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A New Three Parameter Log-Logistic Model for Survival Data Analysis

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Abstract: In this paper, a new three parameter generalized Log-logistic distribution is introduced for modeling survival data. Some properties and characteristics of the newly introduced model are studied. Finally, the initiated model and some other related distributions are fitted to real life data sets of lifetimes, and are compared for their ability to describe the data.

Keywords: Quadratic rank transmutation, Survival Analysis, Reliability, Robust skewness, Medical Sciences.

1 Introduction

Quadratic Rank Transmutation Map (QRTM) given by Shaw and Buckley [1] is one of the techniques for generalizing probability models. Recently, a lot of research has been done in the field of transmutation. Aryal and Tsokos [2, 3] introduced a new generalization of Weibull distribution and developed the transmuted Extreme valve distribution using quadratic rank transmutation map technique. Hussain [4] studied the transmuted Exponentiated Gamma distribution. Merovci [5] proposed the transmuted Lindley distribution. Para and Jan [6] introduced two parameter Transmuted Log-logistic distribution with applications in medical sciences and other applied fields.

In this paper, we explore three parameter transmuted Log-logistic distribution in order to find out its different characteristics as well as the structural properties.
A random variable X is said to have transmuted distribution if its cumulative distribution function is given by

$$F(x) = (1+\lambda)G(x) - \lambda G(x)^2 \qquad ,|\lambda| \le 1 \tag{1}$$

where $G(x)$ is the cdf of the base distribution. If we put $\lambda = 0$ in equation (1), we get the base distribution.

In probability theory, the Log-logistic distribution is a continuous probability distribution used in survival analysis as a parametric model for events whose rate increases initially and decreases later, for example mortality rate from cancer following diagnosis or treatment. The inverse version of Log-logistic model also provides greater flexibility in survival data sets.
The probability density function (pdf) of the Log-logistic (ILL) distribution is defined as

$$g(x;\alpha,\beta) = \frac{\alpha\beta^{\alpha}}{x^{(1-\alpha)}\left(1+\left(\frac{x}{\beta}\right)^{\alpha}\right)^{2}} \qquad x>0, \alpha>0, \beta>0 \tag{2}$$

and its corresponding cumulative distribution function (cdf) is given by

$$G(x;\alpha,\beta) = \frac{\beta^{-\alpha}}{\beta^{-\alpha}+x^{-\alpha}} \qquad x>0, \alpha>0, \beta>0 \tag{3}$$

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where α is the shape parameters and β is the scale parameter.

The rest of this paper is organized as follows: In Section 2, we demonstrate transmuted Log-logistic Distribution. In Section 3, various statistical properties, the distributions of order statistics, moment generating function and the quantile function are summarized. The MLE of the distribution parameters are demonstrated in Section 4 followed by Monte Carlo simulation procedure. Robust measures of skewness and Kurtosis along with graphical exhibition are presented in Section 5. The applicability of this distribution to modeling real life data is illustrated by two real life examples in Section 6.

2 New Generalized Log-logistic Distribution

In this section, we address the transmuted Log-logistic distribution as a new generalization of two parameter Log-logistic distribution and the sub-models of this distribution. Now using (1) and (3), we have the cdf of transmuted Log-logistic distribution given by

$$F(x,\alpha,\beta,\lambda)=\frac{(1+\lambda)+\left(\frac{x}{\beta}\right)^{\alpha}}{\left(\frac{x}{\beta}\right)^{\alpha}\left(1+\left(\frac{x}{\beta}\right)^{-\alpha}\right)^{2}} \qquad x>0,\alpha>0,\beta>0,-1\le\lambda\le 1 \tag{4}$$

Hence the pdf of transmuted Log-logistic distribution with parameters $\alpha,\beta\ and\ \lambda$ is given as

$$f(x,\alpha,\beta,\lambda)=\frac{x^{-(\alpha+1)}\left[\alpha\beta^{\alpha}(1+\lambda)\left(1+\left(\frac{x}{\beta}\right)^{-\alpha}\right)-2\lambda\alpha\beta^{\alpha}\right]}{\left(1+\left(\frac{x}{\beta}\right)^{-\alpha}\right)^{3}}\ ;\quad x>0,\alpha>0,\beta>0,|\lambda|\le 1 \tag{5}$$

Fig.1 and Fig.2 present the pdf plot for (5) for different values of parameters. It is evident that the distribution of the transmuted Log-logistic random variable X is right skewed.

3 Statistical Properties of Transmuted Inverse Log-logistic Distribution

In this section we shall discuss structural properties of transmuted Log-logistic Distribution, specially moments, order statistics, maximum likelihood estimation, moment generating function.

3.1 Moments: The following theorem gives the rth moment of the transmuted Log-logistic Distribution.

Theorem 3.1: If X has the TLLD (α,β,λ) distribution with $|\lambda|\le 1$, then the rth non-central moments are given by

$$\mu_r{}'=\beta^{r}(1+\lambda)\beta\left(1-\frac{r}{\alpha},\frac{r}{\alpha}+1\right)-2\lambda\beta^{r}\beta\left(1-\frac{r}{\alpha},\frac{r}{\alpha}+2\right)$$

Proof: $$\mu_r{}'=\int_0^{\infty}x^{r}\frac{x^{-(\alpha+1)}\left[\alpha\beta^{\alpha}(1+\lambda)\left(1+\left(\frac{x}{\beta}\right)^{-\alpha}\right)-2\lambda\alpha\beta^{\alpha}\right]}{\left(1+\left(\frac{x}{\beta}\right)^{-\alpha}\right)^{3}}dx$$

Substituting $\left(\frac{x}{\beta}\right)^{-\alpha}=t$, we get $x=\beta t^{-\frac{1}{\alpha}}$, and $dx=\frac{\beta}{\alpha}t^{-\frac{1}{\alpha}-1}$

$As\ \ x\to 0,\ \ t\to\infty\ \ \ and\ As\ x\to\infty\ ,\ t\to 0$

After simplification we get

$$\mu_r{}'=\beta^{r}(1+\lambda)\beta\left(1-\frac{r}{\alpha},1+\frac{r}{\alpha}\right)-2\lambda\beta^{r}\beta\left(1-\frac{r}{\alpha},2+\frac{r}{\alpha}\right)$$

where $\beta(a,b)=\dfrac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$

The first two moments about origin for transmuted Log-logistic Distribution are given by

$$\mu_1{}' = \beta\ (1+\lambda)\beta\left(1-\frac{1}{\alpha},1+\frac{1}{\alpha}\right)-2\lambda\beta\ \beta\left(1-\frac{1}{\alpha},2+\frac{1}{\alpha}\right) \tag{6}$$

$$\mu_2{}' = \beta^2(1+\lambda)\beta\left(1-\frac{2}{\alpha},1+\frac{2}{\alpha}\right)-2\lambda\beta^2\beta\left(1-\frac{2}{\alpha},2+\frac{2}{\alpha}\right) \tag{7}$$

Thus the variance of transmuted Log-logistic distribution is given by

$$\mu_2=\left[\beta^2(1+\lambda)\beta\left(1-\frac{2}{\alpha},1+\frac{2}{\alpha}\right)-2\lambda\beta^2\beta\left(1-\frac{2}{\alpha},2+\frac{2}{\alpha}\right)\right]$$
$$-\left\{\beta\ (1+\lambda)\beta\left(1-\frac{1}{\alpha},1+\frac{1}{\alpha}\right)-2\lambda\beta\ \beta\left(1-\frac{1}{\alpha},2+\frac{1}{\alpha}\right)\ \right\}^2$$

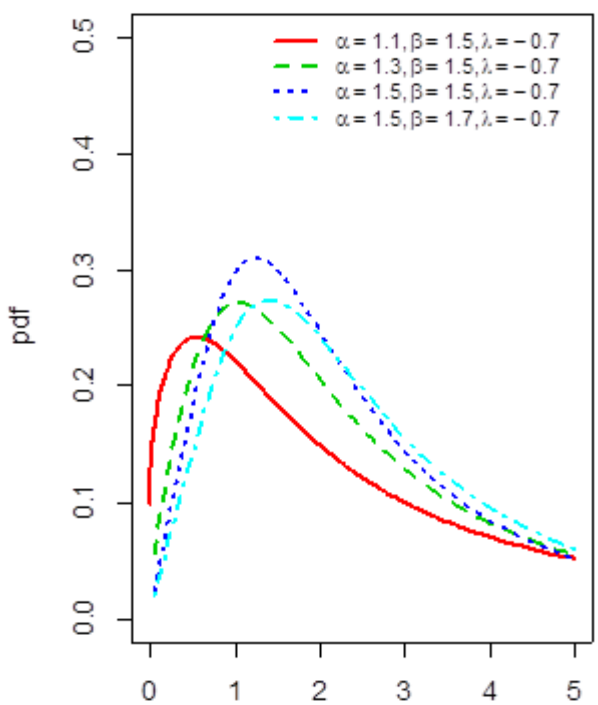

Fig.1: pdf plot for TLLD (α,β,λ)

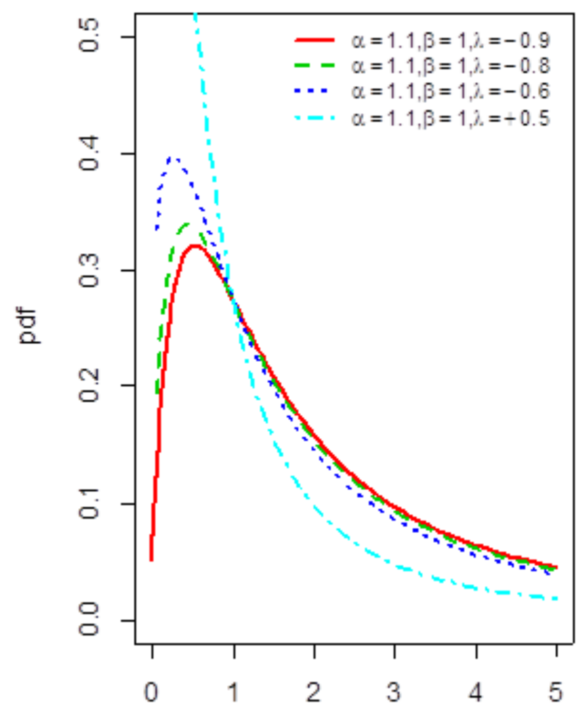

Fig.2: pdf plot for TLLD (α,β,λ)

3.2 Moment Generating Function and Characteristic Function of TLLD

In this sub section we derived the moment generating function of TLLD (α,β,λ) distribution.

Theorem 3.2: If X has the transmuted Log-logistic (α,β,λ) distribution with $|\lambda|\le 1$, then the moment generating function $M_X(t)$ and the characteristic function $\psi_X(t)$ have the following form

$$M_X(t)=\sum_{j=0}^{\infty}\ \frac{t^j}{j!}\left[\beta^j(1+\lambda)\beta\left(1-\frac{j}{\alpha},1+\frac{j}{\alpha}\right)-2\lambda\beta^r\beta\left(1-\frac{j}{\alpha},2+\frac{j}{\alpha}\right)\ \right]$$

and

$$\psi_X(t)=\sum_{j=0}^{\infty}\frac{(it)^j}{j!}\left\{\beta^j(1+\lambda)\beta\left(1-\frac{j}{\alpha},1+\frac{j}{\alpha}\right)-2\lambda\beta^r\beta\left(1-\frac{j}{\alpha},2+\frac{j}{\alpha}\right)\right\}\ \textit{Respectively.}$$

Proof: We begin with the well known definition of the moment generating function given by

$$M_X(t)=E\left(e^{tx}\right)=\int_0^\infty e^{tx}\, f(x;\alpha,\beta,\lambda)\,dx$$

$$=\int_0^\infty\left[1+tx+\frac{(tx)^2}{2!}+....\right]f(x;\alpha,\beta,\lambda)\,dx$$

$$=\int_0^\infty\sum_{j=0}^\infty\frac{t^j}{j!}x^j f(x;\alpha,\beta,\lambda)\,dx$$

$$=\sum_{j=0}^\infty\frac{t^j}{j!}\mu_j{}'$$

$$\Rightarrow M_X(t)=\sum_{j=0}^\infty\frac{t^j}{j!}\left[\beta^j(1+\lambda)\beta\left(1-\frac{j}{\alpha},1+\frac{j}{\alpha}\right)-2\lambda\beta^r\beta\left(1-\frac{j}{\alpha},2+\frac{j}{\alpha}\right)\right] \tag{8}$$

Also we know that $\psi_X(t)=M_X(it)$

Therefore,

$$\psi_X(t)=\sum_{j=0}^\infty\frac{(it)^j}{j!}\left\{\beta^j(1+\lambda)\beta\left(1-\frac{j}{\alpha},1+\frac{j}{\alpha}\right)-2\lambda\beta^r\beta\left(1-\frac{j}{\alpha},2+\frac{j}{\alpha}\right)\right\} \tag{9}$$

3.3 Order Statistics

Order statistics make their appearance in many statistical theory and practice. We know that if $X_{(1)},X_{(2)},\ldots,X_{(n)}$ denotes the order statistics of a random sample $X_1,X_2,\ldots,X_n$ from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$, then the pdf of rth order statistics $X_{(r)}$ is given by

$$f_{X(r)}(x)=\frac{n!}{(r-1)!(n-r)!}f_X(x)\left(F_X(x)\right)^{r-1}\left(1-F_X(x)\right)^{n-r}$$

For $r = 1, 2, \ldots, n$

The pdf of the rth order statistic for a transmuted Log-logistic Distribution is given by

$$f_{X(r)}(x)=\frac{n!}{(r-1)!(n-r)!}\frac{x^{-(\alpha+1)}\left[\alpha\beta^\alpha(1+\lambda)\left(1+\left(\frac{x}{\beta}\right)^{-\alpha}\right)-2\lambda\alpha\beta^\alpha\right]}{\left(1+\left(\frac{x}{\beta}\right)^{-\alpha}\right)^3}\left[\frac{(1+\lambda)+\left(\frac{x}{\beta}\right)^\alpha}{\left(\frac{x}{\beta}\right)^\alpha\left(1+\left(\frac{x}{\beta}\right)^{-\alpha}\right)^2}\right]^{r-1}\left[1-\frac{(1+\lambda)+\left(\frac{x}{\beta}\right)^\alpha}{\left(\frac{x}{\beta}\right)^\alpha\left(1+\left(\frac{x}{\beta}\right)^{-\alpha}\right)^2}\right]^{n-r} \tag{10}$$

Therefore, the pdf of the largest order statistic $X_{(n)}$ is given by

$$f_{X(n)}(x)=\frac{nx^{-(\alpha+1)}\left[\alpha\beta^\alpha(1+\lambda)\left(1+\left(\frac{x}{\beta}\right)^{-\alpha}\right)-2\lambda\alpha\beta^\alpha\right]}{\left(1+\left(\frac{x}{\beta}\right)^{-\alpha}\right)^3}\left[\frac{(1+\lambda)+\left(\frac{x}{\beta}\right)^\alpha}{\left(\frac{x}{\beta}\right)^\alpha\left(1+\left(\frac{x}{\beta}\right)^{-\alpha}\right)^2}\right]^{n-1} \tag{11}$$

and the pdf of the smallest order statistic $X_{(1)}$ is given by

$$f_{X_{(1)}}(x)=\frac{nx^{-(\alpha+1)}\left[\alpha\beta^{\alpha}(1+\lambda)\left(1+\left(\frac{x}{\beta}\right)^{-\alpha}\right)-2\lambda\alpha\beta^{\alpha}\right]}{\left(1+\left(\frac{x}{\beta}\right)^{-\alpha}\right)^{3}}\left[1-\frac{(1+\lambda)+\left(\frac{x}{\beta}\right)^{\alpha}}{\left(\frac{x}{\beta}\right)^{\alpha}\left(1+\left(\frac{x}{\beta}\right)^{-\alpha}\right)^{2}}\right]^{n-1} \tag{12}$$

Note that $\lambda=0$ yields the order statistics of the three parameter transmuted Log-logistic distribution.

3.4 Quantile and Random Number Generation from TLLD

Inverse CDF Method is one of the methods used for the generation of random numbers from a particular distribution. In this method the random numbers from a particular distribution are generated by solving the equation obtained on equating the CDF of a distribution to a number *u*. The number *u* is itself being generated from $U(0,1)$. Thus following the same procedure for the generation of random numbers from the TLLD we will proceed as:

$F(x,\alpha,\beta,\lambda)=u$

$$\frac{(1+\lambda)+\left(\frac{x}{\beta}\right)^{\alpha}}{\left(\frac{x}{\beta}\right)^{\alpha}\left(1+\left(\frac{x}{\beta}\right)^{-\alpha}\right)^{2}}=u \tag{13}$$

Solving the equation (13) for x, we will obtain the required random number from the TLLD. If $p=0.25$, $p=0.5$ and $p=0.75$ the resulting solutions will be the first quartile (Q_1), Median (Q_2) and third quartile (Q_3) respectively. Similarly we will find out the deciles and percentiles of different orders by simply assigning the different values to *u*. Now, the main problem which is being faced while using this method of generating the random numbers is to solve the equations which are usually complex and complicated. To overcome such hindrances we use statistical softwares like R for solving such a complex equation.

3.5 Reliability Measures of TLLD

In this sub-section, we present the reliability function and the hazard function for the proposed transmuted ILL distribution. The reliability function is otherwise known as the survival or survivor function. It is the probability that a system will survive beyond a specified time and it is obtained mathematically as the complement of the cumulative density function (cdf).

The survivor function is given by

$s(x)=1-F(x)$

$$s(x,\alpha,\beta,\lambda)=1-\frac{(1+\lambda)+\left(\frac{x}{\beta}\right)^{\alpha}}{\left(\frac{x}{\beta}\right)^{\alpha}\left(1+\left(\frac{x}{\beta}\right)^{-\alpha}\right)^{2}}\qquad x>0,\alpha>0,\beta>0,-1\le\lambda\le 1 \tag{14}$$

The hazard function is also known as the hazard rate, failure rate, or force of mortality. The hazard rate function is given by

$$h(x)=\frac{f(x)}{s(x)}\Rightarrow h(x)=\frac{\left[(1+\lambda)\alpha\beta^{\alpha}\left(1+\left(\frac{x}{\beta}\right)^{-\alpha}\right)-2\lambda\alpha\beta^{\alpha}\right]x^{-(\alpha+1)}}{\left(1+\left(\frac{x}{\beta}\right)^{-\alpha}\right)\left[\left(1+\left(\frac{x}{\beta}\right)^{-\alpha}\right)^{2}-1+(1+\lambda)\left(\frac{x}{\beta}\right)^{-\alpha}\right]} \tag{15}$$

$$x>0,\alpha>0,\beta>0,-1\le\lambda\le 1$$

4 Maximum Likelihood Estimation

We estimate the parameters of the transmuted Log-logistic distribution using the method of maximum likelihood estimation

(MLE) as follows;

Let $X_1, X_2, ..., X_n$ be a random sample of size n from transmuted Log-logistic distribution. Then the likelihood function is given by

$$L(x\,|\,\alpha,\beta,\lambda)=\prod_{i=1}^{n}\frac{x^{-(\alpha+1)}\left[\alpha\beta^{\alpha}(1+\lambda)\left(1+\left(\frac{x}{\beta}\right)^{-\alpha}\right)-2\lambda\alpha\beta^{\alpha}\right]}{\left(1+\left(\frac{x}{\beta}\right)^{-\alpha}\right)^{3}} \qquad (16)$$

Taking logarithm of (16), we find the log likelihood function

$$l=\sum_{i=1}^{n}\log\left((1+\lambda)\left(1+\left(\frac{x_i}{\beta}\right)^{-\alpha}\right)-2\lambda\right)-(\alpha+1)\sum_{i=1}^{n}\log(x_i)-3\sum_{i=1}^{n}\log\left(1+\left(\frac{x_i}{\beta}\right)^{-\alpha}\right)-n\log\alpha+n\alpha\log\beta \qquad (17)$$

To obtain the MLE's of $\alpha,\beta\, and\, \lambda$, we differentiate loglikelihood with respect to $\alpha,\beta\, and\, \lambda$

$$\frac{\partial l}{\partial\alpha}=3\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\beta}\right)^{-\alpha}\log\left(\frac{x_i}{\beta}\right)}{\left(1+\left(\frac{x_i}{\beta}\right)^{-\alpha}\right)}-\sum_{i=1}^{n}\log(x_i)-\sum_{i=1}^{n}\frac{\left((1+\lambda)\left(\frac{x_i}{\beta}\right)^{-\alpha}\log\left(\frac{x_i}{\beta}\right)\right)}{\left((1+\lambda)\left(1+\left(\frac{x_i}{\beta}\right)^{-\alpha}\right)-2\lambda\right)}+\frac{n}{\alpha}+n\log(\beta) \qquad (18)$$

$$\frac{\partial l}{\partial\beta}=\sum_{i=1}^{n}\frac{1}{\beta^2}\frac{(1+\lambda)\alpha x}{\left((1+\lambda)\left(1+\left(\frac{x_i}{\beta}\right)^{-\alpha}\right)-2\lambda\right)\left(\frac{x_i}{\beta}\right)^{\alpha+1}}-3\sum_{i=1}^{n}\frac{3\alpha x}{\left(1+\left(\frac{x_i}{\beta}\right)^{-\alpha}\right)\left(\frac{x_i}{\beta}\right)^{\alpha+1}\beta^2}+\frac{n\alpha}{\beta} \qquad (19)$$

$$\frac{\partial l}{\partial\lambda}=\sum_{i=1}^{n}\frac{\left\{\left(\frac{x_i}{\beta}\right)^{-\alpha}-1\right\}\left(\frac{x_i}{\beta}\right)^{-(\alpha+1)}}{\left((1+\lambda)\left(1+\left(\frac{x_i}{\beta}\right)^{-\alpha}\right)-2\lambda\right)} \qquad (20)$$

These three derivative equations cannot be solved analytically, so $\hat{\alpha},\hat{\beta}\, and\, \hat{\lambda}$ will be obtained by maximizing the log likelihood function numerically using Newton-Raphson method which is a powerful technique for solving equations iteratively and numerically.

4.1 Simulation

In this section, we investigate the behavior of the ML estimators for a finite sample size n. Simulation study based on different $TILLD(x,\alpha,\beta,\lambda)$ distribution is carried out. The random observations are generated using the inverse cdf method presented in section 3.4 from TLLD (α,β,λ). A simulation study was conducted for each triplet (α,β,λ), taking two parameter combinations as $(\alpha=0.7,\beta=0.9,\lambda=0.2)$, $(\alpha=0.8,\beta=1.1,\lambda=0.6)$ and the process was repeated 500 times by taking different sample sizes n = (25,50,100,150,200).We observe in Table 1 that the agreement between theory and practice improves as the sample size *n* increases. MSE and Variance of the estimators suggest us that the estimators are consistent and the maximum likelihood method performs quite well in estimating the model parameters of the proposed distribution.

Table 1: Average Bias, MSE and Variance for simulated results of ML estimates.

sample size n	Parameters	$(\alpha=0.7,\beta=0.9,\lambda=0.2)$				$(\alpha=0.8,\beta=1.1,\lambda=0.6)$			
		Bias	Variance	MSE	Coverage Probability (95%)	Bias	Variance	MSE	Coverage Probability (95%)
25	α	0.0624	0.0916	0.0955	0.9420	0.0358	0.0296	0.0308	0.9440
	β	0.2590	1.2480	1.3151	0.9440	0.0067	0.2049	0.2050	0.9460
	λ	0.1590	0.1803	0.2056	0.9460	-0.0551	0.1982	0.2012	0.9480
50	α	0.0160	0.0246	0.0249	0.9440	0.0191	0.0212	0.0215	0.9460
	β	0.2207	0.8999	0.9486	0.9560	0.0022	0.1634	0.1634	0.9580
	λ	0.1266	0.1691	0.1851	0.9520	-0.0523	0.1768	0.1796	0.9540
100	α	0.0090	0.0226	0.0227	0.9520	0.0102	0.0161	0.0162	0.9540
	β	0.1507	0.5546	0.5773	0.9580	-0.0221	0.1266	0.1271	0.9600
	λ	0.1255	0.1513	0.1671	0.9560	-0.0551	0.1638	0.1668	0.9580
150	α	0.0080	0.0219	0.0220	0.9580	0.0124	0.0146	0.0147	0.9600
	β	0.1579	0.5342	0.5591	0.9620	-0.0442	0.1058	0.1078	0.9640
	λ	0.1170	0.1478	0.1615	0.9640	-0.0771	0.1501	0.1560	0.9660
200	α	0.0030	0.0216	0.0216	0.9660	0.0110	0.0139	0.0141	0.9680
	β	0.1439	0.5128	0.5335	0.9740	-0.0714	0.0871	0.0922	0.9760
	λ	0.1041	0.1382	0.1490	0.9720	-0.0808	0.1380	0.1445	0.9740

5 Robust Skewness and Kurtosis Measures for TLLD

To illustrate the effect of the parameter α *and* λ on skewness and kurtosis, we consider measures based on quantiles. The shortcomings of the classical kurtosis measure are well known. There are many heavy-tailed distributions for which this measure is infinite, so it becomes uninformative. They are less sensitive to outliers and they exist for the distributions even without defined moments. The Bowley's skewness [7] is one of the earliest skewness measures defined by the average of the quartiles minus the median, divided by half the interquartile range, given by

$$\mathrm{B}=\frac{Q_3-Q_1-2Q_2}{Q_3-Q_1}=\frac{Q\left(\frac{3}{4}\right)-Q\left(\frac{1}{4}\right)-2Q\left(\frac{1}{2}\right)}{Q\left(\frac{3}{4}\right)-Q\left(\frac{1}{4}\right)}$$

and the Moors kurtosis [8] is based on octiles and is given by

$$M=\frac{(E_3-E_1)+(E_7-E_5)}{E_6-E_2}=\frac{Q\left(\frac{3}{8}\right)-Q\left(\frac{1}{8}\right)+Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)}$$

For any distribution symmetrical to 0 the Moors kurtosis reduces to

Table 2: Bowley's Skewness of Transmuted Log-logistic Distribution for different values of parameters.

Parameter		$\beta = 1.2$								
		α								
		0.3	1.1	1.5	2.8	3.5	3.7	4.2	4.7	5.2
λ	-1.0	0.929	0.449	0.353	0.219	0.187	0.180	0.165	0.154	0.144
	-0.9	0.932	0.448	0.350	0.213	0.180	0.173	0.158	0.146	0.137
	-0.8	0.934	0.448	0.348	0.208	0.174	0.166	0.151	0.139	0.129
	-0.5	0.942	0.451	0.345	0.195	0.159	0.151	0.135	0.122	0.111
	0.2	0.948	0.457	0.347	0.191	0.153	0.145	0.128	0.114	0.103
	0.6	0.931	0.418	0.311	0.162	0.127	0.119	0.103	0.090	0.080
	0.7	0.924	0.402	0.296	0.150	0.115	0.108	0.092	0.079	0.069
	0.8	0.916	0.385	0.281	0.137	0.103	0.096	0.080	0.068	0.058
	0.9	0.907	0.368	0.265	0.124	0.090	0.083	0.068	0.056	0.046

$$M = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right)}{Q\left(\frac{6}{8}\right)}$$

It is easy to calculate that for standard normal distribution $E_1 = -E_7 = -1.15$, $E_2 = -E_6 = -0.67$ and $E_3 = -E_5 = -0.32$. Therefore, M = 1.23. Hence, the centered Moor's coefficient is given by:

$$M = \frac{Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right) + Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} - 1.23$$

Table 2 and Table 3 provide the numerically calculated Bowleys skewness and Moors kurtosis of transmuted Log-logistic Distribution for different values of parameters using R studio statistical software.

Table 3: Moors Kurtosis of Transmuted Log-logistic Distribution for different values of parameters.

Parameters		$\beta = 1.9$								
		α								
		0.3	1.1	1.5	2.8	3.5	3.7	4.2	4.7	5.2
λ	-1.0	14.590	2.009	1.702	1.437	1.395	1.387	1.370	1.358	1.349
	-0.9	14.806	2.008	1.700	1.437	1.395	1.387	1.371	1.360	1.351
	-0.8	15.035	2.007	1.697	1.435	1.395	1.387	1.372	1.361	1.353
	-0.5	15.278	2.006	1.693	1.433	1.394	1.386	1.372	1.361	1.353
	0.2	16.367	2.007	1.678	1.413	1.375	1.368	1.355	1.346	1.339
	0.6	12.605	1.863	1.596	1.384	1.355	1.350	1.340	1.333	1.328
	0.7	11.167	1.796	1.554	1.365	1.340	1.336	1.327	1.322	1.318
	0.8	9.738	1.721	1.507	1.342	1.321	1.318	1.311	1.307	1.304
	0.9	8.42	1.64	1.46	1.32	1.30	1.30	1.29	1.29	1.29

6 Applications of Transmuted Log-logistic Distribution in Medical Sciences

In this section, we compare the performance of the three parameter transmuted Log-logistic distribution with some other related distributions on data set representing the survival times of a group of patients suffering from Head and Neck cancer disease reported by Efron [9] as given in Table 4.The data set related to survival times of a group of patients suffering from Head and Neck cancer disease has been studied by Shanker, Fesshaye and Selvaraj [10] using one parameter Akash, Lindley and Exponential distributions. In this section an attempt has been made to fit three parameter transmuted Log-logistic Distribution (TLLD) to data set given in Table 4, in comparison with Lindley, Akash, Exponential, Gamma, Weibull and generalized Lindley distributions (GLD).

We compare the models using AIC (Akaike Information Criterion) given by Akaike [11], AICC (Akaike Information Criterion Corrected) and BIC (Bayesian information criterion) given by Schwarz [12]. Generic functions calculating AIC, AICC and BIC for the model having p number of parameters are given by

$$AIC = 2p - 2\log(l)$$

$$AICC = AIC + \frac{2p(p+1)}{n-p-1}$$

$$BIC = p\log(n) - 2\log(l)$$

Table 4: The data set reported by Efron [9] represent the survival times of a group of patients suffering from Head and Neck cancer disease and treated using radiotherapy (RT).

6.5	7.0	10.4	14.5	16.1	22.7	34.0	41.6
42.0	45.3	49.4	53.6	63.0	64.0	83.0	84.0
91.0	108.0	112.0	129.0	133.0	133.0	139.0	140.0
140.0	146.0	149.0	154.0	157.0	160.0	160.0	165.0
146.0	149.0	154.0	157.0	160.0	160.0	165.0	173.0
176.0	218.0	225.0	241.0	248.0	273.0	277.0	297.0
405.0	417.0	420.0	440.0	523.0	583.0	594.0	1101.0
1146.0	1417.0						

Table 5 exhibits the AIC, AICC, BIC and Negative Loglikelihood values for the models fitted to the data in Table 4. It is obvious that AIC, AICC, and BIC criterion favors discrete transmuted Log-logistic distribution in comparison with Lindley, Akash, Exponential, Gamma, Weibull and generalized Lindley distributions (GLD). Parameter estimates, AIC, AICC, BIC and -2logL values for fitted distributions are calculated using R software [13].

Table 5: AIC, AICC, BIC and -2logL values for fitted distributions to data set of survival times of a group of patients suffering from Head and Neck cancer disease and treated using radiotherapy (RT).

Criteria	Exponential	Lindley	Akash	Gamma	Weibull	TLLD	GLD
-2*logL*	796.5613	763.75	803.96	744.832	744.7922	742.043	745.7118
AIC	798.5613	765.75	805.96	748.832	748.7922	748.043	751.7118
AICC	798.6328	765.82	806.02	749.0502	749.0104	748.4874	752.1562
BIC	800.6218	767.81	810.01	752.9529	752.9131	752.2243	757.8931
Parameter estimation	$\hat{\theta} = 0.012$	$\hat{\theta} = 0.0088$	$\hat{\theta} = 0.013$	$\hat{\alpha} = 1.03$ $\hat{\theta} = 0.004$	$\hat{\alpha} = 0.97$ $\hat{\theta} = 224.5$	$\hat{\alpha} = 1.61$ $\hat{\beta} = 120.8$ $\hat{\lambda} = -0.23$	$\hat{\alpha} = 0.11$ $\hat{\beta} = 0.51$ $\hat{\theta} = 0.004$

7 Conclusion

A new generalization of the two parameter Log-logistic distribution called the transmuted Log-logistic distribution was studied. The subject distribution is generated using the quadratic rank transmutation map technique and taking the two parameter Log-logistic distribution as the base distribution. Finally, the model is examined with real life data from medical science field.

Conflict of Interest

The authors declare that they have no conflict of interest.

References

[1] W. Shaw and I. Buckley, The alchemy of probability distributions: beyond Gram- Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. *Research report*, (2007).

[2] G. R. Aryal and C. P. Tsokos, On the Transmuted Extreme Value Distribution with Application. *Nonlinear Analysis Theory Method and Application*, 71, 1401-1407 (2009).

[3] G. R. Aryal and C. P. Tsokos, Transmuted Weibull distribution: A generalization of the Weibull probability distribution, *Eur J Pure Appl Math.*, 4, 89-102 (2011).

[4] M. A. Hussian, Transmuted Exponentiated Gamma Distribution: A Generalization of the Exponentiated Gamma Probability Distribution. *Applied Mathematical Sciences*, 8(27), 1297-1310 (2014).

[5] F. Merovci, Transmuted Lindley Distribution. *International Journal of Open Problem in Computer Science and Mathematics*, 6(2), 63-72 (2013).

[6] B. A. Para and T. R. Jan, A new generalized version of Log-logistic distribution with applications in medical sciences and other applied fields, *International Journal of Modeling, Simulation, and Scientific Computing*, 9(5), 1850043-22 (2018).

[7] A. L. Bowley, Elements of Statistics, *P. S. King and Son*, London., (1901).

[8] J. A. Moors, A quantile alternative for kurtosis*, Journal of the Royal Statistical Society*, 37, 25-32 (1998).

[9] B. Efron, Logistic regression, survival analysis and the Kaplan-Meier curve, *Journal of the American Statistical Association*, 83 (402), 414-425 (1988).

[10] R. Shanker, H. Fesshaye and S. Selvaraj, On Modeling of Lifetime Data Using One Parameter Akash, Lindley and Exponential Distributions. *Biom Biostat Int J*., 3(2), 00061 (2016).

[11] H. Akaike, H, A new look at the statistical model identification. *IEEE Trans. Autom. Control.,* 19, 716–723 (1974).

[12] G. Schwarz, Estimating the dimension of a model. *Ann. Stat.,* 5, 461–464 (1987).

[13] R Core Team (2019). R version 3.5.3: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria.URL https://www.R-project.org/.