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Common Fixed Point Theorems in Intuitionistic Menger Spaces

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Abstract: The present paper aims to prove some new common fixed point theorems in intutionistic Menger metric spaces. While proving our results, we utilize the idea of compatibility due to Jungck [1] together with conditionally reciprocal continuity due to R. P. Pant and R. K. Bisht [2]. Our results substantially generalize and improve a multitude of relevant common fixed point theorems of the existing literature in metric as well as intutionistic menger metric spaces.

Keywords: Compatible or g- compatible or f- compatible maps; conditionally reciprocal continuous maps; intutionistic menger metric space and common fixed point.

1 Introduction

In 1922, Banach proved the mile stone in the fixed point theory and its applications. Several authors addressed a new class of fixed point problems in metric spaces. They proved fixed point theorem for mappings satisfying certain inequalities involving the altering distances function. There have been a number of generalizations of metric spaces. For example, Menger space was introduced in 1942 by Menger[3]. This space was expanded rapidly with the pioneering works of Schweizer and Sklar[4],[5]. Modifying the idea of Kramosil and Michalek[6], George and Veeramani[7] introduced fuzzy metric spaces which are very similar to that of Menger space. Atanassov[8] introduced and investigated the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. In 2004, Park[9] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous tconorms.
Kutukcu et al. [10] introduced the notion of intuitionistic Menger spaces with the help of t-norms and t-conorms as a generalization of Menger space due to Menger[3]. These observations motivated us to prove a common fixed point theorem in intuitionistic Menger spaces.
In this paper, we prove some new common fixed point theorems in intuitionistic Menger metric spaces. While proving our results, we utilize the idea of compatibility due to Jungck [1] together with conditionally reciprocal continuity due to R. P. Pant and R. K. Bisht [2]. Consequently, our results improve and develop many known common fixed point theorems available in the existing literature of intutionistic menger fixed point theory.

2 Preliminaries

Definition 1.*[4] A binary operation $\star : [0,1]\times[0,1]\to[0,1]$ is continuous t-norm if $\star$ satisfies the following conditions:*
(i) $\star$ is commutative and associative;
(ii) $\star$ is continuous;
(iii) $a\star 1 = a$ for all $a\in[0,1]$; (iv) $a\star b\le c\star d$ whenever $a\le c$ and $b\le d$ for all $a,b,c,d\in[0,1]$.

Definition 2.*[5] A binary operation $\diamond : [0,1]\times[0,1]\to[0,1]$ is continuous t-conorm if $\diamond$ satisfies the following conditions:*
(i) $\diamond$ is commutative and associative;
(ii) $\diamond$ is continuous;
(iii) $a\diamond 0 = a$ for all $a\in[0,1]$; (iv) $a\diamond b\ge c\diamond d$ whenever $a\ge c$ and $b\ge d$ for all $a,b,c,d\in[0,1]$.

Definition 3.*[10] A 5-tuple $(X,F,L,\star,\diamond)$ is said to be an intuitionistic Menger space if X is an arbitrary set, $\star$ is a*

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continuous t-norm, $\diamond$ is continuous t-conorm, F is a probabilistic distance and L is a probabilistic non-distance on X satisfying the following conditions:
for all $x,y,z \in X$ and $t,s > 0$,
(1) $F(x,y,t) + L(x,y,t) \leq 1$,
(2) $F(x,y,0) = 0$,
(3) $F(x,y,t) = H(t)$ if and only if $x = y$,
(4) $F(x,y,t) = F(y,x,t)$,
(5) if $F(x,y,t) = 1$ and $F(y,z,s) = 1$ then $F(x,z,t+s) = 1$,
(6) $F(x,z,t+s) \geq F(x,y,t) \star F(y,z,s)$,
(7) $L(x,y,0) = 1$,
(8) $L(x,y,t) = G(t)$ if and only if $x = y$,
(9) $L(x,y,t) = L(y,x,t)$,
(10) if $L(x,y,t) = 0$ and $L(y,z,s) = 0$ then $L(x,z,t+s) = 0$,
(11) $L(x,z,t+s) \leq L(x,y,t) \diamond L(y,z,s)$.
The function $F(x,y,t)$ and $L(x,y,t)$ denote the degree of nearness and degree of non-nearness between x and y with respect to t, respectively.

Definition 4.[11] *A sequence $\{x_n\}$ in an intuitionistic Menger metric space $(X,F,L,\star,\diamond)$ is said to be a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in N$ such that $F(x_n,x_m,t) > (1-\varepsilon)$, $L(x_n,x_m,t) < \varepsilon$, for each $n,m \geq n_0$.*

Definition 5.[11] *A sequence $\{x_n\}$ in an intuitionistic Menger metric space $(X,F,L,\star,\diamond)$ is said to be convergent to $x \in X$ if for all $t > 0$ $\lim_{n\to\infty} F(x_n,x,t) = 1$, $\lim_{n\to\infty} L(x_n,x,t) = 0$.*

Definition 6.[11] *An intuitionistic Menger metric space $(X,F,L,\star,\diamond)$ is said to be complete if every Cauchy sequence converges to a point of it.*

Definition 7.[11] *Let f and g be self-mappings of an intuitionistic Menger metric space $(X,F,L,\star,\diamond)$. Then, the pair (f,g) is said to be compatible if*

$$\lim_{n\to\infty} F(fgx_n,gfx_n,t) = 1,$$

$$\lim_{n\to\infty} L(fgx_n,gfx_n,t) = 0$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$.

Definition 8.[11] *Let f and g be self-mappings of an intuitionistic Menger metric space $(X,F,L,\star,\diamond)$. Then, the pair (f,g) is said to be non-compatible if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ but either*

$$\lim_{n\to\infty} F(fgx_n,gfx_n,t) \neq 1,$$

$$\lim_{n\to\infty} L(fgx_n,gfx_n,t) \neq 0,$$

or the limit does not exist for all $u \in X$.

Definition 9.[12] *Let f and g be self-mappings of an intuitionistic Menger metric space $(X,F,L,\star,\diamond)$. Then, the pair (f,g) is said to be g-compatible if*

$$\lim_{n\to\infty} F(ffx_n,gfx_n,t) = 1,$$

$$\lim_{n\to\infty} L(ffx_n,gfx_n,t) = 0$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$.

Definition 10.[12] *Let f and g be self-mappings of an intuitionistic Menger metric space $(X,F,L,\star,\diamond)$. Then, the pair (f,g) is said to be f-compatible if*

$$\lim_{n\to\infty} F(fgx_n,ggx_n,t) = 1,$$

$$\lim_{n\to\infty} L(fgx_n,ggx_n,t) = 0$$

for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$.

Definition 11.[2] *A pair of self mappings (f,g) of an intuitionistic Menger metric space $(X,F,L,\star,\diamond)$ is said to be conditionally reciprocally continuous (CRC) if whenever set of sequences $\{x_n\}$ satisfying $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$ is non-empty, there exists a sequence $\{y_n\}$ satisfying $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = u$ (say) such that $\lim_{n\to\infty} fgy_n = fu$ and $\lim_{n\to\infty} gfy_n = gu$.*

Lemma 1.[13] *Let $(X,F,L,\star,\diamond)$ be an intuitionistic Menger metric space and $\{y_n\}$ be a sequence in X. If there exists a number $k \in (0,1)$ such that*

$$F(y_n,y_{n+1},kt) \geq F(y_{n-1},y_n,t),$$
$$L(y_n,y_{n+1},kt) \leq L(y_{n-1},y_n,t)$$

for all $t > 0$ and n = 1,2,3,...
then $\{y_n\}$ is a Cauchy sequence in X.

3 Main Result

Theorem 1.*Let f and g be conditionally reciprocally continuous self-mappings of a complete intutionistic menger metric space $(X,F,L,\star,\diamond)$ satisfying the conditions:*
(3.1) $f(X) \subseteq g(X)$;
(3.2) there exists $k \in (0,1)$ such that
$F(fx,fy,kt) \geq F(gx,gy,t)$ and $L(fx,fy,kt) \leq L(gx,gy,t)$ for any $x,y \in X$, $t > 0$.
If f and g are either compatible or $g-$compatible or $f-$compatible then, f and g have a unique common fixed point.

*Proof.*Let x_0 be any point in X. Then, as $f(X) \subseteq g(X)$, there exists a sequence of points $\{x_n\}$ in X such that $fx_n = gx_{n+1}$.
Also, define a sequence $\{y_n\}$ in X as $y_n = fx_n = gx_{n+1}$.

(3.3)
First, we show that $\{y_n\}$ is a Cauchy sequence in X. For proving this, by (3.2), we have
$F(y_n, y_{n+1}, kt) = F(fx_n, fx_{n+1}, kt)$
$\geq F(gx_n, gx_{n+1}, t) = F(y_{n-1}, y_n, t)$
and $L(y_n, y_{n+1}, kt) = L(fx_n, fx_{n+1}, kt)$
$\leq L(gx_n, gx_{n+1}, t) = L(y_{n-1}, y_n, t)$
and then by Lemma 2.2, $\{y_n\}$ is a Cauchy sequence in X. As X is complete, there exists a point $z \in X$ such that $\lim_{n\to\infty} y_n = z$. Therefore, by (3.3), we have $\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} = z$. Since f and g are conditionally reciprocally continuous and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$, there exists a sequence $\{s_n\}$ satisfying $\lim_{n\to\infty} fs_n = \lim_{n\to\infty} gs_n = u$(say) such that $\lim_{n\to\infty} fgs_n = fu$ and $\lim_{n\to\infty} gfs_n = gu$. Since $f(X) \subseteq g(X)$, for each s_n, there exists z_n in X such that $fs_n = gz_n$. Thus, $\lim_{n\to\infty} fs_n = \lim_{n\to\infty} gs_n = \lim_{n\to\infty} gz_n = u$.
Using (3.2), we get
$F(fs_n, fz_n, kt) \geq F(gs_n, gz_n, t)$
and $L(fs_n, fz_n, kt) \leq L(gs_n, gz_n, t)$.
Taking limit $n \to \infty$, we get
$F(u, \lim_{n\to\infty} fz_n, kt) \geq F(u, u, t) = 1$
and $L(u, \lim_{n\to\infty} fz_n, kt) \leq F(u, u, t) = 0$.
This gives

$$\lim_{n\to\infty} fz_n = u.$$

Hence, $\lim_{n\to\infty} fs_n = \lim_{n\to\infty} gs_n = \lim_{n\to\infty} gz_n = \lim_{n\to\infty} fz_n = u$.
Suppose that f and g are compatible mappings. Then, $\lim_{n\to\infty} F(fgs_n, gfs_n, t) = 1$ and $\lim_{n\to\infty} L(fgs_n, gfs_n, t) = 0$.
This gives $fu = gu$. Moreover, $fgu = ffu = fgu = gfu$.
Using (3.2), we get
$F(fu, ffu, kt) \geq F(gu, gfu, t) = F(fu, ffu, t)$, and $L(fu, ffu, kt) \leq L(gu, gfu, t) = L(fu, ffu, t)$, that is $fu = ffu$. Hence, $fu = ffu = gfu$ and fu is a common fixed point of f and g.
Now, suppose that f and g are g-compatible mappings. Then,
$\lim_{n\to\infty} F(ffs_n, gfs_n, t) = 1$, and $\lim_{n\to\infty} L(ffs_n, gfs_n, t) = 0$.
That is, $\lim_{n\to\infty} ffs_n = \lim_{n\to\infty} gfs_n = gu$. Using (3.2), we get
$F(fu, ffs_n, kt) \geq F(gu, gfs_n, t)$ and $L(fu, ffs_n, kt) \leq L(gu, gfs_n, t)$.
Taking limit $n \to \infty$, we get
$F(fu, gu, kt) \geq F(gu, gu, t) = 1$ and $L(fu, gu, kt) \leq L(gu, gu, t) = 0$.
This gives $fu = gu$. Hence, $fgu = ffu = fgu = gfu$.
Using (3.2), we get
$F(fu, ffu, kt) \geq F(gu, gfu, t) = F(fu, ffu, t)$ and $L(fu, ffu, kt) \leq L(gu, gfu, t) = L(fu, ffu, t)$.
That is $fu = ffu$. Hence, $fu = ffu = gfu$ and fu is a common fixed point of f and g.
Finally, suppose that f and g are f-compatible mappings. Then,
$\lim_{n\to\infty} F(fgz_n, ggz_n, t) = 1$ and $\lim_{n\to\infty} L(fgz_n, ggz_n, t) = 0$.
That is, $\lim_{n\to\infty} fgz_n = \lim_{n\to\infty} ggz_n = gu$. Also, $\lim_{n\to\infty} gfs_n = \lim_{n\to\infty} ggz_n = gu$. Hence, $\lim_{n\to\infty} fgz_n = \lim_{n\to\infty} ggz_n = gu$.
Using (3.2), we get
$F(fu, fgz_n, kt) \geq F(gu, ggz_n, t)$ and $L(fu, fgz_n, kt) \leq L(gu, ggz_n, t)$.
Taking limit $n \to \infty$, we get
$F(fu, gu, kt) \geq F(gu, gu, t) = 1$ and $L(fu, gu, kt) \leq L(gu, gu, t) = 0$.
This gives, $fu = gu$. Furthermore, $fgu = ffu = fgu = gfu$. Using (3.2), we get
$F(fu, ffu, kt) \geq F(gu, gfu, t) = F(fu, ffu, t)$ and $L(fu, ffu, kt) \leq L(gu, gfu, t) = L(fu, ffu, t)$ that is $fu = ffu$.
Hence, $fu = ffu = gfu$ and fu is a common fixed point of f and g.
Uniqueness of the common fixed point theorem follows easily in each of the three cases.

Theorem 2.*Let f and g be non-compatible self-mappings of an intutionistic menger metric space $(X, F, L, \star, \diamond)$ satisfying the conditions:*
(3.4) $f(X) \subseteq g(X)$;
(3.5) $F(fx, fy, t) \geq F(gx, gy, t)$ and $L(fx, fy, t) \leq L(gx, gy, t)$;
(3.6) $F(fx, ffx, t) > F(gx, ggx, t)$ and $L(fx, ffx, t) < L(gx, ggx, t)$ whenever $gx \neq ggx$ for all $x, y \in X$, $t > 0$.
Suppose f and g are conditionally reciprocally continuous. If f and g are either $g-$ compatible or $f-$compatible, then f and g have a common fixed point.

*Proof.*Since f and g are non-compatible maps, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$, but either $\lim_{n\to\infty} F(fgx_n, gfx_n, t) \neq 1$, $\lim_{n\to\infty} L(fgx_n, gfx_n, t) \neq 0$ or the limit does not exist. Also, since f and g are conditionally reciprocally continuous and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$, there exists a sequence $\{y_n\}$ satisfying $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = u$ (say) such that $\lim_{n\to\infty} fgy_n = fu$ and $\lim_{n\to\infty} gfy_n = gu$. Since $f(X) \subseteq g(X)$, for each y_n, there exists z_n in X such that $fy_n = gz_n$. Thus, $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} gz_n = u$. Using (3.5), we get
$F(fy_n, fz_n, t) \geq F(gy_n, gz_n, t)$,
$L(fy_n, fz_n, t) \leq L(gy_n, gz_n, t)$.
Taking limit $n \to \infty$, we get
$F(u, \lim_{n\to\infty} fz_n, t) \geq F(u, u, t) = 1$ and $L(u, \lim_{n\to\infty} fz_n, t) \leq L(u, u, t) = 0$.
This gives $\lim_{n\to\infty} fz_n = u$. Therefore, we have
$\lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} gz_n = \lim_{n\to\infty} fz_n = u$.
Now, Suppose that f and g are g-compatible mappings. Then,
$\lim_{n\to\infty} F(ffy_n, gfy_n, t) = 1$, $\lim_{n\to\infty} L(ffy_n, gfy_n, t) = 0$. That is, $\lim_{n\to\infty} ffy_n = \lim_{n\to\infty} gfy_n = gu$. Using (3.5), we get

$F(fu, ffy_n, t) \geq F(gu, gfy_n, t)$, $L(fu, ffy_n, t) \leq L(gu, gfy_n, t)$.
Taking limit $n \to \infty$, we get
$F(fu, gu, t) \geq F(u, u, t) = 1$, $L(fu, gu, t) \leq L(u, u, t) = 0$.
This gives, $fu = gu$. In addition, $fgu = ffu = fgu = gfu$.
If $fu \neq ffu$, using (3.6), we get
$F(fu, ffu, t) > F(gu, ggu, t) = F(fu, ffu, t)$, $L(fu, ffu, t) < L(gu, ggu, t) = L(fu, ffu, t)$,
a contradiction. Hence, $fu = ffu = gfu$ and fu is a common fixed point of f and g.
Finally, suppose that f and g are $f-$compatible mappings. Then,
$\lim_{n\to\infty} F(fgz_n, ggz_n, t) = 1$, $\lim_{n\to\infty} L(fgz_n, ggz_n, t) = 0$. That is, $\lim_{n\to\infty} fgz_n = \lim_{n\to\infty} ggz_n$. Furthermore, $\lim_{n\to\infty} gfy_n = \lim_{n\to\infty} ggz_n = gu$. Therefore, $\lim_{n\to\infty} fgz_n = \lim_{n\to\infty} ggz_n = gu$.
Using (3.5), we get
$F(fu, fgz_n, t) \geq F(gu, ggz_n, t)$, $L(fu, fgz_n, t) \leq L(gu, ggz_n, t)$.
Taking limit $n \to \infty$, we get
$F(fu, gu, t) \geq F(gu, gu, t) = 1$, $L(fu, gu, t) \leq L(gu, gu, t) = 0$.
This gives $fu = gu$. Also, $fgu = ffu = fgu = gfu$.
If $fu \neq ffu$, using (3.6), we get
$F(fu, ffu, t) > F(gu, ggu, t) = F(fu, ffu, t)$, $L(fu, ffu, t) < L(gu, ggu, t) = L(fu, ffu, t)$,
a contradiction. Hence, $fu = ffu = gfu$ and fu is a common fixed point of f and g.

The following example illustrates Theorem 3.1 and Theorem 3.2.

*Example 1.*Let $(X, F, L, \star, \diamond)$ be an intutionistic menger metric space where $X = [1, 19]$. Define $f, g : X \to X$ by
$fx = 1$ if $x = 1$ or $x > 4$,
$fx = 5$ if $1 < x \leq 4$,
$gx = 1$ if $x = 1, gx = 10$ if $1 < x \leq 4$,
$gx = \frac{(x+1)}{2}$ if $x > 4$.
(i) Let $\{x_n\}$ be a sequence in X such that $x_n = 1$ and $\{y_n\}$ be sequences in X such that $y_n = 4 + \frac{1}{n}$ for each n. Also, f and g are compatible. Then, f and g satisfy all the conditions of Theorem 3.1 and have a unique common fixed point at $x = 1$.
(ii) Also, f and g are non compatible for $x_n = 4 + \frac{1}{n}$ and $\{y_n\}$ be sequences in X such that $y_n = 4 + \frac{1}{n}$ for each n. Then, f and g satisfy all the conditions of Theorem 3.2 and have a common fixed point at $x = 1$.

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