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Transmuted Topp-Leone Power Function Distribution: Theory and Application

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Abstract: In this paper, a new four-parameter (i.e. called the transmuted Topp-Leone power function (TTLPF) distribution) is proposed based on the transmuted Topp-Leone-G family. We derive moments, incomplete moments, probability weighted moments, quantile function, Bonferroni and Lorenz curves, and order statistics. The maximum likelihood and percentiles procedures are used to estimate the model parameters. A simulation study is carried out to evaluate and compare the performance of estimates in terms of their biases, standard errors and mean square errors. Eventually, we empirically prove the importance and flexibility of the new model in modeling two types of lifetime data.

Keywords: Maximum likelihood estimation, Order statistics, Power Function distribution, Transmuted Topp-Leone.

1 Introduction

Literature of lifetime distributions is rich with several continuous univariate distributions and develops rapidly. These distributions arise from different fields of practical studies, such as reliability experiments, clinical trials, infant mortality rate, medicine, finance and engineering applications, ... etc. Several researchers have helped expand known distributions to some new structural forms and develop corresponding probabilistic and statistical properties. In recent times, generated families of probability distributions have attracted the attention of several researchers. These families provide great flexibility in modeling various real data. Some of the generated families are: beta-G [1], Kumaraswamy-G [2], transformed-transformer [3], Weibull-G [4], Kumaraswamy Weibull-G [5], Elgarhy-G [6], exponentiated Weibull-G [7], additive Weibull-G [8], Topp- Leone-G (TL-G) [9], Type II half logistic-G [10], generalized additive Weibull-G [11], transmuted Topp-Leone-G (TTL-G) [12], inverse Weibull-G [13], power Lindley-G [14] and Type II generalized Topp-Leone-G [15].
The TL-G family proposed by [9] has cumulative distribution function (cdf) written by

$$F_{TL-G}(x;\alpha) = (G(x))^{\alpha}\,(2-G(x))^{\alpha} \quad \alpha > 0, \tag{1}$$

where α is the shape parameter. The probability density function (pdf) corresponding to (1) is given by

$$f_{TL-G}(x) = 2\alpha g(x)(1-G(x))(G(x))^{\alpha-1}(2-G(x))^{\alpha-1},$$

where $G(x)$ is the baseline distribution function, $g(x) = \acute{G}(x)$ is the baseline density function. The transmuted class defined by [16] with cdf and pdf is given by

$$F_{TC-G}(x;\lambda) = G(x)[1+\lambda-\lambda G(x)],\ |\lambda| \le 1,\ x \in R, \tag{2}$$

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and

$$f_{TC-G}(x;\lambda) = g(x)[1+\lambda-2\lambda G(x)],$$

respectively. Recently, [12] proposed the TTL-G family which is flexible because its hazard rate shapes can be increasing, decreasing, J, reversed-J, bathtub and upside-down bathtub. They defined the cdf of the TTL-G by inserting cdf (1) in cdf (2), as follows:

$$F_{TTL-G}(x;\alpha,\lambda) = (1+\lambda)\left\{1-[1-G(x)]^2\right\}^{\alpha} - \lambda\left\{1-[1-G(x)]^2\right\}^{2\alpha}. \tag{3}$$

The pdf corresponding to (3) is

$$f_{TTL-G}(x;\alpha,\lambda) = 2\alpha g(x)(1-G(x))\left\{1-[1-G(x)]^2\right\}^{\alpha-1}\left\{1+\lambda-2\lambda\left\{1-[1-G(x)]^2\right\}^{\alpha}\right\}. \tag{4}$$

One of the most useful models is the power function (PF) distribution. The PF distribution has the ability to model various types of data. It is the inverse form of Pareto distribution and it is a special case from beta distribution. The pdf and cdf of the PF distribution with scale parameter γ and shape parameter β are given, respectively, by

$$g(x;\gamma,\beta) = \beta\gamma^{-\beta}x^{\beta-1};\ \ 0<x<\gamma,\ \ \beta>0, \tag{5}$$

and

$$G(x;\gamma,\beta) = \gamma^{-\beta}x^{\beta}. \tag{6}$$

Several authors have addressed many extensions of the PF to increase the flexibility of the baseline model for example, the beta PF distribution [17], Kumaraswamy PF distribution [18], the Weibull PF distribution [19], exponentiated Weibull PF distribution [20], the exponentiated Kumaraswamy PF distribution [21], McDonald PF distribution [22] odd generalized exponential PF distribution [23] and exponentiatd generalized PF [24].

The present paper aims to introduce the TTLPF distribution, discuss its primary properties, estimate the model parameters and provide its applications to real data. The rest of the paper is outlined as follows: In Section 2, we define the TTLPF distribution and provide some plots of its pdf and hazard rate function (hrf). Mixture representations of the pdf and cdf as well as reliability analysis are derived in the same section. We derive some basic properties, including quantile function, probability weighted moments, order statistics, as well as ordinary and incomplete moments in Section 3. Maximum likelihood and percentiles estimation procedures of the model parameters are addressed in Section 4. In Section 5, simulation results to assess the performance of the proposed estimates are discussed. In Section 6, we provide the applications to real data to illustrate the importance of the proposed distribution. Conclusion is presented in Section 7.

2 TTLPE model

Taking the PF distribution as the baseline distribution function in (3), we obtain the TTLPF model with cdf (for $0<x<\gamma$), as follows:

$$F(x;\varpi) = (1+\lambda)\left\{1-\left[1-\left(\frac{x}{\gamma}\right)^{\beta}\right]^2\right\}^{\alpha} - \lambda\left\{1-\left[1-\left(\frac{x}{\gamma}\right)^{\beta}\right]^2\right\}^{2\alpha}, \tag{7}$$

where, $\gamma>0$, $\alpha>0$, $\beta>0$, $|\lambda|\le 1$, and $\varpi=(\gamma,\lambda,\alpha,\beta)$ is the set of parameters. The TTLPF density reduces to

$$f(x;\varpi) = \frac{2\alpha\beta x^{\beta-1}}{\gamma^{\beta}}\left(1-\left(\frac{x}{\gamma}\right)^{\beta}\right)\left\{1-\left[1-\left(\frac{x}{\gamma}\right)^{\beta}\right]^2\right\}^{\alpha-1}\left\{1+\lambda-2\lambda\left\{1-\left[1-\left(\frac{x}{\gamma}\right)^{\beta}\right]^2\right\}^{\alpha}\right\}. \tag{8}$$

Henceforth, we denote by X a random variable having pdf (8). For $\lambda=0$, the pdf (8) provides the Topp-Leone PF (TLPF) density as a new model. Some plots of the density (8) are displayed in Fig 1.They reveal that the pdf of X is quite flexible and can take symmetric, asymmetric and reversed J forms, ... etc. In summary, they reinforce the importance of the proposed model.

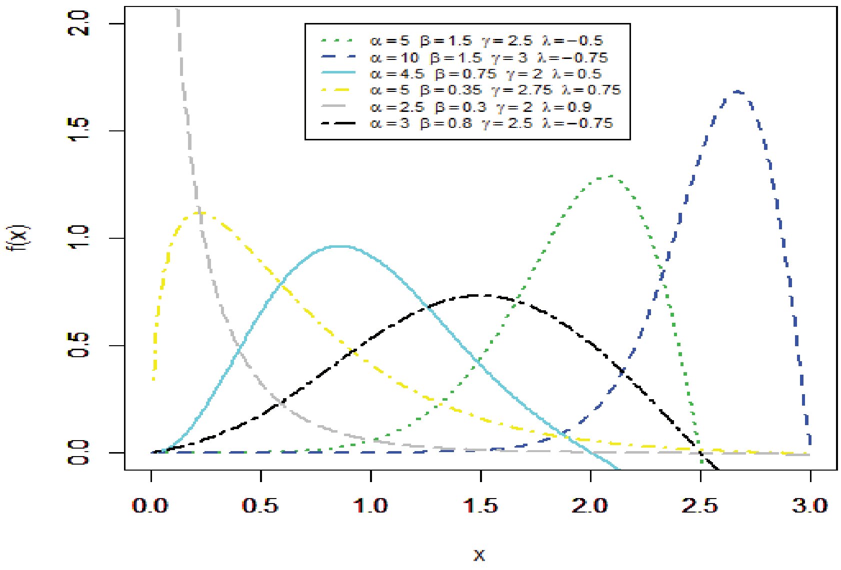

Fig. 1: Plots of density function of the TTLPF distribution for different values of parameter

2.1 Expansion for Density of the TTLPF Model

Here, useful expansions of the pdf and cdf of the TTLPF distribution are derived. Since, the pdf (8) can be rewritten, as follows:

$$f(x;\varpi)=(1+\lambda)\frac{2\alpha\beta x^{\beta-1}}{\gamma^{\beta}}\left[1-\left(\frac{x}{\gamma}\right)^{\beta}\right]\left\{1-\left[1-\left(\frac{x}{\gamma}\right)^{\beta}\right]^{2}\right\}^{\alpha-1}
-\frac{4\alpha\beta\lambda x^{\beta-1}}{\gamma^{\beta}}\left[1-\left(\frac{x}{\gamma}\right)^{\beta}\right]\left\{1-\left[1-\left(\frac{x}{\gamma}\right)^{\beta}\right]^{2}\right\}^{2\alpha-1}. \tag{9}$$

Using the binomial expansion in (9), pdf can be expressed as follows:

$$f(x;\varpi)=\Sigma_{j=0}^{\infty}\Sigma_{k=0}^{2j+1}(-1)^{j+k}\frac{2(1+\lambda)\Gamma(\alpha+1)}{j!\Gamma(\alpha-j)(k+1)}\binom{2j+1}{k}\frac{\beta(k+1)}{\gamma}\left(\frac{x}{\gamma}\right)^{\beta(k+1)-1}
-\Sigma_{j=0}^{\infty}\Sigma_{k=0}^{2j+1}(-1)^{j+k}\frac{4\lambda\Gamma(2\alpha+1)}{j!\Gamma(2\alpha-j)(k+1)}\binom{2j+1}{k}\frac{\beta(k+1)}{\gamma}\left(\frac{x}{\gamma}\right)^{\beta(k+1)-1}. \tag{10}$$

Then, the pdf (10) can be written, as follows:

$$f(x;\varpi)=\Sigma_{j=0}^{\infty}\Sigma_{k=0}^{2j+1}\eta_{j,k}\,w_{\beta(k+1)}(x), \tag{11}$$

$$\eta_{j,k}=(-1)^{j+k}\left[\frac{2(1+\lambda)\Gamma(\alpha+1)}{j!\Gamma(\alpha-j)(k+1)}-\frac{4\lambda\Gamma(2\alpha+1)}{j!\Gamma(2\alpha-j)(k+1)}\right]\binom{2j+1}{k},$$

where, $w_{\beta(k+1)}(x)$ denotes the pdf of the PF distribution with parameters $\beta(k+1)$ and γ. Furthermore, the associated cdf $F^s(x;\varpi)$, where s is an integer, using binomial expansion, can be expressed, as follows:

$$F^s(x;\varpi)=\Sigma_{l=0}^{s}(-1)^l\binom{s}{l}\lambda^l(1+\lambda)^{s-l}\left\{1-\left[1-\left(\frac{x}{\gamma}\right)^{\beta}\right]^2\right\}^{\alpha(l+s)}. \quad (12)$$

Adopting the binomial expansion more than one time, (12) will be

$$F^s(x;\varpi)=\Sigma_{l=0}^{s}\Sigma_{m=0}^{\infty}\Sigma_{q=0}^{2m}\,(-1)^{l+m+q}\binom{s}{l}\binom{\alpha(l+s)}{m}\binom{2m}{q}\lambda^l(1+\lambda)^{s-l}\left(\frac{x}{\gamma}\right)^{\beta q}.$$

Hence, $F^s(x;\varpi)$ can be written, as follows:

$$F^s(x;\varpi)=\Sigma_{l=0}^{s}\Sigma_{m=0}^{\infty}\Sigma_{q=0}^{2m}\,\omega_{l,m,q}\;G_{\beta q}(x) \quad (13)$$

$$\omega_{l,m,q}=(-1)^{l+m+q}\binom{s}{l}\binom{\alpha(l+s)}{m}\binom{2m}{q}\lambda^l(1+\lambda)^{s-l},$$

where, $G_{\beta q}(x)$ is the cdf of PF with parameters βq and γ.

2.2 Reliability Analysis

The expressions for the survival function and hrf are all established in this sub-section. These are particularly important to analyze survival data that involve the time associated to an event of interest such as the time until failure of a certain component and death of a patient or a disease relapse. The survival function of X is given by

$$S(x;\varpi)=1-\left\{(1+\lambda)\left\{1-\left[1-\left(\frac{x}{\gamma}\right)^{\beta}\right]^2\right\}^{\alpha}-\lambda\left\{1-\left[1-\left(\frac{x}{\gamma}\right)^{\beta}\right]^2\right\}^{2\alpha}\right\}.$$

The hrf of X is given by

$$h(x;\varpi)=\frac{2\alpha\beta x^{\beta-1}}{\gamma^{\beta}}\frac{\left(1-\left(\frac{x}{\gamma}\right)^{\beta}\right)\left\{1-\left[1-\left(\frac{x}{\gamma}\right)^{\beta}\right]^2\right\}^{\alpha-1}\left\{1+\lambda-2\lambda\left\{1-\left[1-\left(\frac{x}{\gamma}\right)^{\beta}\right]^2\right\}^{\alpha}\right\}}{\left\{1-(1+\lambda)\left\{1-\left[1-\left(\frac{x}{\gamma}\right)^{\beta}\right]^2\right\}^{\alpha}+\lambda\left\{1-\left[1-\left(\frac{x}{\gamma}\right)^{\beta}\right]^2\right\}^{2\alpha}\right\}}.$$

Some plots of the hrf are displayed in Fig 2. The hrf forms of X can be J and reversed-J, and bathtub shape. This non-monotone form is particularly important because of its great practical applicability.

3 Statistical Properties

Some important statistical properties of the TTLPF distribution, such as quantile, ordinary and incomplete moments, probability weighted moments, moment generating function, and order statistics are given.

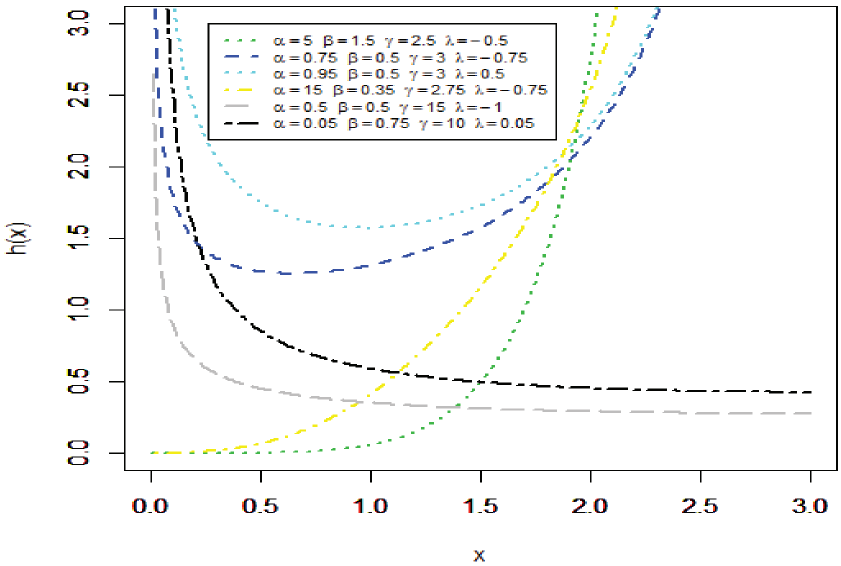

Fig. 2: Plots of the hrf of the TTLPF distribution for different values of parameter

3.1 Quantile function

The TTLPF distribution can be easily simulated by inverting (7), as follows: if $0 < p < 1$ follows uniform distribution, then

$$Q(p) = \gamma \left\{ 1 - \left\{ 1 - (2\lambda)^{-1/\alpha} \left[(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda p} \right]^{1/\alpha} \right\}^{0.5} \right\}^{1/\beta}. \quad (14)$$

This scheme is useful to generate the TTLPF random variates. Also, the median (m) of X is $m = Q(0.5)$.

3.2 Moments

Moments are necessary for any statistical analysis especially in applications. They can be used to study the most important characteristics distribution (e.g. dispersion, skewness, kurtosis and tendency). The r^{th} moment of the TTLPF is derived using pdf (11), as follows:

$$\acute{\mu}_r = \Sigma_{j=0}^{\infty} \Sigma_{k=0}^{2j+1} \eta_{j,k} \int_0^{\gamma} x^r \, w_{\beta(k+1)}(x) \, dx = \Sigma_{j=0}^{\infty} \Sigma_{k=0}^{2j+1} \eta_{j,k} \int_0^{\gamma} x^r \, \frac{\beta(k+1)}{\gamma} \left(\frac{x}{\gamma} \right)^{\beta(k+1)-1} dx.$$

After simplification, the r^{th} moment of the TTLPF is obtained, as follows:

$$\acute{\mu}_r = \Sigma_{j=0}^{\infty} \Sigma_{k=0}^{2j+1} \eta_{j,k} \frac{\gamma^r \beta(k+1)}{r + \beta(k+1)}, \; r = 1, 2, 3, \ldots$$

In particular, the mean and variance of the TTLPF distribution are given by

$$E(X)=\Sigma_{j=0}^{\infty}\Sigma_{k=0}^{2j+1}\,\eta_{j,k}\,\frac{\gamma\beta(k+1)}{\beta(k+1)+1}\,,$$

and

$$Var(X)=\Sigma_{j=0}^{\infty}\Sigma_{k=0}^{2j+1}\,\eta_{j,k}\,\frac{\gamma^{2}\beta(k+1)}{\beta(k+1)+2}-\left[\Sigma_{j=0}^{\infty}\Sigma_{k=0}^{2j+1}\,\eta_{j,k}\,\frac{\gamma\beta(k+1)}{\beta(k+1)+1}\right]^{2}.$$

Furthermore, the moment generating function of the TTLPF distribution is obtained, as follows:

$$M_{x}(t)=\Sigma_{r=0}^{\infty}\frac{t^{r}}{r!}E(X^{r})=\Sigma_{r,j=0}^{\infty}\Sigma_{k=0}^{2j+1}\,\eta_{j,k}\,\frac{(t\gamma)^{r}\beta(k+1)}{r!(r+\beta(k+1))}\,.$$

Table 1 presents the numerical values of some moments (order 1, 2, 3 and 4), the skewness (SK) and the kurtosis (KU) of X for selected values of the parameters (i) $(\gamma=2,\lambda=0.5,\alpha=0.5,\beta=0.5)$, (ii) $(\gamma=5,\lambda=-0.5,\alpha=1.5,\beta=0.5)$, (iii) $(\gamma=5,\lambda=0.8,\alpha=3,\beta=1.5)$, (iv) $(\gamma=0.5,\lambda=-0.8,\alpha=2,\beta=2)$, and (v) $(\gamma=4,\lambda=0.2,\alpha=4,\beta=2.5)$.

Table 1: Some moments, SK and KU of X for the selected parameters values

$\acute{\mu}_s$	(i)	(ii)	(iii)	(iv)	(v)
$\acute{\mu}_1$	0.121	1.394	2.878	0.370	3.153
$\acute{\mu}_2$	0.080	3.202	8.939	0.142	10.143
$\acute{\mu}_3$	0.080	9.103	29.448	0.056	33.195
$\acute{\mu}_4$	0.096	29.216	101.681	0.023	110.262
SK	3.243	0.799	-0.108	-0.742	-0.613
KU	14.863	2.818	2.583	3.401	3.139

3.3 Incomplete Moments

The answers to many important questions in economics require identifying the mean of the distribution and its shape. The a^{th} incomplete moment, say $\kappa_a(t)$, is defined by

$$\kappa_{a}(t)=\int_{-\infty}^{t}x^{a}f(x;\varpi)\;dx. \tag{15}$$

Hence, the a^{th} incomplete moment of TTLPF is derived using pdf (11), as follows:

$$\kappa_{a}(t)=\Sigma_{j=0}^{\infty}\Sigma_{k=0}^{2j+1}\,\eta_{j,k}\int_{0}^{t}x^{a}\frac{\beta(k+1)}{\gamma}\left(\frac{x}{\gamma}\right)^{\beta(k+1)-1}\;dx,$$

which leads to

$$\kappa_{a}(t)=\Sigma_{j=0}^{\infty}\Sigma_{k=0}^{2j+1}\,\eta_{j,k}\,\frac{t^{\beta(k+1)+a}}{\gamma^{\beta(k+1)}}\frac{\beta(k+1)}{\beta(k+1)+a}. \tag{16}$$

In particular, the first incomplete moment of the TTLPF distribution can be obtained by putting $a=1$ in (16), as follows

$$\kappa(t)=\Sigma_{j=0}^{\infty}\Sigma_{k=0}^{2j+1}\,\eta_{j,k}\,\frac{t^{\beta(k+1)+1}}{\gamma^{\beta(k+1)}}\frac{\beta(k+1)}{\beta(k+1)+1}.$$

Another application of the first incomplete moment refers to the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The Lorenz and Bonferroni curves are obtained, respectively, as follows:

$$LO(t) = \Sigma_{j=0}^{\infty}\Sigma_{k=0}^{2j+1}\ \eta_{j,k}\ \frac{t^{\beta(k+1)+1}}{\gamma^{\beta(k+1)}}\frac{\beta(k+1)}{\beta(k+1)+1}\left[\Sigma_{j=0}^{\infty}\Sigma_{k=0}^{2j+1}\ \eta_{j,k}\ \frac{\gamma\beta(k+1)}{\beta(k+1)+1}\right]^{-1},$$

and

$$BO(t) = \Sigma_{j=0}^{\infty}\Sigma_{k=0}^{2j+1}\ \eta_{j,k}\ \frac{t^{\beta(k+1)+1}}{\gamma^{\beta(k+1)}}\frac{\beta(k+1)}{\beta(k+1)+1}\left[F(t;\varpi)\ \Sigma_{j=0}^{\infty}\Sigma_{k=0}^{2j+1}\ \eta_{j,k}\ \frac{\gamma\beta(k+1)}{\beta(k+1)+1}\right]^{-1}.$$

3.4 Probability Weighted Moments

The probability weighted moments (PWM) can be used to derive estimators of the parameters and quantiles of generalized distributions. The PWM of X is defined by

$$\upsilon_{r,s} = E\{X^r[F(x)]^s\} = \int_{-\infty}^{\infty} x^r[F(x)]^s f(x)\ dx, \tag{17}$$

where s and r are positive integers. Inserting pdf (11) and cdf (13) in (17), then the PWM of the TTLPF distribution is obtained, as follows:

$$\upsilon_{r,s} = \Sigma_{l=0}^{s}\Sigma_{m,j=0}^{\infty}\Sigma_{q=0}^{2m}\Sigma_{k=0}^{2j+1}\ \omega_{l,m,q}\ \eta_{j,k}\ \frac{\beta(k+1)}{\gamma}\int_0^{\gamma} x^r\left(\frac{x}{\gamma}\right)^{\beta(q+k+1)-1}\ dx.$$

Therefore, the PWM of TTLPF distribution is given by

$$\upsilon_{r,s} = \Sigma_{l=0}^{s}\Sigma_{m,j=0}^{\infty}\Sigma_{q=0}^{2m}\Sigma_{k=0}^{2j+1}\ \omega_{l,m,q}\ \eta_{j,k}\ \frac{\gamma^r\beta(k+1)}{(r+\beta(q+k+1))}.$$

3.5 Order Statistics

A closed form expression for the pdf of the u^{th} order statistics of the TTLPF distribution is derived. Let $X_{(1)} < X_{(2)} < ... < X_{(n)}$ be the order statistics of a random sample $X_1, X_2, ..., X_n$ of size n from TTLPF distribution. It is known that the pdf of the u^{th} order statistic is defined by

$$f_{(u)}(x) = \frac{1}{B(u,n-u+1)}[F(x)]^{u-1}[1-F(x)]^{n-u} f(x).$$

Using the binomial series expansion for $[1-F(x)]^{n-u}$, $f_{(u)}(x)$ can be written as

$$f_{(u)}(x) = \frac{1}{B(u,n-u+1)}\Sigma_{d=0}^{n-u}\binom{n-u}{d}[F(x)]^{d+u-1} f(x). \tag{18}$$

Hence, the pdf of the u^{th} order statistic from the TTLPF distribution is obtained by inserting (11) and (12), but with $d+u-1$ instead of s in (18), so we obtain

$$f_{(u)}(x) = \Sigma_{d=0}^{n-r}\Sigma_{j,m=0}^{\infty}\Sigma_{k=0}^{2j+1}\Sigma_{l=0}^{d+u-1}\Sigma_{q=0}^{2m}\ \eta_{j,k}\ \phi_{u,d,l,q,m}\frac{\beta(k+1)}{\gamma}\left(\frac{x}{\gamma}\right)^{\beta(k+q+1)-1}, \tag{19}$$

where,

$$\phi_{u,d,l,q,m} = \frac{(-1)^{l+m+q+d}}{B(u,n-u+1)} \binom{n-u}{d} \binom{\alpha(d+u+l-1)}{m} \binom{2m}{q} \binom{d+u-1}{l} \lambda^{l} (1+\lambda)^{d+u-l-1}.$$

The pdf of the first and largest order statistics can be obtained by putting $u = 1$ and $u = n$ in (19), so at $u = 1$

$$f_{(1)}(x) = \Sigma_{d=0}^{n-1} \Sigma_{j,m=0}^{\infty} \Sigma_{k=0}^{2j+1} \Sigma_{l=0}^{d} \Sigma_{q=0}^{2m} \, \eta_{j,k} \, \phi_{1,d,l,q,m} \frac{\beta(k+1)}{\gamma} \left(\frac{x}{\gamma}\right)^{\beta(k+q+1)-1},$$

where,

$$\phi_{1,d,l,q,m} = (-1)^{l+m+q+d} \, n \binom{n-1}{d} \binom{\alpha(d+l)}{m} \binom{2m}{q} \binom{d}{l} \lambda^{l} (1+\lambda)^{d-l}.$$

At $u = n$

$$f_{(n)}(x) = \Sigma_{j,m=0}^{\infty} \Sigma_{k=0}^{2j+1} \Sigma_{l=0}^{d+n-1} \Sigma_{q=0}^{2m} \, \eta_{j,k} \, \phi_{n,d,l,q,m} \frac{\beta(k+1)}{\gamma} \left(\frac{x}{\gamma}\right)^{\beta(k+q+1)-1},$$

and,

$$\phi_{n,d,l,q,m} = (-1)^{l+m+q+d} \, n \binom{\alpha(d+n+l-1)}{m} \binom{2m}{q} \binom{d+n-1}{l} \lambda^{l} (1+\lambda)^{d+n-l-1}.$$

4 Parameter Estimation

In this section, the maximum likelihood (ML) and percentiles methods of estimation for the population parameters of the TTLPF distribution are addressed.

4.1 Maximum Likelihood Estimator

We discuss the estimation of the population parameters of the TTLPF distribution using the ML method. Let $x_1, x_2, ..., x_n$ be the observed values from the TTLPF distribution with set of parameters $(\varpi = \gamma, \lambda, \alpha, \beta)^T$. The total log-likelihood function, denoted by $\ln L$, based on complete sample for the vector of parameters ϖ, can be expressed as

$$\ln L = n\ln 2 + n\ln\alpha + n\ln\beta - n\beta\ln\gamma + (\beta-1)\Sigma_{i=1}^{n}\ln(x_i) + \Sigma_{i=1}^{n}\ln b_i + (\alpha-1)\Sigma_{i=1}^{n}\ln(1-b_i^2)$$
$$+ \Sigma_{i=1}^{n}\ln\left[1+\lambda-2\lambda(1-b_i^2)^{\alpha}\right],$$

where, $b_i = \left[1-\left(\frac{x_i}{\gamma}\right)^{\beta}\right]$. It is known that the estimator of γ is the sample maxima, i.e. $\hat{\gamma} = X_{(n)}$. The partial derivatives of the log-likelihood function with respect to λ, β and α components of the score vector $U_L = (U_\alpha, U_\lambda, U_\beta)^T$ can be obtained, as follows:

$$U_\lambda = \Sigma_{i=1}^{n} \frac{\left[1-2(1-b_i^2)^{\alpha}\right]}{\left[1+\lambda-2\lambda\left(1-b_i^2\right)^{\alpha}\right]},$$

$$U_\beta = \frac{n}{\beta} - n\ln\gamma + \Sigma_{i=1}^{n}\ln(x_i) - \Sigma_{i=1}^{n}\left(\frac{1}{b_i}\right)\left(\frac{x_i}{\gamma}\right)^{\beta}\ln\left(\frac{x_i}{\gamma}\right) - 2(\alpha-1)\Sigma_{i=1}^{n}\frac{b_i}{(1-b_i^2)}\left(\frac{x_i}{\gamma}\right)^{\beta}\ln\left(\frac{x_i}{\gamma}\right)$$
$$- \Sigma_{i=1}^{n}\frac{4\lambda\alpha(b_i)\left(1-b_i^2\right)^{\alpha-1}}{\left[1+\lambda-2\lambda\left(1-b_i^2\right)^{\alpha}\right]}\left(\frac{x_i}{\gamma}\right)^{\beta}\ln\left(\frac{x_i}{\gamma}\right),$$

and,

$$U_\alpha = \frac{n}{\alpha} + \Sigma_{i=1}^{n}\ln(1-b_i^2) - 2\lambda\Sigma_{i=1}^{n}\frac{\left(1-b_i^2\right)^{\alpha}\ln\left(1-b_i^2\right)}{\left[1+\lambda-2\lambda\left(1-b_i^2\right)^{\alpha}\right]}.$$

The non-linear equations U_α, U_λ and U_β are solved numerically via iterative technique, to get the ML estimators of α, λ and β.

4.2 Percentiles Estimator

Here, the percentiles method (PM) will be used to obtain the percentiles estimators (PE) of α, λ and β denoted by $\bar{\alpha}$, $\bar{\lambda}$ and $\bar{\beta}$ of the TTLPF distribution. Let $X_1, X_2, ..., X_n$ be a random sample from the TTLPF distribution and $X_{(i)}$ denote the i^{th} order statistic, i.e. $X_{(1)} < X_{(2)} < ... < X_{(n)}$. Based on the quantile function (14), the PE estimates of population parameters are obtained by minimizing the following equation

$$\Sigma_{i=1}^{n}\left\{x_{(i)}-\left(\gamma\left\{1-\left\{1-(2\lambda)^{-1/\alpha}\left[(1+\lambda)-\sqrt{(1+\lambda)^2-4\lambda p_i}\right]^{1/\alpha}\right\}^{0.5}\right\}^{1/\beta}\right)\right\}^2,$$

with respect to λ, α and β. The formula $p_i = \left[\frac{i}{(n+1)}\right]$ is an estimate of $F(x_{(i)};\alpha,\lambda,\beta)$.

5 Simulation Study

As mentioned in the previous section, the derived expressions for the estimators are too complicated to be studied analytically. Thus, a numerical study will be conducted to obtain and compare the proposed estimates via MathCAD 14. The performances of the different estimates are compared in terms of their mean square errors (MSEs) and standard errors (SEs). 1000 random samples of sizes 10, 20, 30, 40, 50 and 100 are generated from TTLPF distribution. Assuming the scale parameter γ is known and four sets of parameters are considered as Case I $\equiv (\alpha = 0.5, \beta = 0.25, \lambda = 0.25)$, Case II $\equiv (\alpha = 0.75, \beta = 0.25, \lambda = 1.5)$ Case III $\equiv (\alpha = 0.75, \beta = 0.25, \lambda = 0.5)$ and Case IV $\equiv (\alpha = 1, \beta = 0.25, \lambda = 0.75)$. ML estimator and PE of the population parameters are obtained. We compute MSEs and SEs of estimates and the results are listed in Tables 2 and 3. We notice the following concerning the performance of estimates:

–For both methods of estimation, it is clear that MSEs and SEs decrease as sample size increases for all cases of parameters (see Tables 2-3).
–The MSEs of ML estimates are smaller than the corresponding for PE in almost all cases. The MSEs of ML and PE estimates for unknown parameters decrease as the sample size increases for different values of parameters (see Tables 2-3).
–For fixed value of β and as the value of α and λ increases, the MSEs and SEs of estimates based on both methods increase (see Tables 2-3).
–For fixed value of β and as the value of α and λ decrease, the MSEs and SEs for estimates based on both methods decrease (see Tables 2-3).
–The SEs for ML estimates for parameters values are smaller than the corresponding of PE in almost all cases (see Tables 2-3).
–The MSEs of the ML estimates in Case III have the smallest values corresponding to the MSEs for the other cases of parameters for different sample sizes (see Tables 2-3).
–The MSEs of the PE estimates in Case I have the smallest values corresponding to the MSEs for the other cases of parameters for different sample sizes (see Tables 2-3).

6 Applications

In this section, the utility of TTLPF distribution is demonstrated with the help of the following two data sets.

Data Set 1: It is discussed in [22]. The first data contain 40 observations and are listed as: 1.6, 2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.0, 5.1, 5.3, 5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3,6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3,7.3, 7.7, 7.7, 7.8, 7.9, 8.0, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.0

Table 2: MSEs and SEs of estimates for Case I and Case II of parameters for TTLPF distribution

n	Method	Properties	Case I			Case II		
			$\alpha=0.5$	$\beta=0.25$	$\lambda=0.25$	$\alpha=0.75$	$\beta=0.25$	λ=1.5
10	ML	MSE	0.2389	1.5354	0.2822	0.5239	3.2985	3.0076
		SE	0.0146	0.1585	0.1512	0.0450	0.2271	0.3080
	PM	MSE	0.2479	1.9546	0.0664	0.5566	4.2992	2.2434
		SE	0.1213*	0.1735	0.0227	1.4497*	0.2395	0.0253
20	ML	MSE	0.2360	0.8881	0.1121	0.5103	1.9201	2.2547
		SE	0.0200	0.0744	0.0628	0.0351	0.1087	0.0560
	PM	MSE	0.2471	1.1090	0.0641	0.5537	2.5270	2.2392
		SE	0.1944*	0.0789	0.0122	0.1408*	0.1075	0.0146
30	ML	MSE	0.2266	0.6298	0.0625	0.4953	1.5622	2.2393
		SE	0.9115*	0.0480	0.5241*	0.0183	0.0727	0.8525*
	PM	MSE	0.2478	0.8173	0.0636	0.5535	1.9952	2.2304
		SE	0.0787*	0.0513	0.9887*	0.1436*	0.0720	0.0215
40	ML	MSE	0.2263	0.5474	0.0630	0.4794	1.2415	2.2201
		SE	0.0156	0.0382	0.0183	0.0200	0.0589	0.0136
	PM	MSE	0.2472	0.6692	0.0628	0.5532	1.6327	2.2282
		SE	0.0598*	0.0402	0.2922*	0.0838*	0.0560	0.4050*
50	ML	MSE	0.2226	0.5216	0.0624	0.4633	1.1661	2.2143
		SE	0.0132	0.0323	0.6506*	0.0184	0.0480	0.9557*
	PM	MSE	0.2464	0.6152	0.0627	0.5527	1.4968	2.2224
		SE	0.0765*	0.0335	0.5105*	0.0542*	0.0457	0.1843*
100	ML	MSE	0.2211	0.3274	0.0621	0.4505	0.8098	2.0249
		SE	0.8705*	0.0177	0.5091*	0.0125	0.0272	0.6116*
	PM	MSE	0.2460	0.3812	0.0623	0.5514	1.0392	2.1470
		SE	0.0791*	0.0177	0.4796*	0.0542*	0.0246	0.3995*

* Indicate that the value multiply 10^{-3}

Data Set 2: This data set consists of 63 observations of the strengths of 1.5 cm glass fibers, originally obtained by workers at the UK National Physical Laboratory. The data have been discussed recently by [25],[26] and [27]. The data are: 0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28,1.29,1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50,1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63,1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68,1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82,1.84, 1.84, 2.00, 2.01, 2.24

We obtain the ML estimates of the parameters of TTLPF model for the above-mentioned two data sets. We compare the fitting of the following models: the TTL exponential (TTLEx), TTL BurrXII (TTLBxii), TTL Lomax (TTLLo), TTL Rayleigh (TTLR), and generalized exponential (GE). We estimate the unknown parameters of the distributions using the ML method. We compute certain information, such as Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (AICc), Hannan-Quinn information criterion (HQIC) and minus log-likelihood (-LL) in case of each fitted models. The smaller values of these information criterions are, the better the fit is. The numerical values of the ML estimates of the fitted models are listed in Table 4. In Table 5, we compare the fits of the TTLPF model with the TTLEx, TTLBxii, TTLLo, TTLR and GE distributions.

Furthermore, for graphical comparison, we obtain the cdf and pdf plots corresponding to each model as given in Fig. 3 and Fig. 4 for the first and second data.

Table 5, Fig. 3 and Fig. 4 indicate that TTLPF model gives relatively better fit to each of the data sets compared to existing models.

7 Concluding Remarks

In this paper, we investigated a new four-parameter model, i.e. the transmuted Topp-Leone power function distribution. We presented some statistical properties of the TTLPF distribution, including ordinary moment, moment generating, order statistics, quantile functions as well as Bonferroni and Lorenz curves. We discuss the parameter estimation by maximum likelihood and percentiles methods. We provided simulation study to evaluate the maximum likelihood and percentiles

Table 3: MSEs and SEs of estimates for Case III and Case IV of parameters for TTLPF distribution

n	Method	Properties	Case III			Case IV		
			$\alpha = 0.75$	$\beta = 0.25$	$\lambda = 0.5$	$\alpha = 1$	$\beta = 0.25$	λ=0.75
10	ML	MSE	0.5491	2.9624	0.2828	0.9830	4.2074	3.6580
		SE	0.0511	0.2029	0.0679	0.0806	0.2208	0.5731
	PM	MSE	0.5580	3.8189	0.2525	0.9888	5.4018	0.5699
		SE	0.1887*	0.2174	0.0315	0.1937	0.2248	0.0285
20	ML	MSE	0.5397	1.7289	0.6111	0.9303	2.7190	1.0413
		SE	0.0355	0.0954	0.1407	0.0503	0.1216	0.1716
	PM	MSE	0.5564	2.1807	0.2507	0.9885	3.5610	0.5638
		SE	0.0885*	0.0992	0.7077*	0.1443*	0.1140	0.0253
30	ML	MSE	0.5196	1.3875	0.2502	0.9106	2.2201	0.5662
		SE	0.0193	0.0653	0.0155	0.0307	0.0875	0.0376
	PM	MSE	0.5552	1.7366	0.2494	0.9880	2.8292	0.5633
		SE	0.0873*	0.0655	0.7835*	0.0782*	0.0778	0.0102
40	ML	MSE	0.5220	1.1924	0.2474	0.8959	1.9540	0.5610
		SE	0.0176	0.0505	0.0146	0.0368	0.0705	0.0678
	PM	MSE	0.5536	1.4368	0.2490	0.9800	2.4822	0.5611
		SE	0.0593*	0.0511	0.1587*	0.1612*	0.0592	0.0202
50	ML	MSE	0.5198	1.1200	0.2470	0.8907	1.8121	0.5606
		SE	0.0143	0.0426	0.0110	0.0332	0.0594	0.0428
	PM	MSE	0.5514	1.3143	0.2490	0.9759	2.2677	0.5608
		SE	0.1292*	0.0426	0.9042*	0.0758*	0.0481	0.9095*
100	ML	MSE	0.5168	0.8369	0.2451	0.8782	1.2915	0.5305
		SE	0.0128	0.0248	0.0103	0.0255	0.0365	0.0185
	PM	MSE	0.5473	0.9537	0.2488	0.9693	1.6043	0.5524
		SE	0.0513*	0.0229	0.6490*	0.0958*	0.0269	0.8264*

* Indicate that the value multiply 10^{-3}

Table 4: The ML estimates of models for both data

	TTLPF	TTLEx	TTLLBxii	TTLLo	TTLR	GE
Data1	$\hat{\alpha} = 0.15257$ $\hat{\lambda} = 0.0373$ $\hat{\beta} = 16.461$ $\hat{\theta} = 9.132$ $\hat{\gamma} = 9$	$\hat{\alpha} = 9.5146$ $\hat{\lambda} = 0.000001$ $\hat{\theta} = 0.2249$	$\hat{\alpha} = 171.4979$ $\hat{\lambda} = 0.000001$ $\hat{\beta} = 1.771$ $\hat{\theta} = 0.7586$	$\hat{\alpha} = 171.4979$ $\hat{\lambda} = 0.0000001$ $\hat{\beta} = 1000.756$ $\hat{\theta} = 0.00128$	$\hat{\alpha} = 2.3847$ $\hat{\lambda} = 0.0000001$ $\hat{\theta} = 0.0188$	$\hat{\alpha} = 9.5146$ $\hat{\theta} = 0.4498$
Data2	$\hat{\alpha} = 2.0219$ $\hat{\lambda} = 0.9084$ $\hat{\beta} = 2.7178$ $\hat{\theta} = 2.289$ $\hat{\gamma} = 2.24$	$\hat{\alpha} = 31.349$ $\hat{\lambda} = 0.00001$ $\hat{\theta} = 1.3057$	$\hat{\alpha} = 7.5028$ $\hat{\lambda} = 0.000001$ $\hat{\beta} = 2.637$ $\hat{\theta} = 0.9412$	$\hat{\alpha} = 30.149$ $\hat{\lambda} = 0.000001$ $\hat{\beta} = 400.333$ $\hat{\theta} = 0.0032$	$\hat{\alpha} = 5.486$ $\hat{\lambda} = 0.000001$ $\hat{\theta} = 0.4869$	$\hat{\alpha} = 31.349$ $\hat{\theta} = 2.646$

estimates of the model parameters. Applications to real data indicated that the TTLPF distribution provides a good fit and can be used as a competitive model to fit real data.

Conflict of Interest

The authors declare that they have no conflict of interest.

Table 5: Information criterion for both data

	Distribution	-LL	AIC	AICc	BIC	HQIC
Data1	TTLPF	77.450	162.906	164.049	169.661	165.346
	TTLE	90.143	186.285	186.952	191.352	188.173
	TTLBxii	98.745	205.489	206.632	212.245	207.932
	TTLLo	90.148	188.297	189.439	195.052	190.739
	TTLR	85.796	177.593	178.669	182.425	179.4245
	GE	90.143	184.285	184.610	187.663	185.507
Data2	TTLPF	16.008	40.016	40.706	48.590	43.389
	TTLE	31.383	68.767	69.174	75.196	71.296
	TTLBxii	35.553	79.107	79.796	87.679	82.478
	TTLLo	31.431	70.862	71.552	79.435	74.234
	TTLR	23.929	53.857	54.264	60.287	56.386
	GE	31.383	66.767	66.967	71.053	68.453

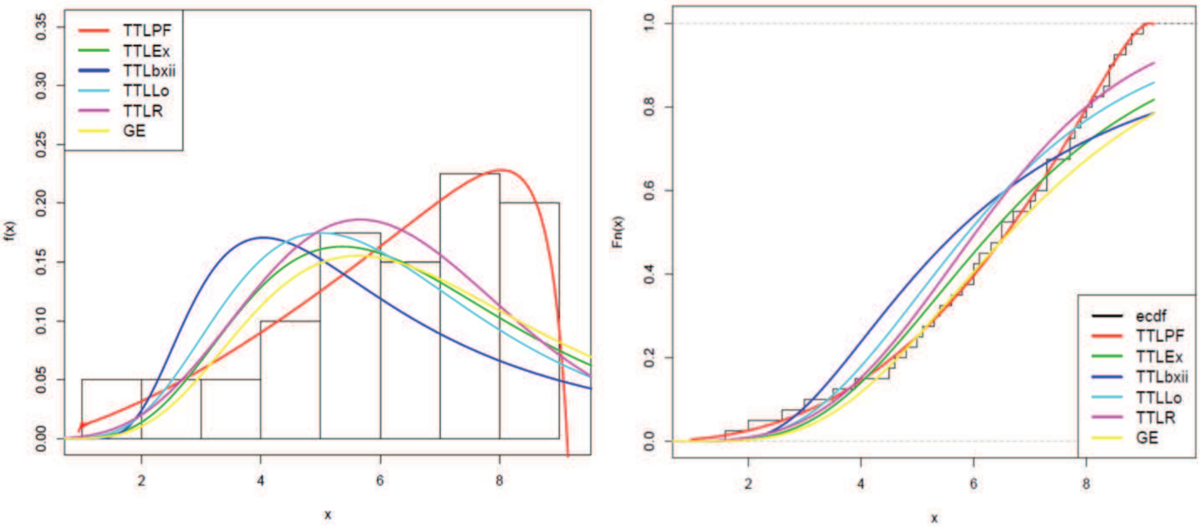

Fig. 3: Fitted densities and cdf of fitted models for first data

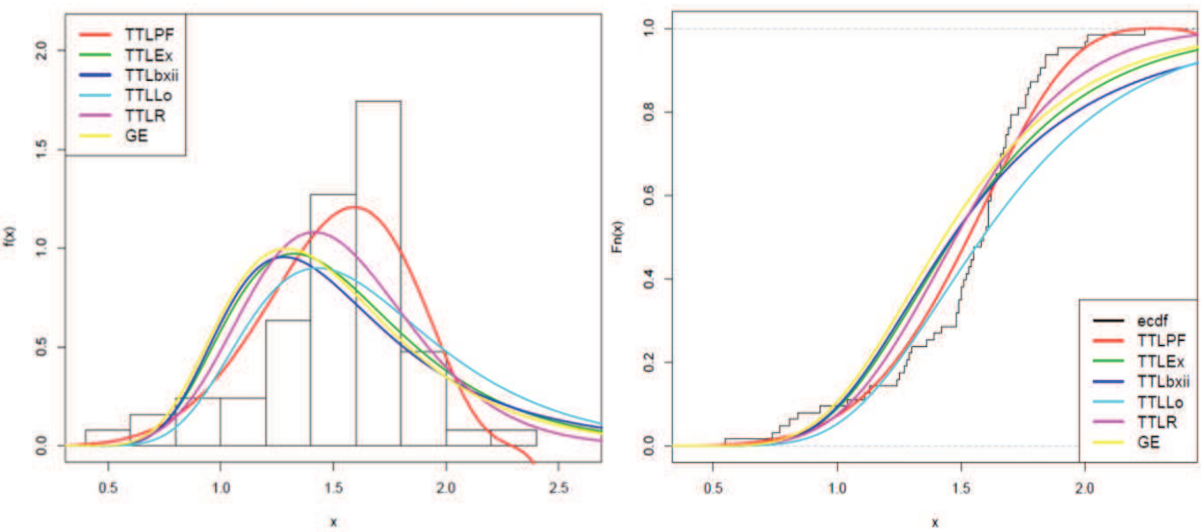

Fig. 4: Fitted densities and cdf of fitted models for second data

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