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A Review on Application of the Local Fractal Calculus

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Abstract: In this paper, we review local fractal calculus. We present and review the summery of applications in classical mechanics, quantum mechanics, and optics.

Keywords: F^{α}-calculus, staircase function, fractal optics, fractal mechanics, fractal schrödinger equation

1 Introduction

Fractals are useful for modelling in nature: complex, irregular objects may be described by jaggy, irregular figures. Fractals have complicated structure at a wide range of scales, self-similarity properties, and often fractional dimensions [1,2,3]. Fractals were defined as geometrical shapes such that their fractal dimension (box dimension) is more than their topological dimension [4,5,6,7,8,9].

Analysis on fractals has an important role in applications and modeling of processes with fractal structures. Researchers have formulated different methods to suggest fractal analysis, such as probabilistic method, harmonic analysis, fractional spaces, fractional calculus, and time scale methods [10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30].

In the seminal papers [31,32,33,34,35,36], a type of Riemann-like calculus called F^{α}-calculus (F^{α}-C) was formulated for functions supported on fractal totally disconnected sets such as fractal Cantor sets, fractal Cantor cubes, fractal Koch curve, fractal Cesáro curve, and Cantor tartan spaces. F^{α}-C is a simple, constructive, and algorithmic approach to performing analysis on fractals. It is a generalization of ordinary calculus which applies in cases where standard calculus is not applicable. Its advantages include the following properties.

1. The fractal derivative is local, which is very important in physics: firstly so as not to violate causality, secondly because measurement in physics is local.
2. The order of the fractal derivative is non-integer and has geometrical meaning, being equal to the dimension of the support of the function.
3. The order of the fractal derivative also has physical meaning by having a relationship with the spectral dimension [37,38].

The fractal calculus was expanded to include a wider class of functions to be F^{α}-integrable [39]. Analogues of existence and uniqueness theorems, stability of solutions, and Sumudu transforms were given in [40,41,42].

Fractal calculus has found many applications in physics, which are summarized in [43,44,47,48,49,50,51,52,53,54,55].

2 Basic Tools

Fractal calculus is a generalization of ordinary calculus which gives a new framework for constructing differential equations on fractal sets. In this section, we give a brief summary of fractal calculus which involves functions defined on thin Cantor-like sets with Lebesgue measure zero [31,32,33,34,35,36].

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2.1 Thin Cantor Sets

In this we subsection, We review thin Cantor sets. The thin Cantor sets (middle-$\mathfrak{k}$ Cantor sets) has Lebesgue measure zero. [56,57,58]. Thin Cantor sets are uncountable, self-similar, perfect compact metric space and totally disconnected.

Thin Cantor Sets
We review the stags which set up the thin Cantor set/ middle-$\mathfrak{k}$ Cantor set [56].

Consider a interval $\mathbf{I}=[0,1]$, then delete an open interval of length $0<\mathfrak{k}<1$ from the middle of $\mathbf{I}$. Continuing, the same schemes we form thin Cantor set as follows:
Stage 1.

$$C_1^{\mathfrak{k}}=[0,\frac{1}{2}(1-\mathfrak{k})]\cup[\frac{1}{2}(1+\mathfrak{k}),1]. \tag{1}$$

Stage 2.

$$C_2^{\mathfrak{k}}=[0,\frac{1}{4}(1-\mathfrak{k})^2]\cup[\frac{1}{4}(1-\mathfrak{k}^2),$$
$$\frac{1}{2}(1-\mathfrak{k})]\cup[\frac{1}{2}(1+\mathfrak{k})+\frac{1}{2}((1+\mathfrak{k})$$
$$,\frac{1}{2}(1-\mathfrak{k})^2)]\cup[\frac{1}{2}(1+\mathfrak{k})(1+\frac{1}{2}(1-\mathfrak{k})),1].$$

$$\vdots$$

After $\mathbf{n}$ stages, we have

$$C^{\mathfrak{k}}=\bigcap_{\mathbf{n}=1}^{\infty}C_{\mathbf{n}}^{\mathfrak{k}}, \tag{2}$$

which is called thin Cantor set.
The Hausdorff dimension of thin Cantor set is defined by

$$\mathfrak{D}_H(C^{\mathfrak{k}})=\frac{\log(2)}{\log(2)-\log(1-\mathfrak{k})}, \tag{3}$$

which is base of Hausdorff measure [2,56].
In Figure 1 we show the processes that established the middle-$\mathfrak{a}$ Cantor set.

In the next section, we preset a short review to fractal calculus that was formulated for the function with thin Cantor set support [31,32,33,34,35,36].

2.2 Fractal Calculus on Thin Cantor Sets

Fractal calculus is a generalization of Riemann-like calculus. Fractal Calculus was adopted to standard calculus to includes function with fractal sets since we can not use standard calculus [31,32,33,34,35,36].
The fractal calculus is start by defining **the flag function** for the every thin Cantor set $C^{\mathfrak{k}}$ as follows:

$$\mathbf{F}(C^{\mathfrak{k}},\mathbf{J})=\begin{cases}1 & \text{if } \; C^{\mathfrak{k}}\cap\mathbf{J}\neq\emptyset\\ 0 & \text{otherwise.}\end{cases} \tag{4}$$

where $\mathbf{J}=[c_1,c_2]$.
Let $\mathbf{Q}_{[c_1,c_2]}=\{c_1=x_0,x_1,x_2,\dots,x_n=c_2\}$ be a subdivision of $\mathbf{J}$. Then, $\mathbf{G}^{\alpha}[C^{\mathfrak{k}},\mathbf{Q}]$ is defined in [34,31,32] by

$$\mathbf{G}^{\alpha}[C^{\mathfrak{k}},\mathbf{Q}]=\sum_{i=1}^{n}\Gamma(\alpha+1)(x_i-x_{i-1})^{\alpha}\mathbf{F}(C^{\mathfrak{k}},[x_{i-1},x_i]), \tag{5}$$

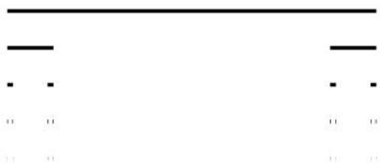

Fig. 1: Thin Cantor set with $\mathfrak{k} = 3/4$

where $0 < \alpha \leq 1$ and $\Gamma(.)$ is gamma function.
The mass function $\mathbf{M}^{\alpha}(C^{\mathfrak{k}}, c_1, c_2)$ is given in [31,32] by

$$\begin{aligned}\mathbf{M}^{\alpha}(C^{\mathfrak{k}}, c_1, c_2) &= \lim_{\delta \to 0} \mathbf{M}^{\alpha}_{\delta}(C^{\mathfrak{k}}, c_1, c_2)\\ &= \lim_{\delta \to 0} \left(\inf_{\mathbf{Q}_{[c_1,c_2]}:|\mathbf{Q}|\leq\delta} \mathbf{G}^{\alpha}[C^{\mathfrak{k}}, \mathbf{Q}] \right), \qquad (6)\end{aligned}$$

where, we take infimum over all subdivisions $\mathbf{Q}$ of $[c_1, c_2]$ satisfying $|\mathbf{Q}| := \max_{1\leq i\leq n}(x_i - x_{i-1}) \leq \delta$.
The integral staircase function of the fractal sets is defined in [31,32] by

$$S^{\alpha}_{C^{\mathfrak{k}}}(x) = \begin{cases} \mathbf{M}^{\alpha}(C^{\mathfrak{k}}, x_0, x) & \text{if} \quad t \geq x_0 \\ -\mathbf{M}^{\alpha}(C^{\mathfrak{k}}, x_0, x) & \text{otherwise}, \end{cases} \qquad (7)$$

where x_0 is an arbitrary real and fixed number.
In Figure 2, we have sketched Eq.(7) thin Cantor set by letting $\mathfrak{l} = 3/4$.
The γ-dimension of $C^{\mathfrak{l}} \cap [c_1, c_2]$ is

$$\begin{aligned}\dim_{\gamma}(C^{\mathfrak{l}} \cap [c_1, c_2]) &= \inf\{\alpha : \mathbf{M}^{\alpha}(C^{\mathfrak{l}}, c_1, c_2) = 0\}\\ &= \sup\{\alpha : \mathbf{M}^{\alpha}(C^{\mathfrak{l}}, c_1, c_2) = \infty\}. \qquad (8)\end{aligned}$$

In Figure 3, we have obtained γ-dimension in view of Eq.(8). Note that for every chosen thin Cantor set we have

$$\dim_{\gamma}(C^{\mathfrak{l}}) = \alpha(\mathfrak{l}) = \mathfrak{D}_H(C^{\mathfrak{k}}) = \frac{\log(2)}{\log(2) - \log(1-\mathfrak{k})}.$$

The characteristic function $\chi_{C^{\mathfrak{a}}}(\alpha, x)$ for a given thin Cantor set is defined by

$$\chi_{C^{\mathfrak{l}}}(\alpha, x) = \begin{cases} \frac{1}{\Gamma(\alpha+1)}, & x \in C^{\mathfrak{l}}; \\ 0, & \text{otherwise}. \end{cases} \qquad (9)$$

In Figure 4, we have shown characteristic function for thin Cantor set by choosing $\mathfrak{l} = 3/4$.
The C^{α}-Limit of $h(x) : C^{\mathfrak{l}} \to \Re$ is defined by

$$x, z \in C^{\mathfrak{l}} \quad \text{and} \quad |z - x| < \delta \Rightarrow |h(z) - \ell| < \varepsilon, \qquad (10)$$

if ℓ exists, namely

$$l = \mathrm{C}^{\alpha}\text{-}\lim_{z \to x} h(x). \qquad (11)$$

The C^{α}-Continuity of $h(x)$ is defined by

$$h(z) = \mathrm{C}^{\alpha}\text{-}\lim_{z \to x} h(x). \qquad (12)$$

The C^{α}-Differentiation of $h(x)$ on α-perfect set, is defined by [31,32,34],

$$D^{\alpha}_{C^{\mathfrak{l}}} h(x) = \begin{cases} \mathrm{C}^{\alpha}\text{-}\lim_{z \to x} \frac{h(z) - h(x)}{S^{\alpha}_{C^{\mathfrak{l}}}(z) - S^{\alpha}_{C^{\mathfrak{l}}}(x)}, & \text{if } z \in C^{\mathfrak{l}}, \\ 0, & \text{otherwise}. \end{cases} \qquad (13)$$

The C^{α}-Integral of $h(x)$ on $[c_1, c_2]$ is denoted by $\int_{c_1}^{c_2} h(x) d^{\alpha}_{C^{\mathfrak{l}}} x$ and approximately given by [31,32,34]

$$\int_{c_1}^{c_2} h(x) d^{\alpha}_{C^{\mathfrak{l}}} x \approx \sum_{i=1}^{n} h_i(x)(S^{\alpha}_{C^{\mathfrak{l}}}(t_j) - S^{\alpha}_{C^{\mathfrak{l}}}(x_{j-1})). \qquad (14)$$

For more details we refer the reader to [31,32,34].

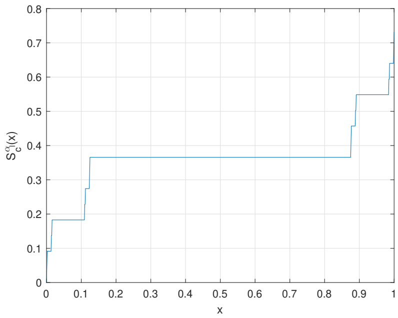

Fig. 2: Staircase function corresponding to thin Cantor set with $\mathfrak{l} = 3/4$

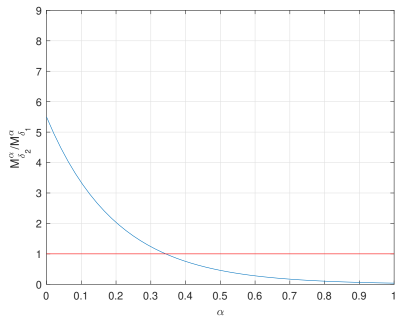

Fig. 3: The γ-dimension gives $\alpha = 0.33$ to thin Cantor set with $\mathfrak{l} = 3/4$

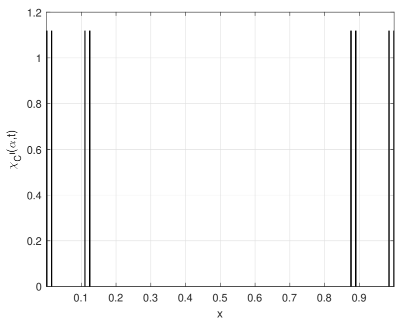

Fig. 4: Characteristic function for thin Cantor set with $\mathfrak{l} = 3/4$

2.3 Scale properties of the fractal calculus

A function $g(S^\alpha_{C^\mathfrak{l}}(x))$ is called **the fractal homogenous of degree-$n\alpha$** if we have

$$g(S^\alpha_{C^\mathfrak{l}}(\mathfrak{l}x)) = \mathfrak{l}^{n\alpha} g(S^\alpha_{C^\mathfrak{l}}(x)), \ \ \exists \ n, \ \ \forall \ \mathfrak{l}, \tag{15}$$

where

$$x \to \lambda x \Rightarrow S^\alpha_{C^\mathfrak{l}}(\lambda x) = \lambda^\alpha S^\alpha_{C^\mathfrak{l}}(x). \tag{16}$$

For example, if we choose $(\mathfrak{l} = 3/4)$, $m = 1$ and $\lambda = (3/4)^n, n = 1,2,...,$ then we have

$$g\left(S^\alpha_{C^\mathfrak{l}}((\frac{3}{4})^n x)\right) = (\frac{3}{4})^{n\alpha} g(S^\alpha_{C^\mathfrak{l}}(x)). \tag{17}$$

The scale change of the local fractal derivative is given by

$$D^{\alpha}_{C^{\mathfrak{l}}} g(S^{\alpha}_{C^{\mathfrak{l}}}(\lambda x)) = \lambda^{n\alpha-\alpha} D^{\alpha}_{F} g(S^{\alpha}_{C^{\mathfrak{l}}}(x)). \tag{18}$$

3 Applications of Fractal Calculus In Physics

In this section we present mathematical model for the processes with fractal structure[59,60,61,62,63,64,65,66,68,69,70,71,72,73].

3.1 Classical Mechanics on Fractal Sets

In this section we review Newton, Lagrange and Hamilton mechanics, on the thin fractal sets [45,46].
Fractal Newtonian Mechanics: If a point particle with mass m moves in the fractal-time space. Newton's second law is suggested as follows:

$$f = m_K (D^{\alpha}_{K,t})^2 x(t), \quad t \in C^{\mathfrak{l}} = K, \tag{19}$$

where $x(t)$ the position of particle, f is a force applied on particle and the physical dimension of $[m_K] = (Mass)(Time)^{\alpha}$, meanwhile $m_K = m$.
Consider the **fractal simple harmonic oscillator** with force $f = -\tau x$ where τ is constant. Then using Eq.(19) we obtain the following fractal differential equation

$$-\tau x = m_K (D^{\alpha}_{K,t})^2 x(t), \;\; x(0) = A, \quad D^{\alpha}_{K} x(t)|_0 = 0. \tag{20}$$

The solution to Eq.(20) is

$$x(t) = A\cos(\omega S^{\alpha}_{K}(t)), \tag{21}$$

where $\omega = \sqrt{\tau/m_K}$ is constant. In view of upper bound of the staircase function $S^{\alpha}_{K}(t) < t^{\alpha}$, we can write

$$x(t) \approx A\cos(\omega t^{\alpha}). \tag{22}$$

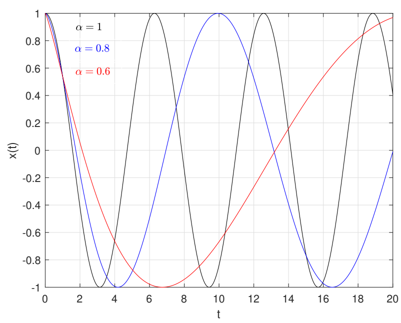

Fig. 5: Graph of the position function $x(t)$ for several values of α

In Figure 5, we have plotted the position function defined by Eq. (22) for different values of α.
Fractal Kepler's Third Law: Consider the following fractal differential equation

$$(D^{\alpha}_{K,t})^2 x(t) + \kappa x(t)^{-2} = 0. \tag{23}$$

By applying scaling transformation, namely

$$\begin{aligned} &t \to \varsigma t, \quad x \to \xi x, \quad S^{\alpha}_{K}(\varsigma t) \to \varsigma^{\alpha} S^{\alpha}_{K}(t) \\ &D^{\alpha}_{K,t} \xi x(\varsigma t) \to \frac{\xi}{\varsigma^{\alpha}} D^{\alpha}_{K,t} x(t). \end{aligned} \tag{24}$$

Therefore, we get

$$\varsigma = \xi^{3/2\alpha}, \tag{25}$$

where ς is the orbital period and ξ is the semi-major axis of its elliptic orbit. If we set $\alpha = 1$ in Eq.(25), we lead to Kepler's Third Law.

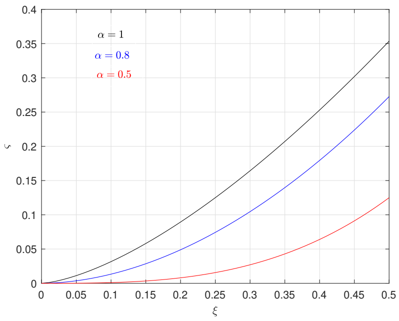

Fig. 6: Graph of the orbital period ς versus the dimension α

In Figure 6 we have plotted Eq.(25) in terms of different values of α.

Fractal Lagrangian and Hamiltonian Mechanics: Let $L(t,x(t),D^{\alpha}_{K,t}x)$ be the Lagrangian of a particle on the thin Cantor set. Then the Euler-Lagrange equation corresponding to L is

$$D^{\alpha}_{K,x}L - D^{\alpha}_{K,t}\frac{\partial L}{\partial D^{\alpha}_{K,t}x} = 0. \tag{26}$$

Using the fractal Legendre transform, it is easy to shows that the fractal Hamilton equations is

$$D^{\alpha}_{K,t}x = \frac{\partial H}{\partial P_K}, \quad D^{\alpha}_{K,t}P_K = D^{\alpha}_{K,x}H, \tag{27}$$

and

$$P_K = \frac{\partial L}{\partial D^{\alpha}_{K,t}x}. \tag{28}$$

where P_K is called fractal conjugate momentum.

3.2 Quantum Mechanics on Fractal Sets

In this section, the schrödinger equation on fractal-time space is reviewed [47,53].

Fractal Schrödinger Equation Consider the fractal schrödinger equation for a particle with mass m is as follows [74]:

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + v(x)\psi(x,t) = i\hbar D^{\alpha}_{K,t}\psi(x,t), \quad x \in C^{\omega}, \tag{29}$$

where $v(x)$ is potential energy and $\psi(x,t)$ is wave function. The solution of Eq.(29) is

$$\begin{aligned}\psi(x,t) &= \phi(x)e^{\frac{-iES^{\alpha}_{K}(t)}{\hbar}} \\ &\approx \phi(x)e^{\frac{-iEt^{\alpha}}{\hbar}}.\end{aligned} \tag{30}$$

The fractal energy spectrum for a particle in fractal well is given by

$$\begin{aligned}E^{\alpha}_{n} &= \frac{\pi^2\hbar^2n^2}{2mS^{\alpha}_{K}(1)} \\ &= \frac{\pi^2\hbar^2n^2}{2m\Gamma(\alpha+1)}, \quad n = 1,2,...,\end{aligned} \tag{31}$$

where $\hbar$ is the Planck constant.
Fractal Klein-Gordon Equation The Klein-Gordon Equation of fractal-time is given by

$$-(D^{\alpha}_{K,t})^2\psi(x,t)+\frac{\partial^2\psi(x,t)}{\partial x^2}=m^2\psi(x,t), \tag{32}$$

where m is the mass of particle [74]. The solution of Eq.(32) is

$$\begin{aligned}\psi(x,t) &\approx e^{ikx-\omega S^{\alpha}_{K}(t)}\\ &\approx e^{ikx-\omega t^{\alpha}}\end{aligned} \tag{33}$$

where k wave number and is angular frequency [74].

3.3 Optic on Fractal Sets

In this section we review fractal gratings and its diffraction fringes by the Fraunhofer approximation model [69,70,71, 49].
If plane wave incident to the fractal gratings, then for finding diffraction fringes in the sense of the Fraunhofer approximation we consider

$$f(t)=\begin{cases} e^{iS^{\alpha}_{K}(\omega_0)S^{\alpha}_{K}(t)}, & -\iota_0/2<t<\iota_0/2;\\ 0, & \text{otherwise.}\end{cases} \tag{34}$$

where ι_0 is constant and ω the frequency and t is the time.

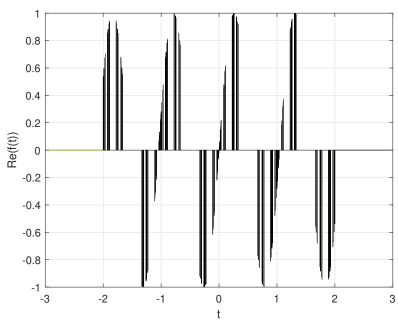

Fig. 7: Graph of real part of $f(t)$ setting $\iota=2$.

In Figure 7 we have plotted real part wave function given by Eq.(34).
Then by using fractal Fourier transform we obtain [49].

$$g(\omega)=\sqrt{\frac{2}{\pi}}\frac{\sin\left(S^{\alpha}_{K}(\iota_0/2)(S^{\alpha}_{K}(\omega)-S^{\alpha}_{K}(\omega_0))\right)}{S^{\alpha}_{K}(\omega)-S^{\alpha}_{K}(\omega_0)} \tag{35}$$

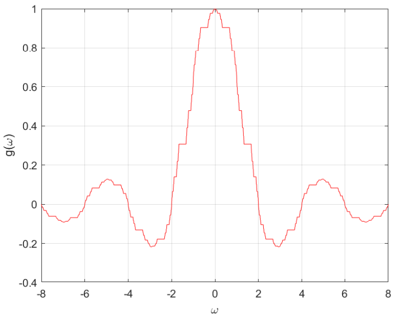

Fig. 8: Graph of $g(\omega)$ in the case of $\alpha = 0.63$, and $\omega_0 = 0$.

In Figure 8 we have present the diffraction fringes corresponding to thin Cantor set choosing $\mathfrak{l} = 1/3$

4 Conclusion

In this paper, we have given a summary of fractal calculus and presented some applications in classical mechanics, quantum mechanics, and optics.

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