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Some Results on Multipoint Integral Boundary Value Problems for Fractional Integro-Differential Equations

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Abstract: This paper investigates nonlinear fractional integro-differential equation involving caputo fractional derivative with non local multi point integral boundary conditions. Main results are obtained by applying the Banach contraction principle and Krasnoselskii fixed point theorems. An example is presented to illustrate the results.

Keywords: Existence, fixed point theorems, fractional integro-differential equation, integral boundary condition, Caputo fractional derivative, uniqueness.

1 Introduction

Fractional-order models have been more sufficient than integer-order models in the real-world problems. In addition, for the explanation of memory and hereditary properties, the fractional derivative is acts as excellent instrument. The fractional differential equations have gained more attention in different fields such as applied mathematics, engineering, physics, polymer rheology, etc. For more information, see [2–5, 7–10].

Boundary value problems with integral boundary conditions are attractive and applied to find make sense important class of applied engineering mathematics. These problems have been included with multipoints and nonlocal boundary conditions. Moreover, integral boundary conditions appear in blood flow problems, chemical engineering, thermo elasticity, etc. See [14, 17, 19, 20, 23–25].

In [3], A. Anguraj et al. investigated the new existence results for frational integro-diffrentioal equations with integral boundarey conditions. In [8], D. Baleanu, et al. studied the time dependent of the three point boundary value problems for fractional integro differential equations. Existence of solutions, which was proved using the new method for reliable mixed method for non-integer order of singular integro-differential equations was discussed in [11]. Existence of weak solutions for fractional integro-differential equations with multipoint boundary condition was addressed in [14]. In [23], Y. Wang et al. discussed the existence of solutions for Fractional Integro-Differential Equations with integral and multi-point Boundary Conditions. In [24], P. Zhang et al. explored the existence and uniqueness of the global solution for a class of nonlinear fractional integro-differential equations in a Banach space.

In this paper, we study the existence of solutions for fractional integro-differential equations involving multi-point integral boundary conditions:

$$ {}^{c}D^{\alpha}x(t) = g(t, x(t), \Psi x(t)), 0 < t < 1, 2 < \alpha \leq 3, \tag{1} $$

$$ \begin{cases} x(0) = 0, \\ x(1) = \kappa \int_0^{\nu} x(s)ds, 0 < \nu < 1, \\ [J^p x](\upsilon) = x(1), 0 < \upsilon < 1, \end{cases} \tag{2} $$

where ${}^{c}D^{\alpha}$ denotes the caputo fractional derivative of order α, J^p denotes the Riemann-Liouville fractional integral of order p, $g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and $\kappa \in \mathbb{R}$ with the condition that $\kappa \neq \frac{2}{\nu^2}$.

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$\Psi x(t) = \int_0^t k(t,s,x(s))ds$ and $k : \Delta \times X \to X, \Delta = \{(t,s) : 0 \le s \le t \le 1\}$. This work has been inspired by that of G. Akram et al. [2]. The fractional differential equation without involving integral term was studied. Here, we extend the results. The main results, identifications and example are presented in subsequent part.

2 Preliminaries

Let $C([0,1],\mathbb{R})$ be a Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm defined by $\|x\| = \sup\{|x(t)|; t \in [0,1]\}$.

Definition 1. *[22] The Riemann-Liouville fractional integral of order α for a continuous function $g(t)$ is defined as*

$$J^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{x(p)}{(t-s)^{\alpha-1}} g(s)dp, \quad \alpha > 0.$$

Definition 2. *[22] For a continuous function $g(t)$,the Riemann-Liouville fractional derivative of order α is defined as*

$${}^{c}D^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{\alpha-1} g(s)dp, n = [\alpha]+1.$$

Definition 3. *[22] For a continuous function $g : [0,\infty) \to \mathbb{R}$, the caputo derivative of fractional order α is defined as*

$${}^{c}D^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} g^n(s)ds, n-1 < \alpha < n, n = [\alpha]+1,$$

provided that $g^{(n)}(t)$ exists, where $[\alpha]$ denotes the integer part of the real number α.

Lemma 1. *[2] Consider the following differential equation and for $\alpha > 0$:*

$${}^{c}D^{\alpha} x(t) = 0. \tag{3}$$

The solution of Eq.(3) can be written in the following form:

$$x(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_{n-1} t^{n-1},$$

where $a_i \in \mathbb{R}, i = 0,1,2,\ldots,n-1, n = [\alpha]+1$.

Lemma 2. *[2] Assuming $\alpha > 0$, then*

$$J^{\alpha C} D^{\alpha} x(t) = x(t) + a_0 + a_1 t + a_2 t^2 + \ldots + a_{n-1} t^{n-1},$$

For some $a_i \in \mathbb{R}, i = 0,1,2,\ldots,n-1$ where n is the smallest integer greater than or equal to α.

Lemma 3. *[2] Consider the boundary value problem*

$$\begin{cases} {}^{c}D^{\alpha} x(t) = h(t), 0 < t < 1, 2 < \alpha \le 3, \\ x(0) = 0, \\ x(1) = \kappa \int_0^{\nu} x(s)ds, 0 < \nu < 1, \\ [J^p x](\upsilon) = x(1), 0 < \upsilon < 1, \end{cases} \tag{4}$$

where $(h \in C[0,1],\mathbb{R})$. Define

$$A = \left(\frac{1}{-2\upsilon^{p+1}(\kappa\nu^2 - 3)(p+2) - 6\upsilon^{p+2}(2 - \kappa\nu^2) + (p+2)!\kappa\nu^2(2\nu - 3)} \right)$$

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such that $A \neq 0$.
The solution of BVP (4) is

$$
\begin{aligned}
x(t) &= \frac{1}{\Gamma(p)} \int_0^t (t-s)^{\alpha-1} h(s) ds - [\frac{2t}{(2-\kappa\nu^2)\Gamma(\alpha)} \\
&+ \frac{(p+2)!(-2t(\kappa\nu^2-3)-3t^2(2-\kappa\nu^2)(\kappa\nu^2)}{A\Gamma(\alpha)(2-\kappa\nu^2)}] \int_0^1 (1-s)^{\alpha-1} h(s) ds \\
&+ \left[\frac{2t\kappa}{(2-\kappa\nu^2\Gamma(\alpha)} + \frac{2\kappa(p+2)!(-2t(\kappa\nu^3-3)-3t^2(2-\kappa\nu^2)}{A\Gamma(\alpha)(2-\kappa\nu^2}\right] \\
&\times \int_0^\nu \left(\int_0^s (s-m)^{\alpha-1} h(m) dm\right) - \frac{(p+2)!}{A\Gamma(p)\Gamma(\alpha)} \\
&\times (-2t(\kappa\nu^3-3)-3t^2(2-\kappa\nu^2)) \int_0^\upsilon \int_0^s (\upsilon-s)^{p-1} (s-r)^{\alpha-1} h(r) ds \\
&+ \left[\frac{2\upsilon^{p+1}(p+1)(-2t(\kappa\nu^3-3)-3t(2-\kappa\nu^2)}{A\Gamma(p)\Gamma(\alpha)(2-\kappa\nu^2)}\right] \\
&\times \int_0^\upsilon \int_0^1 (\upsilon-s)^{p-1} (1-r)^{\alpha-1} h(r) drds \\
&- \left[\frac{2\kappa\upsilon^{p+1}(p+2)(-2t(\kappa\nu^3-3)-3t^2(2-\kappa\nu^2)}{A\Gamma(p)\Gamma(\alpha)(2-\kappa\nu^2)}\right] \\
&\times \int_0^\nu \left(\int_0^\upsilon \int_0^r (\upsilon-s)^{p-1} (r-m)^{\alpha-1} h(m) dm\right) drds. \qquad (5)
\end{aligned}
$$

Let us take

$$
\begin{aligned}
\zeta = [&\frac{1}{\Gamma(\alpha+1)} + \left|\frac{2}{(2-\kappa\nu^2\Gamma(\alpha+1)} + \frac{(p+2)!(((\kappa\nu^2)^2)(-2\nu+3))}{A\Gamma(\alpha+1)(2-\kappa\nu^2)}\right| \\
&+ \frac{\nu^{\alpha+1}}{\Gamma(\alpha+2)} \left|\frac{2\kappa}{(2-\kappa\nu^2} + \frac{2\kappa(p+2)!((\kappa\nu^2)(-2\nu+3))}{A(2-\kappa\nu^2)}\right| \\
&+ \frac{\upsilon^{p+\alpha}}{\Gamma(p+\alpha+1)} \left|\frac{(p+2)!(\kappa\nu^2(-2\nu+3))}{A}\right| \\
&+ \frac{\upsilon^p}{\Gamma(p+1)\Gamma(\alpha+1)} \left|\frac{2\upsilon^{p+1}(p+2)(\kappa\nu^2(-2\nu+3))}{2-\kappa\nu^2)A}\right| \\
&+ \frac{1}{\Gamma(p+1)\Gamma(\alpha+1)} \left|\frac{2\upsilon^{p+1}(p+2)(\kappa\nu^2(-2\nu+3))}{(2-\kappa\nu^2)A}\right| (-(-\upsilon-\nu)^p+\upsilon^p)]. \qquad (6)
\end{aligned}
$$

Theorem 1. *[13] (Krasnoselkii's fixed point theorem)*
Let K be a closed convex, bounded and nonempty subset of a Banach space X. Let A_1, A_2, be two operators such that
(i)$A_1x + A_2y \in K$ for any $x, y \in K$.
(ii)A_1 is completely continuous operator.
(iii) A_1 is contraction operator.
Then there exists one fixed point $z_1 \in K$ such that $z_1 = A_1z_1 + A_2z_1$,atleast.

3 Main results

The main results are proved with the support of following hypothesis.
(H_1) g is continuous function and there exists a constant $L_1, L_2 > 0$ such that:

$$|g(t,u,v) - g(t,\overline{u},\overline{v})| \leq L_1 |u-\overline{u}| + L_2 |v-\overline{v}|.$$

For any $u, v, \overline{u}, \overline{v} \in \mathbb{R}$ and $t \in [0,1]$.
(H_2) There exists a constant $M > 0$ such that:

$$|k(t,s,u) - k(t,s,v)| \leq M |u-v|$$

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(H_3) There exists a non decreasing function $\upsilon(t), \iota(t) > 0$ such that for $t \in J$
$|g(t,x,y)| \leq \upsilon(t), \ \ |k(t,x,y)| \leq \iota(t), \ \forall (t,x,y) \in [0,1] \times \mathbb{R} \times \mathbb{R}$ with $\upsilon, \iota \in C([0,T],\mathbb{R})$.

Theorem 2. *Suppose that the hypothesis $(H_1)-(H_2)$ is satisfied. If*

$$[(L_1 + L_2 M)\zeta] < 1 \tag{7}$$

then the problem (1) − (2) has a unique solution on $[0,1]$

Proof. Consider the operator $\Upsilon : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$, as

$$\begin{aligned}
\Upsilon(x)(t) = {} & \frac{1}{\Gamma(p)} \int_0^t (t-s)^{\alpha-1} g(s,x(s),\psi x(s))ds - [\frac{2t}{(2-\kappa\nu^2)\Gamma(\alpha)} \\
& + \frac{(p+2)!(-2t(\kappa\nu^2-3) - 3t^2(2-\kappa\nu^2)(\kappa\nu^2)}{A\Gamma(\alpha)(2-\kappa\nu^2)}] \int_0^1 (1-s)^{\alpha-1} g(s,x(s),\psi x(s))ds \\
& + \left[\frac{2t\kappa}{(2-\kappa\nu^2\Gamma(\alpha)} + \frac{2\kappa(p+2)!(-2t(\kappa\nu^3-3) - 3t^2(2-\kappa\nu^2)}{A\Gamma(\alpha)(2-\kappa\nu^2}\right] \\
& \times \int_0^\nu \left(\int_0^s (s-m)^{\alpha-1} g(m,x(m),\psi x(m))dm \right) \\
& - \frac{(p+2)!}{A\Gamma(p)\Gamma(\alpha)} \times (-2t(\kappa\nu^3-3) - 3t^2(2-\kappa\nu^2)) \\
& \int_0^\upsilon \int_0^s (\upsilon-s)^{p-1} (s-r)^{\alpha-1} g(r,x(r),\psi x(r))drds \\
& + \left[\frac{2\upsilon^{p+1}(p+1)(-2t(\kappa\nu^3-3) - 3t(2-\kappa\nu^2)}{A\Gamma(p)\Gamma(\alpha)(2-\kappa\nu^2)}\right] \\
& \times \int_0^\upsilon \int_0^1 (\upsilon-s)^{p-1} (1-r)^{\alpha-1} g(r,x(r),\psi x(r))drds \\
& - \left[\frac{2\kappa\upsilon^{p+1}(p+2)(-2t(\kappa\nu^3-3) - 3t^2(2-\kappa\nu^2)}{A\Gamma(p)\Gamma(\alpha)(2-\kappa\nu^2)}\right] \\
& \times \int_0^\nu \left(\int_0^\upsilon \int_0^r (\upsilon-s)^{p-1} (r-m)^{\alpha-1} g(m,x(m),\psi x(m))dm \right) drds.
\end{aligned}$$

The fixed point of operator Υ is the solution of BVP (1) with boundary conditions (2).
Let $x, y \in C([0,1],\mathbb{R}) \ \forall \ \ t \in J$, then

$$\begin{aligned}
|\Upsilon(x)(t) - \Upsilon(y)(t)| \leq {} & \frac{1}{\Gamma(p)} \int_0^t (t-s)^{\alpha-1} |g(s,x(s),\psi x(s)) - g(s,y(s),\psi y(s))| \, ds \\
& + |[\frac{2t}{(2-\kappa\nu^2)\Gamma(\alpha)} + \frac{(p+2)!(-2t(\kappa\nu^2-3) - 3t^2(2-\kappa\nu^2)(\kappa\nu^2)}{A\Gamma(\alpha)(2-\kappa\nu^2)}]| \\
& \int_0^1 (1-s)^{\alpha-1} |g(s,x(s),\psi x(s)) - g(s,y(s),\psi y(s))| \, ds \\
& + \left| \left[\frac{2t\kappa}{(2-\kappa\nu^2\Gamma(\alpha)} + \frac{2\kappa(p+2)!(-2t(\kappa\nu^3-3) - 3t^2(2-\kappa\nu^2)}{A\Gamma(\alpha)(2-\kappa\nu^2}\right] \right| \\
& \times \int_0^\nu \left(\int_0^s (s-m)^{\alpha-1} |g(m,x(m),\psi x(m)) - g(m,y(m),\psi y(m))| \, dm \right) \\
& + \left| \frac{(p+2)!}{A\Gamma(p)\Gamma(\alpha)} \right| \times (-2t(\kappa\nu^3-3) - 3t^2(2-\kappa\nu^2))
\end{aligned}$$

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$$\int_0^{\upsilon}\int_0^{s}(\upsilon-s)^{p-1}(s-r)^{\alpha-1}|g(r,x(r),\psi x(r))-g(r,y(r),\psi y(r))|\,drds$$
$$+\left|[\frac{2\upsilon^{p+1}(p+1)(-2t(\kappa\nu^3-3)-3t(2-\kappa\nu^2)}{A\Gamma(p)\Gamma(\alpha)(2-\kappa\nu^2)}]\right|$$
$$\times\int_0^{\upsilon}\int_0^{1}(\upsilon-s)^{p-1}(1-r)^{\alpha-1}|g(r,x(r),\psi x(r))-g(r,y(r),\psi y(r))|\,drds$$
$$+\left|\left[\frac{2\kappa\upsilon^{p+1}(p+2)(-2t(\kappa\nu^3-3)-3t^2(2-\kappa\nu^2)}{A\Gamma(p)\Gamma(\alpha)(2-\kappa\nu^2)}\right]\right|$$
$$\times\int_0^{\nu}(\int_0^{\upsilon}\int_0^{r}(\upsilon-s)^{p-1}(r-m)^{\alpha-1}|g(m,x(m),\psi x(m))-g(m,y(m),\psi y(m))|\,dm)drds.$$

$$|\Upsilon(x)(t)-\Upsilon(y)(t)|\leq(L_1+L_2M)\frac{1}{\Gamma(p)}\int_0^t(t-s)^{\alpha-1}|x(s)-y(s)|\,ds+$$
$$+(L_1+L_2M)\left|\left[\frac{2t}{(2-\kappa\nu^2)\Gamma(\alpha)}+\frac{(p+2)!(-2t(\kappa\nu^2-3)-3t^2(2-\kappa\nu^2)(\kappa\nu^2)}{A\Gamma(\alpha)(2-\kappa\nu^2)}\right]\right|$$
$$\int_0^1(1-s)^{\alpha-1}|x(s)-y(s)|\,ds+(L_1+L_2M)|[\frac{2t\kappa}{(2-\kappa\nu^2\Gamma(\alpha)}$$
$$+\frac{2\kappa(p+2)!(-2t(\kappa\nu^3-3)-3t^2(2-\kappa\nu^2)}{A\Gamma(\alpha)(2-\kappa\nu^2}]|$$
$$\times\int_0^{\nu}\left(\int_0^{s}(s-m)^{\alpha-1}|x(m)-y(m)|\,dm\right)+(L_1+L_2M)\left|\frac{(p+2)!}{A\Gamma(p)\Gamma(\alpha)}\right|$$
$$\times(-2t(\kappa\nu^3-3)-3t^2(2-\kappa\nu^2))\int_0^{\upsilon}\int_0^{s}(\upsilon-s)^{p-1}(s-r)^{\alpha-1}|x(r)-y(r)|\,drds$$
$$+(L_1+L_2M)\left|\left[\frac{2\upsilon^{p+1}(p+2)(-2t(\kappa\nu^3-3)-3t(2-\kappa\nu^2)}{A\Gamma(p)\Gamma(\alpha)(2-\kappa\nu^2)}\right]\right|$$
$$\times\int_0^{\upsilon}\int_0^{1}(\upsilon-s)^{p-1}(1-r)^{\alpha-1}|x(r)-y(r)|\,drds$$
$$+(L_1+L_2M)\left|\left[\frac{2\kappa\upsilon^{p+1}(p+2)(-2t(\kappa\nu^3-3)-3t^2(2-\kappa\nu^2)}{A\Gamma(p)\Gamma(\alpha)(2-\kappa\nu^2)}\right]\right|$$
$$\times\int_0^{\nu}\left(\int_0^{\upsilon}\int_0^{r}(\upsilon-s)^{p-1}(r-m)^{\alpha-1}|x(m)-y(m)|\,dm\right)drds.$$

$$\leq(L_1+L_2M)\|x-y\|[[\frac{1}{\Gamma(\alpha+1)}$$
$$+\left|\frac{2}{(2-\kappa\nu^2\Gamma(\alpha+1)}+\frac{(p+2)!(((\kappa\nu^2)^2)(-2\nu+3))}{A\Gamma(\alpha+1)(2-\kappa\nu^2)}\right|$$
$$+\frac{\nu^{\alpha+1}}{\Gamma(\alpha+2)}\left|\frac{2\kappa}{(2-\kappa\nu^2}+\frac{2\kappa(p+2)!((\kappa\nu^2)(-2\nu+3))}{A(2-\kappa\nu^2)}\right|$$
$$+\frac{\upsilon^{p+\alpha}}{\Gamma(p+\alpha+1)}\left|\frac{(p+2)!(\kappa\nu^2(-2\nu+3))}{A}\right|$$
$$+\frac{\upsilon^{p}}{\Gamma(p+1)\Gamma(\alpha+1)}\left|\frac{2\upsilon^{p+1}(p+2)(\kappa\nu^2(-2\nu+3))}{2-\kappa\nu^2)A}\right|$$
$$+\frac{1}{\Gamma(p+1)\Gamma(\alpha+1)}\left|\frac{2\upsilon^{p+1}(p+2)(\kappa\nu^2(-2\nu+3))}{(2-\kappa\nu^2)A}\right|(-(-\upsilon-\nu)^p+\upsilon^p)]]$$

$$\leq[(L_1+L_2M)\zeta]\|x-y\|.$$

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which shows that ε depends on parameters taken in the problem. Therefore,Υ is a contraction. As a consequence of the Banach fixed point theorem, Υ has a unique solution of BVP with boundary condition.

Theorem 3. *Suppose that the Hypothesis (H_3) is fulfilled, then the problem (1)-(2) has at least one solution on* $[0,1]$.

Proof. Choose

$$r \geq \upsilon(t)\iota(t)\zeta \tag{8}$$

where $\upsilon(t), \iota(t)$ are mentioned in (H_3) and ζ is defined equation (6)
Consider, $B_r = \{x \in C(J,\mathbb{R}) : |x| \leq r\}$.
Let A_1 and A_2 be the two operators defined on B_r by,

$$(A_1x)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s,x(s),\psi x(s))ds. \tag{9}$$

$$\begin{aligned}
(A_2x)(t) := &\left[\frac{2t}{(2-\kappa\nu^2)\Gamma(\alpha)} + \frac{(p+2)!(-2t(\kappa\nu^2-3)-3t^2(2-\kappa\nu^2)(\kappa\nu^2)}{A\Gamma(\alpha)(2-\kappa\nu^2)}\right] \int_0^1 (1-s)^{\alpha-1} f(s,x(s),\psi x(s))ds \\
&+ \left[\frac{2t\kappa}{(2-\kappa\nu^2\Gamma(\alpha)} + \frac{2\kappa(p+2)!(-2t(\kappa\nu^3-3)-3t^2(2-\kappa\nu^2)}{A\Gamma(\alpha)(2-\kappa\nu^2}\right] \\
&\times \int_0^\nu \left(\int_0^s (s-m)^{\alpha-1} g(m,x(m),\psi x(m))dm\right) - \frac{(p+2)!}{A\Gamma(p)\Gamma(\alpha)} \\
&\times (-2t(\kappa\nu^3-3)-3t^2(2-\kappa\nu^2)) \int_0^\upsilon \int_0^s (\upsilon-s)^{p-1}(s-r)^{\alpha-1} g(r,x(r),\psi x(r))ds \\
&+ \left[\frac{2\upsilon^{p+1}(p+1)(-2t(\kappa\nu^3-3)-3t(2-\kappa\nu^2)}{A\Gamma(p)\Gamma(\alpha)(2-\kappa\nu^2)}\right] \\
&\times \int_0^\upsilon \int_0^1 (\upsilon-s)^{p-1}(1-r)^{\alpha-1} f(r,x(r),\psi x(r))drds \\
&- \left[\frac{2\kappa\upsilon^{p+1}(p+2)(-2t(\kappa\nu^3-3)-3t^2(2-\kappa\nu^2)}{A\Gamma(p)\Gamma(\alpha)(2-\kappa\nu^2)}\right] \\
&\times \int_0^\nu \left(\int_0^\upsilon \int_0^r (\upsilon-s)^{p-1}(r-m)^{\alpha-1} g(m,x(m),\psi x(m))dm\right) drds
\end{aligned} \tag{10}$$

respectively.
Note that $x,y \in B_r$ then $A_1x + A_2x \in B_r$.
Indeed it is easy to check inequality.

$$\begin{aligned}
|A_1x + A_2y| = &\left|\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s,x(s),\psi x(s))ds + \left[\frac{2t}{(2-\kappa\nu^2)\Gamma(\alpha)}\right.\right. \\
&\left. + \frac{(p+2)!(-2t(\kappa\nu^2-3)-3t^2(2-\kappa\nu^2)(\kappa\nu^2)}{A\Gamma(\alpha)(2-\kappa\nu^2)}\right] \int_0^1 (1-s)^{\alpha-1} g(s,x(s),\psi x(s))ds \\
&+ \left[\frac{2t\kappa}{(2-\kappa\nu^2\Gamma(\alpha)} + \frac{2\kappa(p+2)!(-2t(\kappa\nu^3-3)-3t^2(2-\kappa\nu^2)}{A\Gamma(\alpha)(2-\kappa\nu^2}\right] \\
&\times \int_0^\nu \left(\int_0^s (s-m)^{\alpha-1} g(m,x(m),\psi x(m))dm\right) - \frac{(p+2)!}{A\Gamma(p)\Gamma(\alpha)} \\
&\times (-2t(\kappa\nu^3-3)-3t^2(2-\kappa\nu^2)) \int_0^\upsilon \int_0^s (\upsilon-s)^{p-1}(s-r)^{\alpha-1} g(r,x(r),\psi x(r))drds \\
&+ \left[\frac{2\upsilon^{p+1}(p+1)(-2t(\kappa\nu^3-3)-3t(2-\kappa\nu^2)}{A\Gamma(p)\Gamma(\alpha)(2-\kappa\nu^2)}\right] \\
&\times \int_0^\upsilon \int_0^1 (\upsilon-s)^{p-1}(1-r)^{\alpha-1} g(r,x(r),\psi x(r))drds \\
&- \left[\frac{2\kappa\upsilon^{p+1}(p+2)(-2t(\kappa\nu^3-3)-3t^2(2-\kappa\nu^2)}{A\Gamma(p)\Gamma(\alpha)(2-\kappa\nu^2)}\right] \\
&\left.\times \int_0^\nu \left(\int_0^\upsilon \int_0^r (\upsilon-s)^{p-1}(r-m)^{\alpha-1} g(m,x(m),\psi x(m))dm\right) drds\right|
\end{aligned}$$

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$$\leq |\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,x(s),\psi x(s))ds| + |[\frac{2t}{(2-\kappa\nu^2)\Gamma(\alpha)}$$
$$+\frac{(p+2)!(-2t(\kappa\nu^2-3)-3t^2(2-\kappa\nu^2)(\kappa\nu^2)}{A\Gamma(\alpha)(2-\kappa\nu^2)}]\int_0^1 (1-s)^{\alpha-1} g(s,x(s),\psi x(s))ds|$$
$$+|\left[\frac{2t\kappa}{(2-\kappa\nu^2\Gamma(\alpha)}+\frac{2\kappa(p+2)!(-2t(\kappa\nu^3-3)-3t^2(2-\kappa\nu^2)}{A\Gamma(\alpha)(2-\kappa\nu^2}\right]$$
$$\times\int_0^\nu \left(\int_0^s (s-m)^{\alpha-1} g(m,x(m),\psi x(m))dm\right)ds| + |\frac{(p+2)!}{A\Gamma(p)\Gamma(\alpha)}$$
$$\times(-2t(\kappa\nu^3-3)-3t^2(2-\kappa\nu^2))\int_0^\upsilon\int_0^s (\upsilon-s)^{p-1}(s-r)^{\alpha-1} g(r,x(r),\psi x(r))ds|$$
$$+|\left[\frac{2\upsilon^{p+1}(p+1)(-2t(\kappa\nu^3-3)-3t(2-\kappa\nu^2)}{A\Gamma(p)\Gamma(\alpha)(2-\kappa\nu^2)}\right]$$
$$\times\int_0^\upsilon\int_0^1 (\upsilon-s)^{p-1}(1-r)^{\alpha-1} g(r,x(r),\psi x(r))drds|$$
$$+|\left[\frac{2\kappa\upsilon^{p+1}(p+2)(-2t(\kappa\nu^3-3)-3t^2(2-\kappa\nu^2)}{A\Gamma(p)\Gamma(\alpha)(2-\kappa\nu^2)}\right]$$
$$\times\int_0^\nu\left(\int_0^\upsilon\int_0^r (\upsilon-s)^{p-1}(r-m)^{\alpha-1} g(m,x(m),\psi x(m))dm\right)drds|$$

$$\leq \frac{1}{\Gamma(p)}\int_0^t (t-s)^{\alpha-1}|g(s,x(s),\psi x(s))|ds + [\frac{2t}{(2-\kappa\nu^2)\Gamma(\alpha)}$$
$$+\frac{(p+2)!(-2t(\kappa\nu^2-3)-3t^2(2-\kappa\nu^2)(\kappa\nu^2)}{A\Gamma(\alpha)(2-\kappa\nu^2)}]\int_0^1 (1-s)^{\alpha-1}|g(s,x(s),\psi x(s))|ds$$
$$+\left[\frac{2t\kappa}{(2-\kappa\nu^2\Gamma(\alpha)}+\frac{2\kappa(p+2)!(-2t(\kappa\nu^3-3)-3t^2(2-\kappa\nu^2)}{A\Gamma(\alpha)(2-\kappa\nu^2}\right]$$
$$\times\int_0^\nu\left(\int_0^s (s-m)^{\alpha-1}|g(m,x(m),\psi x(m))|dm\right)+\frac{(p+2)!}{A\Gamma(p)\Gamma(\alpha)}$$
$$\times(-2t(\kappa\nu^3-3)-3t^2(2-\kappa\nu^2))\int_0^\upsilon\int_0^s (\upsilon-s)^{p-1}(s-r)^{\alpha-1}|g(r,x(r),\psi x(r))|drds$$
$$+\left[\frac{2\upsilon^{p+1}(p+1)(-2t(\kappa\nu^3-3)-3t(2-\kappa\nu^2)}{A\Gamma(p)\Gamma(\alpha)(2-\kappa\nu^2)}\right]$$
$$\times\int_0^\upsilon\int_0^1 (\upsilon-s)^{p-1}(1-r)^{\alpha-1}|g(r,x(r),\psi x(r))|drds$$
$$+\left[\frac{2\kappa\upsilon^{p+1}(p+2)(-2t(\kappa\nu^3-3)-3t^2(2-\kappa\nu^2)}{A\Gamma(p)\Gamma(\alpha)(2-\kappa\nu^2)}\right]$$
$$\times\int_0^\nu\left(\int_0^\upsilon\int_0^r (\upsilon-s)^{p-1}(r-m)^{\alpha-1}|g(m,x(m),\psi x(m))|dm\right)drds.$$

$$\leq \gamma(t)\upsilon(t)[\frac{1}{\Gamma(\alpha+1)}$$
$$+\left|\frac{2}{(2-\kappa\nu^2\Gamma(\alpha+1)}+\frac{(p+2)!(((\kappa\nu^2)^2)(-2\nu+3))}{A\Gamma(\alpha+1)(2-\kappa\nu^2)}\right|$$
$$+\frac{\nu^{\alpha+1}}{\Gamma(\alpha+2)}\left|\frac{2\kappa}{(2-\kappa\nu^2}+\frac{2\kappa(p+2)!((\kappa\nu^2)(-2\nu+3))}{A(2-\kappa\nu^2)}\right|$$
$$+\frac{\upsilon^{p+\alpha}}{\Gamma(p+\alpha+1)}\left|\frac{(p+2)!(\kappa\nu^2(-2\nu+3))}{A}\right|$$

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$$+\frac{\upsilon^p}{\Gamma(p+1)\Gamma(\alpha+1)}\left|\frac{2\upsilon^{p+1}(p+2)(\kappa\nu^2(-2\nu+3))}{2-\kappa\nu^2)A}\right|$$
$$+\frac{1}{\Gamma(p+1)\Gamma(\alpha+1)}\left|\frac{2\upsilon^{p+1}(p+2)(\kappa\nu^2(-2\nu+3))}{(2-\kappa\nu^2)A}\right|(-(-\upsilon-\nu)^p+\upsilon^p)]$$
$$\leq \iota(t)\upsilon(t)\zeta$$
$$\leq r$$

It is also clear that A_1 is a contraction mapping. From continuity of x, the operator $(A_1x)(t)$ is continuous in accordance with g. Also we observe that

$$|(A_1x)(t)| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|g(s,x(s),\psi x(s))|\,ds$$
$$\leq \frac{1}{\Gamma(\alpha+1)}\upsilon(t)\iota(t)$$

A_1 is uniformly bounded on B_r.
Now, let's prove that $(A_1x)(t)$ is equicontinuous.
Let $t_1,t_2 \in J, t_2 < t_1$ and $x \in B_r$. Using the fact that g is bounded on the compact set

$$\sup_{(t,x,y)\in[0,1]\times B_r} |g(t,x,\Psi x(t))| = C_0 < \infty$$

we will get

$$|(A_1x)(t_2)-(A_1x)(t_1)| = \Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}g(s,x(s),\psi x(s))ds$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}g(s,x(s),\psi x(s))ds\Big|$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\right]|g(s,x(s),\psi x(s))ds|$$
$$+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|g(s,x(s),\psi x(s))|\,ds$$
$$\leq \frac{C_0}{\Gamma(\alpha+1)}[2(t_2-t_1)^\alpha+|(t_2)^\alpha-(t_2)^\alpha|]$$

Which does not depend on x and tends to be zero as $t_2 \to t_1$. Thus $A_1(B_r)$ is relatively compact. Hence by the Ascoli-Arzela theorem, A_1 is compact on B_r. We conclude that problem (1)-(2) has at least one fixed point on J.

4 Example

Consider the boundary value problem:

$$\begin{cases} {}^cD^{\frac{5}{2}}x(t) = \frac{1}{5e^{t+2}(1+|y(t)|}+\int_0^t \frac{e^{-(s-t)}}{10}y(s)ds \\ x(0)=0; \\ x(1)=\int_0^{\frac{1}{3}}x(s)ds \\ (J^{\frac{3}{2}x})(\frac{1}{4})=x(1) \end{cases} \tag{11}$$

where set $\nu=\frac{1}{3},\alpha=\frac{5}{2},p=\frac{3}{2},\kappa=1,\upsilon=\frac{1}{4},L_1=L_2=\frac{1}{5e^3},M=\frac{1}{10},\frac{2}{\nu^2}=98.0392 \Rightarrow \kappa \neq \frac{2}{\nu^2}$ and

$$(L_1+L_2M)\zeta = (L_1+L_2M)\Big[\frac{1}{\Gamma(\alpha+1)}+\left|\frac{2}{(2-\kappa\nu^2\Gamma(\alpha+1)}+\frac{(p+2)!(((\kappa\nu^2)^2)(-2\nu+3))}{A\Gamma(\alpha+1)(2-\kappa\nu^2)}\right|$$
$$+\frac{\nu^{\alpha+1}}{\Gamma(\alpha+2)}\left|\frac{2\kappa}{(2-\kappa\nu^2}+\frac{2\kappa(p+2)!((\kappa\nu^2)(-2\nu+3))}{A(2-\kappa\nu^2)}\right|$$

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$$+\frac{\upsilon^{p+\alpha}}{\Gamma(p+\alpha+1)}\left|\frac{(p+2)!(\kappa\nu^2(-2\nu+3))}{A}\right|$$
$$+\frac{\upsilon^{p}}{\Gamma(p+1)\Gamma(\alpha+1)}\left|\frac{2\upsilon^{p+1}(p+2)(\kappa\nu^2(-2\nu+3))}{2-\kappa\nu^2)A}\right|$$
$$+\frac{1}{\Gamma(p+1)\Gamma(\alpha+1)}\left|\frac{2\upsilon^{p+1}(p+2)(\kappa\nu^2(-2\nu+3))}{(2-\kappa\nu^2)A}\right|(-(-\upsilon-\nu)^p+\upsilon^p)].$$

where

$$A=\left(\frac{1}{-2\upsilon^{p+1}(\kappa\nu^2-3)(p+2)-6\upsilon^{p+2}(2-\kappa\nu^2)+(p+2)!\kappa\nu^2(2\nu-3)}\right)$$

Furthermore, if $[(L_1+L_2M)\zeta]<1\approx 0.0885$
Thus, the boundary value problem (11) has a unique solution on $[0,1]$.

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