})}{1+Exp(e(x+y)-\frac{2e^2 t}{Re})}],$$

again, if we take $A = -1$, then we obtain:

$$u(x,y,t) = -v(x,y,t) = \frac{a_1 e}{4} csch^2(\frac{e(x+y)-\frac{2e^2}{Re}t}{2}),$$

$$p(x,y,t) = \frac{ca_1 csch^2(\frac{ex+ky-\frac{(e^2+k^2)}{Re})t}{2})}{2(1+Exp(e(x+y)-\frac{2e^2}{Re}t))}$$

$$\omega(x,y,t) = \frac{-a_1 e^2}{2} csch^2(\frac{e(x+y)-\frac{2e^2 t}{Re}}{2})[\frac{1+Exp(e(x+y)-\frac{2e^2 t}{Re})}{1-Exp(e(x+y)-\frac{2e^2 t}{Re})}],$$

case 2 :$e = -k, c = \frac{2k^2}{Re}, \frac{\partial p}{\partial y} \neq -\frac{\partial p}{\partial x}, \xi \to -\infty$,

$$u(x,y,t) = v(x,y,t) = \frac{-Aka_1 Exp(k(y-x)-\frac{2k^2}{Re}t)}{(1+AExp(k(y-x)-\frac{2k^2}{Re}t)^2}$$

$$p(x,y,t)=t-\frac{Aka_1[-4AExp(k(y-x)-\frac{2k^2}{Re})+8AExp(2k(y-x)-\frac{4k^2}{Re})]x}{(1+AExp(k(y-x)-\frac{2k^2}{Re}))^4}$$

or,

$$p(x,y,t)=t-\frac{Aka_1[-4Exp(k(y-x)-\frac{2k^2}{Re})+8AExp(2k(y-x)-\frac{4k^2t}{Re})]y}{(1+AExp(k(y-x)-\frac{2k^2t}{Re}))^4}$$

$$\omega=\frac{2a_1k^2[AExp(k(y-x)-\frac{2k^2t}{Re})-A^2Exp(2k(y-x)-\frac{4k^2t}{Re})]}{(1+AExp(k(y-x)-\frac{k^2t}{Re}))^3},$$

if $A=1$, :

$$u(x,y,t)=v(x,y,t)=-\frac{a_1k}{4}sech^2(\frac{k(y-x)-\frac{2k^2t}{Re}}{2}),$$

$$p(x,y,t)=t+ka_1sech^2(\frac{k(y-x)-\frac{2k^2t}{Re}}{2})[\frac{1}{(1+Exp(k(y-x)-\frac{2k^2t}{Re}))^2}$$
$$-\frac{1}{2}sech^2(\frac{k(y-x)-\frac{2k^2t}{Re}}{2})]x.$$

Or,

$$p(x,y,t)=t+ka_1sech^2(\frac{k(y-x)-\frac{2k^2t}{Re}}{2})[\frac{1}{(1+Exp(k(y-x)-\frac{2k^2t}{Re}))^2}$$
$$-\frac{1}{2}sech^2(\frac{k(y-x)-\frac{2k^2t}{Re}}{2})]y.$$

$$\omega(x,y,t)=\frac{a_1k^2}{2}sech^2(\frac{k(y-x)-\frac{2k^2t}{Re}}{2})[\frac{1-Exp(k(y-x)-\frac{2k^2t}{Re})}{1+Exp(k(y-x)-\frac{2k^2t}{Re})}],$$

and if $A=-1$, the results can be obtained as follows:

$$u(x,y,t)=v(x,y,t)=\frac{a_1k}{4}csch^2(\frac{k(y-x)-\frac{2k^2t}{Re}}{2}),$$

$$p(x,y,t)=t+ka_1csch^2(\frac{k(y-x)-\frac{2k^2t}{Re}}{2})[\frac{1}{(1-Exp(k(y-x)-\frac{2k^2t}{Re}))^2}$$
$$+\frac{1}{2}csch^2(\frac{k(y-x)-\frac{2k^2t}{Re}}{2})]x.$$

Or,

$$p(x,y,t)=t+ka_1csch^2(\frac{k(y-x)-\frac{2k^2t}{Re}}{2})[\frac{1}{(1-Exp(k(y-x)-\frac{2k^2t}{Re}))^2}$$
$$+\frac{1}{2}csch^2(\frac{k(y-x)-\frac{2k^2t}{Re}}{2})]y.$$

$$\omega(x,y,t)=\frac{-a_1k^2}{2}csch^2\left(\frac{k(y-x)-\frac{2ek^2t}{Re}}{2}\right)\left[\frac{1+Exp(k(y-x)-\frac{2e^2t}{Re})}{1-Exp(k(y-x)-\frac{2k^2t}{Re})}\right],$$

- **Two -dimensional Navier -Stokes and heat equations**

The Kudryashov method is applied for the two -dimensional Navier -Stokes and heat equations. The traveling wave transformation can be rewritten;

$$\psi(x,y,t)=H(\xi),T(x,y,t)=F(\xi), \tag{39}$$

using these new traveling wave forms into Equations (24) and (25) respectively, $H(\xi)$ and $F(\xi)$ satisfies;

$$c(e^2+k^2)\frac{d^3H(\xi)}{d\xi^3}+(e^2+k^2)^2pr\frac{d^4H(\xi)}{d\xi^4}=prRae\frac{dF(\xi)}{d\xi} \tag{40}$$

$$c\frac{dF(\xi)}{d\xi}+(e^2+k^2)\frac{d^2F(\xi)}{d\xi^2}=0, \tag{41}$$

from Equation (40) and Equation (41),

$$c^2(e^2+k^2)\frac{d^3H(\xi)}{d\xi^3}+(c(e^2+k^2)^2+c(e^2+k^2)^2pr)\frac{d^4H(\xi)}{d\xi^4}+(e^2+k^2)^3pr\frac{d^5H(\xi)}{d\xi^5}=0, \tag{42}$$

Integrating Equation (42) three times with respect to ξ and the chosen the constants of integration as zero to obtain the following ordinary differential equation:

$$c^2(e^2+k^2)H(\xi)+(c(e^2+k^2)^2+c(e^2+k^2)^2pr)\frac{dH(\xi)}{d\xi}+(e^2+k^2)^3pr\frac{d^2H(\xi)}{d\xi^2}=0, \tag{43}$$

Now, homogeneous balance of the order derivative $\frac{d^2H(\xi)}{d\xi^2}$ and the term $H^0(\xi)\frac{d^0H(\xi)}{d\xi^0}$ with apply Equation (6) and (7) satisfies;

$$D(\frac{d^2H(\xi)}{d\xi^2})=N+2, \tag{44}$$

$$D(H^0(\xi)\frac{d^0H(\xi)}{d\xi^0})=2N, \tag{45}$$

from Equations (44) and (45) , we get $N=2$, therefore Equation (3) reduces to

$$H(\xi)=a_0+a_1Q+a_2Q^2, \tag{46}$$

$$\frac{dH(\xi)}{d\xi}=-a_1Q+a_1Q^2-2a_2Q^2+2a_2Q^3, \tag{47}$$

$$\frac{d^2H(\xi)}{d\xi^2}=a_1Q-3a_1Q^2+4a_2Q^2-10a_2Q^3+2a_2Q^3+6a_2Q^4, \tag{48}$$

substitution of Equations (46), (47)and (48) in Equation (43), and equating the coefficients of $Q^k, k=0,1,2,3,4$ to zero, a system of algebraic equations can be obtained as follows:

$$Q^0: c^2(e^2+k^2)a_0=0, \tag{49}$$

$$Q^1: c^2(e^2+k^2)a_1-c(e^2+k^2)^2(pr+1)a_1+pr(e^2+k^2)^3a_1=0, \tag{50}$$

$$Q^2: c(e^2+k^2)^2a_1+pr(e^2+k^2)^2a_1-2c(e^2+k^2)^2=0,$$

$$(pr+1)a_2-3pr(e^2+k^2)^3a_1+4pr(e^2+k^2)^3a_2=0, \tag{51}$$

$$Q^3=2c(e^2+k^2)^2(pr+1)a_2-10pr(e^2+k^2)^3a_2+2pr(e^2+k^2)^3a_1=0, \tag{52}$$

$$Q^4: 6pr(e^2+k^2)^3a_2=0, \tag{53}$$

From Equation (49)$c^2(e^2+k^2)\neq 0, a_0=0$, if we impose $a_1\neq 0$ then the following relations can be obtain from Equation(50):

$$c=pr(e^2+k^2), \tag{54}$$

$$c=(e^2+k^2), \quad (55)$$

substitution of (45) and (55) in Equation (51) yield

$$a_1=a_2, \quad a_1,a_2\neq 0, \quad (56)$$

other values can be obtain from substitution of Equations (54) and (55) in the Equation(52) respectively as follows

$$a_1=\frac{2cpr-8c}{-2c}a_2, \quad (57)$$

$$a_1=\frac{-8cpr+2c}{-2cpr}a_2, \quad (58)$$

Now, the solutions can be analyzed in two sets based on the values of the parameter a_1, a_2 and pr.

Set 1, when $a_0=0, a_1=a_2, c=pr(e^2+k^2)$:

$$H(\xi)=a_1Q+a_2Q^2, \quad (59)$$

$$H(\xi)=\frac{a_1(2+AExp(\xi))}{(1+AExp(\xi))^2}, \quad (60)$$

$$\psi(x,y,t)=\frac{a_1(2+AExp(ex+ky-ct))}{(1+AExp(ex+ky-ct))^2}, \quad (61)$$

$$F(\xi)=\frac{ca_1A(e^2+k^2)[-6Exp(\xi)+18AExp(\xi)]}{(1+AExp(\xi))^5}, \quad (62)$$

$$T(x,y,t)=\frac{ca_1A(e^2+k^2)[-6Exp(ex+ky-ct)+18AExp(2ex+2ky-2ct)]}{(1+AExp(ex+ky-ct))^5}, \quad (63)$$

Set 2, when $a_0=0, a_1=3a_2, pr=1, c=(e^2+k^2)$:

$$H(\xi)=\frac{a_2(4+3AExp(\xi))}{(1+AExp(\xi))^2}, \quad (64)$$

$$\psi(x,y,t)=\frac{a_2(4+3AExp(ex+ky-ct))}{(1+AExp(ex+ky-ct))^2}, \quad (65)$$

$$F(\xi)=\frac{ca_2A(e^2+k^2)[-10Exp(\xi)+32AExp(2\xi)+8A^2ExP(3\xi)]}{(1+AExp(\xi))^5}, \quad (66)$$

$$T(x,y,t)=\frac{-ca_2A(e^2+k^2)Exp(ex+ky-ct)}{prRae(1+AExp(\xi))^5}$$

$$+\frac{ca_2A(e^2+k^2)[32AExp(2ex+2ky-2ct)+8A^2ExP(3ex+3ky-3ct)]}{prRae(1+AExp(ex+ky-ct))^5}, \quad (67)$$

There are several cases for exact traveling wave solutions which can be summarized as:

Case 1: $a_0=0, a_1=a_2, k=-e, c=2e^2pr, \xi\to-\infty$,

$$u(x,y,t)=v(x,y,t)=\frac{3a_1eAExp(e(x-y)-2e^2prt)+a_1eA^2Exp(e(x-y)-4e^2ct)}{(1+AExp(e(x-y)-2e^2prt))^3},$$

$$p(x,y,t)=\frac{6a_1cAExp(e(x-y)-2e^2prt)}{(1+AExp(e(x-y)-2e^2prt))^4}$$

$$T(x,y,t)=\frac{2a_1ceA[-6Exp(e(x-y)-2e^2prt)+18AExp(2e(x-y)-4e^2t)]}{prRa(1+AExp(e(x-y)-2e^2prt))5},$$

$$\omega=\frac{2a_1e^2A[3Exp(e(x-y)-2e^2prt)-4AExp(2e(x-y)-4e^2prt)-A^2Exp(3e(x-y))-6e^2t]}{(1+AExp(e(x-y)-2e^2prt))^4}$$

if $A=1$, we obtain:

$$u(x,y,t)=v(x,y,t)=\frac{a_1 e}{4}sech2(\frac{e(x-y)-2e^2prt}{2})[\frac{3+Exp(e(x-y)-2e^2prt)}{1+Exp(e(x-y)-2e^2prt)}],$$

$$p(x,y,t)=\frac{3a_1c}{2}\frac{sech^2(\frac{e(x-y)-2e^2prt}{2})}{(1+Exp(e(x-y)-2e^2prt))^2},$$

$$T(x,y,t)=-\frac{2ca_1e}{prRa}[\frac{3sech^2(\frac{e(x-y)-2e^2prt}{2})}{2(1+Exp(e(x-y)-2e^2prt))^2}-\frac{9sech^4(\frac{e(x-y)-2e^2prt}{2})}{8(1+Exp(e(x-y)-2e^2prt))}],$$

$$\omega(x,y,t)=\frac{a_1e^2}{2}sech^2(\frac{e(x-y)-2e^2prt}{2})[\frac{3}{(1+Exp(e(x-y)-2e^2prt))^2}$$

$$-sech^2(\frac{e(x-y)-2e^2prt}{2})-\frac{Exp(e(x-y)-2e^2prt)}{4}],$$

and if $A=-1$, we get

$$u(x,y,t)=v(x,y,t)=-\frac{a_1k}{4}csch^2(\frac{e(x-y)-2e^2prt}{2})[\frac{3-Exp(e(x-y)-2e^2prt)}{1-Exp(e(x-y)-2e^2prt)}],$$

$$p(x,y,t)=-\frac{3a_1c}{2}\frac{csch^2(\frac{ex+ky-(e^2+k^2)}{2})}{(1-Exp(e(x-y)-2e^2prt))^2},$$

$$T(x,y,t)=\frac{2ca_1e}{prRa}[\frac{3sech^2(\frac{e(x-y)-2e^2prt}{2})}{2(1-Exp(e(x-y)-2e^2prt))^2}+\frac{9sech^4(\frac{e(x-y)-2e^2prt}{2})}{8(1-Exp(e(x-y-2e^2prt))}],$$

$$\omega(x,y,t)=\frac{-a_1e^2}{2}csch^2(\frac{e(x-y)-2e^2prt}{2})[\frac{3}{(1-Exp(e(x-y)-2e^2prt))^2}$$

$$+csch^2(\frac{e(x-y)-2e^2prt}{2})-\frac{Exp(e(x-y)-2e^2prt)}{4}],$$

Case 2 :$a_0=0, a_1=a_2, e=k, c=2k^2pr, \xi\to-\infty,$

$$u(x,y,t)=-v(x,y,t)=\frac{-3a_1AkExp(e(x+y)-2t)-a_1A^2kExp(2k(x+y)-2k^2prt)}{(1+AExp(k(x+y)-2k^2prt))3},$$

$$p(x,y,t)=t-\frac{6a_1AcExp(k(x+y)-2k^2prt)}{(1+AExp(k(x+y)-k^2prt)))^4},$$

$$T(x,y,t)=\frac{2ca_1Ak[-6Exp(k(x+y)-2K^2prt)+18AExp(2k(x+y)-4k^2prt)]}{prRa(1+AExp(k(x+y)-2k^2prt))^4}$$

$$\omega=\frac{2a_1Ak^2[5Exp(k(x+y)-2k^2prt)-4AExp(2k(x+y)-4k^2prt)-3A^2Exp(3k(x+y)-6k^2prt)]}{(1+AExp(k(x+y)-2k^2prt))^4}.$$

Case 3:$a_0=0, a_1=3a_2, pr=1, k=-e, c=2e^2, \xi\to-\infty,$

$$u(x,y,t)=v(x,y,t)=\frac{5a_2eAExp(e(x-y)-2e^2t)+3a_2eA^2Exp(2e(x-y)-4e^2t)}{(1+AExp(e(x-y)-2e^2t))^3},$$

$$p(x,y,t)=\frac{a_2c[10AExp(e(x-y)-2e^2t)+4A^2Exp(2e(x-y)-4e^2t)]}{(1+AExp(e(x-y)-2e^2t))^4},$$

$$T(x,y,t)=\frac{2ca_2eA[-10Exp(e(x-y)-2e^2t)+32AExp(2e(x-y)-4e^2t)]}{prRa(1+AExpe(e(x-y)-2e^2t))^5}$$

$$+\frac{16ca_2eA^3Exp(3e(x-y)-6e^2t)}{prRa(1+AExp(e(x-y)-2e^2t))5},$$

$$\omega = \frac{2a_1Ae^2[5Exp(e(x-y)-2e^2t)-4AExp(2e(x-y)-4e^2t)-3A^2Exp(3e(x-y)-6e^2t)]}{(1+AExp(e(x-y)-2e^2t))^4},$$

Case 4: $a_0 = 0, a_1 = 3a_2, pr = 1, k = e, c = 2k^2, \xi \to -\infty,$

$$u(x,y,t) = -v(x,y,t) = \frac{-5a_2AkExp(k(x+y)-2k^2t)-3a_2A^2kExp(2k(x+y)-2k^2t)}{(1+AExp(k(x+y)-2k^2t))^3},$$

$$p(x,y,t) = t - \frac{a_2c[10AExp(k(x+y)-2k^2t)+4A^2Exp(2k(x+y)-2k^2t)]}{(1+AExp(k(x+y)-2k^2t))^4},$$

$$T(x,y,t) = \frac{2ca_2kA[-10Exp(K(x+y)-2k^2t)+32AExp(2k(x+y)-4k^2t)]}{prRa(1+AExp(k(x+y)-2k^2t))^5} + \frac{16ca_2kA^3Exp(3k(x+y)-6k^2t)}{prRa(1+AExp(k(x+y)-2k^2t))^5},$$

$$\omega = \frac{2a_1Ak^2[5Exp(k(x+y)-2k^2t)-4AExp(2k(x+y)-4k^2t)-3A^2Exp(3k(x+y)-6k^2t)]}{(1+AExp(k(x+y)-2k^2t))^4}.$$

Remark: Remained solutions in terms of sech and csch functions for the case 2,3 and 4 can be obtained in the same procedure that is obtained in case 1, with assume $A = 1, -1$.

3 Conclusions

In summary, the Kudryashov method is used for obtaining exact travelling wave solutions of two nonlinear system of equations which include two-dimensional Navier-Stokes equations, Navier-Stokes and heat equations. The solutions that were obtained in this paper are new and not found elsewhere. In addition we have shown that all solutions satisfy the original equations. In results, the Kudryashov method is a reliable and effective for finding exact solutions.

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