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Statistical Inference for Weibull-Exponential Distribution Using Adaptive Type-II Progressive Censoring

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Abstract: The main purpose of the paper is to investigate the classical and Bayesian estimation for the unknown parameters of the Weibull-exponential distribution (WED) based on Adaptive Type-II progressive censoring (A-II-PRO-C). Maximum likelihood (ML), percentile bootstrap and Bayes methods are used to estimate the unknown parameters of WED. Moreover, the approximate confidence intervals (ACIs) and asymptotic variance-covariance matrix have been obtained. Markov Chain Monte Carlo (MCMC) technique is applied to estimate the unknown parameters of WED. The Metropolis–Hastings algorithm is the MCMC method that's to generate samples from the posterior density functions. An example is applied to different estimation methods. Finally, a Monte Carlo simulation study is carried out to compare the performance of the different methods.

Keywords: Adaptive Type-II progressive censoring; Maximum likelihood; Percentile bootstrap; MCMC approach.

1 Introduction

The WED is a distribution generated from the Weibull G-family of distributions by considering baseline distribution is the exponential distribution. For more details about WED and its properties see Bourguignon et al. [2] and Oguntunde et al. [13]. If a random variable X has a WED, with probability density function (pdf) and cumulative distribution function (cdf), respectively given by

$$f\ (x) = \alpha\gamma\beta e^{\gamma x} \left(e^{\gamma x}-1\right)^{\beta-1} e^{-\alpha\left(e^{\gamma x}-1\right)^{\beta}} \ ,x>0;\alpha,\gamma,\beta>0, \tag{1}$$

and

$$F\ (x) = 1-e^{-\alpha\left(e^{\gamma x}-1\right)^{\beta}} ,x>0;\alpha,\gamma,\beta>0, \tag{2}$$

where β is the shape parameter , α and γ are the scale parameters. Also, according to Oguntunde et al. [13], the WED is useful as a life-testing model for data indicating unimodal failure rates and it is more flexible than the exponential distribution. The most common censoring schemes are the conventional Type-I censoring, where the experiment is terminated at a prefixed time T, and the conventional Type- II censoring, where the experiment is terminated after the prefixed failure r occurs. But, failure may not occur until time T or the experiment alot of time after failure r occurs. So, Balakrishnan and Sandhu [1] introduced a more general type of censoring scheme called a progressive Type-II censoring (PRO-II-C). It can be described as follows: suppose that n independent and identical units are put on the life test. When the first and second failure occur at time x_1, R_1 and/or x_2, R_2 , respectively, the surviving units are randomly selected and withdrawn from the test. The test is continued until the $(m-1)^{th}$ failure occurs at time x_{m-1}, R_{m-1}, and the surviving units are randomly selected and withdrawn from the test. This test is terminated when the $m^{\ th}$ failure occurs at time x_m and the remaining surviving units $R_m = n-m-(R_1+R_2+...+R_{m-1})$ are all withdrawn from the test. The disadvantage of PRO-II-C scheme is that the effective sample size m is random and it may be a very small number (even equal to zero), so that usual statistical inference procedures may not be applicable or may be have low efficiency. Therefore, Ng et al.[12] suggested an A-II-PRO-C scheme in which the effective sample size m is fixed in advance and the progressive

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censoring scheme $(R_1, R_2, ..., R_m)$ is allow to depend on the failure times. For more details about A-II-PRO-C scheme, readers may refer to Ng et al. [12] and Cramer and Iliopoulos [3]. Many authors have discussed inference under an A-II-PRO-C for different lifetime distributions, see, for example, Mahmoud et al. [10] who estimated the generalized Pareto under an A-II-PRO-C, EL-Sagheer et al. [6] discussed statistical inferences for new Weibull-Pareto distribution under an A-II-PRO-C data and Sobhi and Soliman [14] discussed the problem of estimating parameters of the exponentiated Weibull distribution with A-II-PRO-C schemes. The rest of this paper is organized as follow: Section 2 gives a description of an A-II-PRO-C scheme. Section 3 presents the likelihood inference and information matrix used to estimate the unknown parameters under consideration and the confidence intervals for each parameter. Section 4 introduces a parametric percentile bootstrap procedures to construct the boot CIs for the unknown parameters. Section 5 deals with the Bayesian estimation computed by using MCMC technique. A numerical example is developed to explain the theoretical results in Section 6. Monte Carlo simulation results of different estimation methods are presented in Section 7. The conclusion is reported in Section 8.

2 Adaptive Type-II Progressive Censoring

A mixture of Type-I censoring and Type-II progressive censoring schemes, called an A-II-PRO-C scheme. It can be described as follows: Suppose that n independent units are placed in the life test simultaneously at the initial time $t_0 = 0$ and let $X_1, X_2, ..., X_n$ be their corresponding lifetimes. suppose a time T, which is an ideal total test time, but we may allow the experiment to run over time T. If the m^{th} progressively censored observed failure occurs before time T (i.e. $X_{m:m:n} < T$), we will have a usual PRO-II-C with the prefixed progressive censoring scheme $(R_1, R_2, ..., R_m)$ and the experiment stops at the time $X_{m:m:n}$. Otherwise, once the experimental time passes time T but the number of observed failures has not reached m, then we adopt the number of units progressively and they are removed from the experiment upon failure by setting $R_{J+1}, R_{J+2}, ..., R_{m-1} = 0$ and $R_m = n - m - \sum_{i=1}^{J} R_i$. Suppose J is the number of failures observed before time T, i.e.

$$X_{J:m:n} < T < X_{J+1:m:n}, \quad J = 0, 1, ..., m,$$

Thus the effectively applied scheme is $R_1, R_2, ..., R_J, 0, 0, ..., 0, R_m$. This formula leads to terminate the experiment as soon as possible if the $(J+1)^{th}$ failure time is greater than T for $J+1 < m$. An A-II-PRO-C can be reduced to well-known types of censoring scheme as the following extreme cases:

1. If $T \longrightarrow \infty$, then A-II-PRO-C reduces to the PRO-II-C.
2. If $T = 0$, then A-II-PRO-C reduces to the Type-II censoring.

3 Maximum Likelihood Estimation

Let $x_{1:m:n}, x_{2:m:n}, \ldots, x_{m:m:n}$ be an A-II-PRO-C sample from WED(α, γ, β), with censoring scheme R. Then the likelihood function based on the A-II-PRO-C for given $J = j$ is given by:

$$L(\alpha,\gamma,\beta|J=j) = d_j \prod_{i=1}^{m} [f(x_{i;m,n})] \prod_{i=1}^{j} [1 - F(x_{i;m,n})]^{R_i} \times [1 - F(x_{m;m,n})]^{n-m-\sum_{i=1}^{j} R_i},$$

where

$$d_j = \prod_{i=1}^{m} \left[n - i + 1 - \sum_{k=1}^{\max\{i-1,j\}} R_k \right].$$

From Equations (1)-(2), $L(\alpha, \gamma, \beta)$ can be written as

$$L(\alpha,\gamma,\beta|J=j) = d_j \left[\prod_{i=1}^{m} \alpha\gamma\beta e^{\gamma x_i} (e^{\gamma x_i} - 1)^{\beta-1} e^{-\alpha(e^{\gamma x_i}-1)^{\beta}} \right] \times \left[\prod_{i=1}^{j} \left[e^{-\alpha(e^{\gamma x_i}-1)^{\beta}} \right]^{R_i} \right] \times \left[e^{-\alpha(e^{\gamma x_m}-1)^{\beta}} \right]^{n-m-\sum\limits_{i=1}^{j} R_i}, \tag{3}$$

where x_i is used instead of $x_{i:m:n}$.Then the log-likelihood function $\ell(\alpha, \gamma, \beta)$ can be written as

$$\ell(\alpha,\gamma,\beta) \propto m(\log\alpha + \log\gamma + \log\beta) + \gamma \sum_{i=1}^{m} x_i + (\beta - 1) \sum_{i=1}^{m} \log(e^{\gamma x_i} - 1)$$
$$-\alpha \sum_{i=1}^{m} (e^{\gamma x_i} - 1)^{\beta} - \alpha \sum_{i=1}^{j} R_i (e^{\gamma x_i} - 1)^{\beta} - \alpha (n - m - \sum_{i=1}^{j} R_i)(e^{\gamma x_m} - 1)^{\beta}, \tag{4}$$

Equating the partial derivatives of the log-likelihood function with respect to α,γ and β ,respectively, by zero, gives:

$$\frac{\partial \ell}{\partial \alpha}=\frac{m}{\alpha}-\sum_{i=1}^{m}(e^{\gamma x_i}-1)^{\beta}-\sum_{i=1}^{j}R_i(e^{\gamma x_i}-1)^{\beta}-(n-m-\sum_{i=1}^{j}R_i)(e^{\gamma x_m}-1)^{\beta}=0, \tag{5}$$

$$\frac{\partial \ell}{\partial \gamma}=\frac{m}{\gamma}+\sum_{i=1}^{m}x_i+(\beta-1)\sum_{i=1}^{m}\frac{x_i e^{\gamma x_i}}{(e^{\gamma x_i}-1)}-\alpha\beta\sum_{i=1}^{m}x_i e^{\gamma x_i}(e^{\gamma x_i}-1)^{\beta-1}$$
$$-\alpha\beta\sum_{i=1}^{j}R_i x_i e^{\gamma x_i}(e^{\gamma x_i}-1)^{\beta-1}-\alpha\beta x_m e^{\gamma x_m}(n-m-\sum_{i=1}^{j}R_i)(e^{\gamma x_m}-1)^{\beta-1}=0, \tag{6}$$

and

$$\frac{\partial \ell}{\partial \beta}=\frac{m}{\beta}+\sum_{i=1}^{m}\log(e^{\gamma x_i}-1)-\alpha\sum_{i=1}^{m}(e^{\gamma x_i}-1)^{\beta}\log(e^{\gamma x_i}-1)$$
$$-\alpha\sum_{i=1}^{j}R_i(e^{\gamma x_i}-1)^{\beta}\log(e^{\gamma x_i}-1)-\alpha(n-m-\sum_{i=1}^{j}R_i)(e^{\gamma x_m}-1)^{\beta}\log(e^{\gamma x_m}-1)=0. \tag{7}$$

A system of equations in three unknowns variables α, γ and β is a system of nonlinear equations. The exact solution of these nonlinear equations cannot be obtained in closed form. So, the Newton-Raphson iteration method must be used to get approximate solution of the system of these nonlinear equations see, EL-Sagheer [5].

3.1 Approximate confidence intervals

In this section, the ACIs of the parameters α, γ and $\beta$are constructed based on the asymptotic variance–covariance matrix for the MLE. The asymptotic variance–covariance matrix for the MLE can be obtained by using the inverse of the asymptotic Fisher information matrix, as follows:

$$F^{-1}=\begin{bmatrix}-\frac{\partial^2\ell}{\partial\alpha^2} & -\frac{\partial^2\ell}{\partial\alpha\partial\gamma} & -\frac{\partial^2\ell}{\partial\alpha\partial\beta}\\ -\frac{\partial^2\ell}{\partial\gamma\partial\alpha} & -\frac{\partial^2\ell}{\partial\gamma^2} & -\frac{\partial^2\ell}{\partial\gamma\partial\beta}\\ -\frac{\partial^2\ell}{\partial\beta\partial\alpha} & -\frac{\partial^2\ell}{\partial\beta\partial\gamma} & -\frac{\partial^2\ell}{\partial\beta^2}\end{bmatrix}^{-1}_{\downarrow(\hat{\alpha},\hat{\gamma},\hat{\beta})}=\begin{bmatrix}var(\hat{\alpha}) & Cov(\hat{\alpha},\hat{\gamma}) & Cov(\hat{\alpha},\hat{\beta})\\ Cov(\hat{\gamma},\hat{\alpha}) & var(\hat{\gamma}) & Cov(\hat{\gamma},\hat{\beta})\\ Cov(\hat{\beta},\hat{\alpha}) & Cov(\hat{\beta},\hat{\gamma}) & var(\hat{\beta})\end{bmatrix}, \tag{8}$$

where the second partial derivatives of the log-likelihood function with respect to α,γ and β respectively, are obtained as

$$\frac{\partial^2\ell}{\partial\alpha^2}=-\frac{m}{\alpha^2}, \tag{9}$$

$$\frac{\partial^2\ell}{\partial\alpha\partial\gamma}=-\beta\sum_{i=1}^{m}x_i e^{\gamma x_i}(e^{\gamma x_i}-1)^{\beta-1}-\beta\sum_{i=1}^{j}R_i x_i e^{\gamma x_i}(e^{\gamma x_i}-1)^{\beta-1}$$
$$-\beta x_m e^{\gamma x_m}(n-m-\sum_{i=1}^{j}R_i)(e^{\gamma x_m}-1)^{\beta-1}, \tag{10}$$

$$\frac{\partial^2\ell}{\partial\alpha\partial\beta}=-\sum_{i=1}^{m}(e^{\gamma x_i}-1)^{\beta}\log(e^{\gamma x_i}-1)-\sum_{i=1}^{j}R_i(e^{\gamma x_i}-1)^{\beta}\log(e^{\gamma x_i}-1)$$
$$-(n-m-\sum_{i=1}^{j}R_i)(e^{\gamma x_m}-1)^{\beta}\log(e^{\gamma x_m}-1), \tag{11}$$

$$\frac{\partial^2\ell}{\partial\gamma^2}=-\frac{m}{\gamma^2}-(\beta-1)\sum_{i=1}^{n_1}\frac{x_i^2 e^{\gamma x_i}}{(e^{\gamma x_i}-1)^2}-\alpha\beta\sum_{i=1}^{m}x_i^2 e^{\gamma x_i}(e^{\gamma x_i}-1)^{\beta-2}((e^{\gamma x_i}-1)+(\beta-1)e^{\gamma x_i})$$
$$-\alpha\beta\sum_{i=1}^{j}R_i x_i^2 e^{\gamma x_i}(e^{\gamma x_i}-1)^{\beta-2}((e^{\gamma x_i}-1)+(\beta-1)e^{\gamma x_i})$$
$$-\alpha\beta x_m^2 e^{\gamma x_m}(n-m-\sum_{i=1}^{j}R_i)(e^{\gamma x_m}-1)^{\beta-2}((e^{\gamma x_m}-1)+(\beta-1)e^{\gamma x_m}), \tag{12}$$

$$\frac{\partial^2\ell}{\partial\gamma\partial\beta}=\sum_{i=1}^{m}\frac{x_ie^{\gamma x_i}}{(e^{\gamma x_i}-1)}-\alpha\sum_{i=1}^{m}x_ie^{\gamma x_i}(e^{\gamma x_i}-1)^{\beta-1}(1+\beta\log(e^{\gamma x_i}-1))$$
$$-\alpha\sum_{i=1}^{j}R_ix_ie^{\gamma x_i}(e^{\gamma x_i}-1)^{\beta-1}(1+\beta\log(e^{\gamma x_i}-1))$$
$$-\alpha x_me^{\gamma x_m}(n-m-\sum_{i=1}^{j}R_i)(e^{\gamma x_m}-1)^{\beta-1}(1+\beta\log(e^{\gamma x_m}-1)). \quad (13)$$

and

$$\frac{\partial^2\ell}{\partial\beta^2}=-\frac{m}{\beta^2}-\alpha\sum_{i=1}^{m}(e^{\gamma x_i}-1)^{\beta}\log(e^{\gamma x_i}-1)^2-\alpha\sum_{i=1}^{j}R_i(e^{\gamma x_i}-1)^{\beta}\log(e^{\gamma x_i}-1)^2$$
$$-\alpha(n-m-\sum_{i=1}^{j}R_i)(e^{\gamma x_m}-1)^{\beta}\log(e^{\gamma x_m}-1)^2. \quad (14)$$

Thus, $(1-\zeta)\,100\%$ ACIs for the parameters α,γ and β, can be obtained by

$$\begin{aligned}(\hat{\alpha}_L,\hat{\alpha}_U)&=\hat{\alpha}\pm z_{\frac{\zeta}{2}}\sqrt{var(\hat{\alpha})},\\(\hat{\gamma}_L,\hat{\gamma}_U)&=\hat{\gamma}\pm z_{\frac{\zeta}{2}}\sqrt{var(\hat{\gamma})},\\(\hat{\beta}_L,\hat{\beta}_U)&=\hat{\beta}\pm z_{\frac{\zeta}{2}}\sqrt{var(\hat{\beta})},\end{aligned} \quad (15)$$

where $z_{\frac{\zeta}{2}}$ is the percentile of the standard normal distribution with right-tail probability $\frac{\zeta}{2}$ and $var(\hat{\alpha}), var(\hat{\gamma})$,and $var(\hat{\beta})$ represent asymptotic variances of maximum likelihood estimates.

4 Bootstrap Confidence Intervals

In this section, we propose the confidence intervals of the parameters α, γ and β based on percentile bootstrap method ($Boot_p$) using the idea of Efron [4]. The algorithm for constructing the CIs using $Boot_p$ method is illustrated as follow:

(1)Based on A-II-PRO-C sample $\underline{x}=x_{1:m:n}\leq x_{2:m:n}\leq\cdots\leq x_{m:m:n}$, compute the MLEs of the parameters α, γ and β say, $\hat{\alpha},\hat{\gamma}$ and $\hat{\beta}$.

(2)Use the point estimate $\hat{\alpha},\hat{\gamma}$ and $\hat{\beta}$ to generate a bootstrap sample $\underline{x}^*$ with the same values of $R_m,\ i=1,2,...,m$ using Balakrishnan and Sandhu [1] algorithm.

(3)Based on ordered bootstrap sample $\underline{x}^*$, compute the bootstrap sample estimates $\hat{\alpha}^*$,γ^* and $\hat{\beta}^*$.

(4)Repeat the steps (2) and (3) $B=200$ times and arrange all $\hat{\alpha}^*$,$\hat{\gamma}^*$ and $\hat{\beta}^*$ in ascending order to obtain the bootstrap sample $\left(\Omega_j^{[1]},\Omega_j^{[2]},...,\Omega_j^{[B]}\right)$, $j=1,2,3$. where $\Omega_1=\hat{\alpha}^*$, $\Omega_2=,\hat{\gamma}^*$, $\Omega_3=\hat{\beta}^*$.

Let $\Phi(z)=P(\Omega_j\leq z)$ be the cumulative distribution function of Ω_j , $j=1,2,3$. Then $\Omega_{jBoot_p}=\Phi^{-1}(z)$ for given z. The approximate bootstrap-p $100(1-\zeta)\%$ confidence interval of Ω_j is given by

$$\left[\,\Omega_{jBoot_p}(\tfrac{\zeta}{2})\ ,\ \Omega_{jBoot_p}(1-\tfrac{\zeta}{2})\,\right]. \quad (16)$$

5 Bayesian Estimation Using MCMC Technique

In this section, Bayesian estimates and their corresponding credible intervals of the unknown parameters α,γ and β are obtained. An important sup-class of MCMC techniques are the Gibbs sampler which was introduced by Geman and Geman [7], and the M-H algorithm developed by Metropolis et al.[11] and later extended by Hastings [9]. Gibbs sampler is used to generate a sequence of samples from the full conditional probability distributions of two or more random variables. Gibbs sampler requires decomposing the joint posterior distribution into full conditional distributions for each parameter and then sampling from them. Let us assume the parameters α, γ and β, are independent and follow the gamma prior distributions as

$$\begin{aligned}\pi(\alpha)&\propto\alpha^{a_1-1}e^{-\alpha b_1},\ \ \alpha>0,\\\pi(\gamma)&\propto\gamma^{a_2-1}e^{-\gamma b_2},\ \ \gamma>0,\\\pi(\beta)&\propto\beta^{a_3-1}e^{-\beta b_3},\ \ \beta>0,\end{aligned} \quad (17)$$

where a_1, a_2, a_3, b_1, b_2 and b_3 are the hyperparameters and they are non negative. Then, the joint prior of α, γ and β can be written as

$$\pi(\alpha,\gamma,\beta) \propto \alpha^{a_1-1}\,\gamma^{a_2-1}\,\beta^{a_3-1}\,e^{-\alpha b_1-\gamma b_2-\beta b_3}\,,\alpha>0,\gamma>0,\ \beta>0. \tag{18}$$

The joint posterior density function of α,γ and β denoted by $\pi^*(\alpha,\gamma,\beta|\underline{x})$ can be written as

$$\pi^*(\alpha,\gamma,\beta|\underline{x}) = \frac{L(\alpha,\gamma,\beta)\times\pi(\alpha,\gamma,\beta)}{\int_0^\infty\int_0^\infty\int_0^\infty L(\alpha,\gamma,\beta)\times\pi(\alpha,\gamma,\beta)\,d\alpha d\gamma d\beta}. \tag{19}$$

Using Equations (3) and (18) to substitute in Equation (19), then the joint posterior density function of α,γ and β is given by

$$\begin{aligned}\pi^*(\alpha,\gamma,\beta|\underline{x}) = {} & K^{-1}\alpha^{m+a_1-1}\,\gamma^{n+a_2-1}\,\beta^{m+a_3-1}\,e^{-\alpha b_1-\gamma b_2-\beta b_3}\\ & \times\left[\prod_{i=1}^{m}e^{\gamma x_i}\,(e^{\gamma x_i}-1)^{\beta-1}\,e^{-\alpha(e^{\gamma x_i}-1)^{\beta}}\right]\\ & \times\left[\prod_{i=1}^{j}\left[e^{-\alpha(e^{\gamma x_i}-1)^{\beta}}\right]^{R_i}\right]\times\left[e^{-\alpha(e^{\gamma x_m}-1)^{\beta}}\right]^{n-m-\sum_{i=1}^{j}R_i},\end{aligned} \tag{20}$$

where K^{-1} is the normalizing constant, which is equal to

$$\begin{aligned}K^{-1} = {} & \int_0^\infty\int_0^\infty\int_0^\infty \alpha^{m+a_1-1}\,\gamma^{n+a_2-1}\,\beta^{m+a_3-1}\,e^{-\alpha b_1-\gamma b_2-\beta b_3}\\ & \times\left[\prod_{i=1}^{m}e^{\gamma x_i}\,(e^{\gamma x_i}-1)^{\beta-1}\,e^{-\alpha(e^{\gamma x_i}-1)^{\beta}}\right]\\ & \times\left[\prod_{i=1}^{j}\left[e^{-\alpha(e^{\gamma x_i}-1)^{\beta}}\right]^{R_i}\right]\times\left[e^{-\alpha(e^{\gamma x_m}-1)^{\beta}}\right]^{n-m-\sum_{i=1}^{j}R_i} d\alpha d\gamma d\beta.\end{aligned} \tag{21}$$

Therefore, the Bayes estimate of any function of the parameters α,γ and β such as $g(\alpha,\gamma,\beta)$ under squared error loss function should be the posterior mean, i.e.

$$g(\alpha,\gamma,\beta) = \frac{\int_0^\infty\int_0^\infty\int_0^\infty g(\alpha,\gamma,\beta)\times L(\alpha,\gamma,\beta)\times\pi(\alpha,\gamma,\beta)\,d\alpha d\gamma d\beta}{\int_0^\infty\int_0^\infty\int_0^\infty L(\alpha,\gamma,\beta)\times\pi(\alpha,\gamma,\beta)\,d\alpha d\gamma d\beta}. \tag{22}$$

The integrals given by Equation (22) cannot be obtained in explicit form, so the MCMC technique applied to obtain an approximate value of this integrals and then the Bayes estimates of the parameters α,γ and β are computed, also their credible intervals. From Equation (20) the conditional posterior densities of α,γ and β can be given respectively, as

$$\begin{aligned}\pi_1^*(\alpha|\gamma,\beta,\underline{x}) \propto {} & \alpha^{m+a_1-1}\\ & \times exp[-\alpha(b_1+\sum_{i=1}^{m}(e^{\gamma x_i}-1)^{\beta}+\sum_{i=1}^{j}(e^{\gamma x_i}-1)^{\beta}R_i+(n-m-\sum_{i=1}^{j}R_i)(e^{\gamma x_m}-1)^{\beta}],\end{aligned} \tag{23}$$

$$\begin{aligned}\pi_2^*(\gamma|\alpha,\beta,\underline{x}) \propto {} & \gamma^{n+a_2-1}\prod_{i=1}^{m}(e^{\gamma x_i}-1)^{\beta-1}\times exp(\gamma\sum_{i=1}^{m}x_i)\times exp(-\gamma b_2)\\ & \times exp[-\alpha(\sum_{i=1}^{m}(e^{\gamma x_i}-1)^{\beta}+\sum_{i=1}^{j}(e^{\gamma x_i}-1)^{\beta}R_i+(n-m-\sum_{i=1}^{j}R_i)(e^{\gamma x_m}-1)^{\beta})],\end{aligned} \tag{24}$$

and

$$\begin{aligned}\pi_3^*(\beta|\alpha,\gamma,\underline{x}) \propto {} & \beta^{m+a_3-1}\times\prod_{i=1}^{m}(e^{\gamma x_i}-1)^{\beta-1}\times exp(-\beta b_3)\\ & \times exp[-\alpha(\sum_{i=1}^{m}(e^{\gamma x_i}-1)^{\beta}+\sum_{i=1}^{j}(e^{\gamma x_i}-1)^{\beta}R_i+(n-m-\sum_{i=1}^{j}R_i)(e^{\gamma x_m}-1)^{\beta})].\end{aligned} \tag{25}$$

It can be seen that, in Equation (23) there is a gamma density function with shape and scale parameters as $m+a_1$and $(b_1+\sum_{i=1}^{m}(e^{\gamma x_i}-1)^{\beta}+\sum_{i=1}^{j}(e^{\gamma x_i}-1)^{\beta}R_i+(n-m-\sum_{i=1}^{j}R_i)(e^{\gamma x_m}-1)^{\beta})$, respectively. Therefore, it is easy to generate the samples of α by using gamma-generating routine. Furthermore, both of the full conditional posterior distributions for γ and β can't be reduced analytically to well-known distributions and it is difficult to sample directly by standard methods. To solve this problem we apply a Metropolis – Hastings algorithm with Gibbs sampling scheme.

Now, the following steps illustrate the method of the Metropolis – Hastings algorithm with Gibbs sampling see [8] to generate the posterior samples as suggested by Tierney [15] and in turn obtain the Bayes estimates and the corresponding credible intervals

(1)Start with an $\left(\alpha^{(0)}=\hat{\alpha},\ \gamma^{(0)}=\hat{\gamma}\text{ and }\beta^{(0)}=\hat{\beta}\right)$.

(2)Put $i=1$.

(3)Generate $\alpha^{(i)}$ from

$$Gamma\left[m+a_1, b_1+\sum_{i=1}^{m}(e^{\gamma x_i}-1)^{\beta}+\sum_{i=1}^{j}(e^{\gamma x_i}-1)^{\beta}R_i+(n-m-\sum_{i=1}^{j}R_i)(e^{\gamma x_m}-1)^{\beta}\right]$$

(4)Using the following Metropolis-Hastings method, generate $\gamma^{(\iota)}$and $\beta^{(\iota)}$ from Equations (24) and (25) with the normal suggested distribution

$$N(\gamma^{(\iota-1)}, var(\gamma))\text{and } N(\beta^{(\iota-1)}, var(\beta)),\text{respectively,}$$

Where $var(\gamma)$ and $var(\beta)$ can be obtained from the main diagonal in asymptotic inverse Fisher information matrix (8) .

i-Generate a proposal γ^* from $N(\gamma^{(\iota-1)}, var(\gamma))$ and β^* from $N(\beta^{(\iota-1)}, var(\beta))$.

ii-Evaluate the acceptance probabilities fo γ and β

$$\rho_{\gamma}=\min\left[1, \frac{\pi_2^*(\gamma^*|\alpha^{(i)},\beta^{(i-1)},\underline{x})}{\pi_2^*(\gamma^{(i-1)}|\alpha^{(i)},\beta^{(i-1)},\underline{x})}\right],$$

$$\rho_{\beta}=\min\left[1, \frac{\pi_3^*(\beta^*|\alpha^{(i)},\gamma^{(i)},\underline{x})}{\pi_3^*(\beta^{(i-1)}|\alpha^{(i)},\gamma^{(i)},\underline{x})}\right],$$

iii-Generate $u_1$and u_2 from a Uniform (0,1) distribution.

iv-If $u_1\le\rho_{\gamma}$ accept the proposal and set $\gamma^{(i)}=\gamma^*$,else set $\gamma^{(i)}=\gamma^{(i-1)}$.

v-If $u_2\le\rho_{\beta}$ accept the proposal and set $\beta^{(i)}=\beta^*$,else set $\beta^{(i)}=\beta^{(i-1)}$.

(5)Compute $\gamma^{(\iota)}$ and $\beta^{(\iota)}$.

(6)Put $i=i+1$.

(7)Repeat Steps $(3-6)Q=12000$ times.

(8)Disregard the first M simulated varieties. Then the selected samples are $\alpha^{(i)},\beta^{(i)}$ and $\gamma^{(i)}$, $\iota=M+1,...,Q$, for sufficiently large Q forming an approximate posterior samples which can be used to obtain the Bayes MCMC point estimates of α,γ and β as

$$\alpha_{MCMC}=\frac{1}{N-M}\sum_{i=M+1}^{N}\alpha^{(i)},$$
$$\gamma_{MCMC}=\frac{1}{N-M}\sum_{i=M+1}^{N}\gamma^{(i)},$$
$$\beta_{MCMC}=\frac{1}{N-M}\sum_{i=M+1}^{N}\beta^{(i)}.$$

(9)To calculate the credible intervals (CRIs) of Ω_k where $\Omega_1=\alpha$, $\Omega_2=\gamma$ and $\Omega_3=\beta$,we take the quantiles of the sample as the endpoints of the intervals. Sort $\left\{\Omega_k^{M+1},\Omega_k^{M+2},...,\Omega_k^{N}\right\}$ as $\left\{\Omega_k^{(1)},\Omega_k^{(2)},...,\Omega_k^{(N-M)}\right\}$. Hence the 100 $(1-\zeta)\%$ credible interval (CRI) of Ω_k is

$$\left[\Omega_{k\left((N-M)(\frac{\zeta}{2})\right)}, \Omega_{k\left((N-M)\left(1-\frac{\zeta}{2}\right)\right)}\right].$$

6 Illustrative Example

In this section, a simulation example is illustrated to evaluate the estimation procedures. In this example, by using the algorithm described in Ng et al. [12]. It generates an A-II-PRO-C sample from the WED with parameters $(\alpha,\gamma,\beta) = (0.1,1.5,2.5)$ and $T = 0.7$ by using censored scheme $n = 30$, $m = 20$ and $R = (1,0,0,1,1,0,0,1,1,1,0,1,0,0,1,0,0,0,0,2)$. An A-II-PRO-C sample is

0.31414	0.39798	0.47821	0.51557	0.52206	0.68564	0.71601
0.72081	0.73082	0.77218	0.89375	0.89638	0.99134	0.77635
0.80277	0.81326	0.83696	0.86216	0.87204	0.88994	

Based on an A-II-PRO-C sample, the point estimates of the parameters using ML, $Boot_p$ are presented in Table 1. Based on the MCMC samples of size 25000 with 5000 as burn-in, the Bayes estimates relative to the informative prior gamma functions of α,γ and β with hyperparameters a_i = 0.2 and b_i= 0.01 where $i = 1,2,3$ are presented in Table 1. The corresponding 95% ACIs and PBCIs and also the 95% credible intervals (CRIs) of α,γ and β are reported in Table 2.

Table (1). Different point estimates for α,γ and β

Parameter	$(\cdot)_{ML}$	$(\cdot)_{Boot_p}$	$(\cdot)_{MCMC}$
α	0.0782	0.0746	0.0702
γ	1.7547	2.0253	1.7792
β	2.2776	2.3503	2.2977

Table (2). 95% confidence intervals for α,γ and β

Parameter	ACI	PBCI	CRI
α	[-0.3434,0.4998]	[0.0150,0.4586]	[0.0431,0.1036]
Length	0.8432	0.4435	0.0604
γ	[-1.2878,4.7972]	[1.1681,2.9226]	[1.7609,1.8094]
Length	6.0849	1.7544	0.0485
β	[0.2616,4.2937]	[1.4932,2.9458]	[2.2870,2.3119]
Length	4.0321	1.4526	0.0249

7 Monte Carlo Simulation Study

In order to compare the different estimators of the parameters, we have simulated 1000 an A-II-PRO-C samples from WED with the values of parameters $(\alpha,\gamma,\beta) = (0.1,1.5,2.5)$ and different censoring schemes R for $T = 0.8$ and 0.9. The samples are simulated by using the algorithm described in Ng et al. [12]. All computations are performed using *MATHEMATICA ver. 9*. The performance of ML, $Boot_p$ and Bayes estimates with respect to the SEL function has been considered in terms of their Average Estimates (AVG) and the Mean Squared Errors (MSEs), computed for θ_k , $k = 1,2,3$ and $(\theta_1 = \alpha, \theta_2 = \gamma, \theta_3 = \beta)$, as

$$AVG(\theta) = \frac{1}{N}\sum_{i=1}^{N}\hat{\theta}_k^{(i)} \quad \text{and} \quad MSE(\theta) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_k^{(i)} - \theta_k)^2$$

where $N = 1000$ is the number of simulated samples. Also, the comparison between different estimation methods are made in terms of the average CI Lengths (ACL) and Coverage Percentages (CP). For each simulated sample, we compute 95% CIs and check whether the true value lies within the interval and we record the length of the CI. The estimated coverage percentage is computed as the number of CIs that covering the true values is divided by 1000. Three types of Censoring Scheme (CS) are applied as following:

CS A :$R_1 = n-m$, $R_i = 0$ for $i \neq 1$

CS B : $R_{\frac{m}{2}} = R_{\frac{m}{2}+1} = \frac{n-m}{2}$, $R_i = 0$ for $i \neq \frac{m}{2}$ and $i \neq \frac{m}{2}+1$

CS C : $R_m = n-m$, $R_i = 0$ for $i \neq m$.

Based on 1000 replications, the results of the AVG estimates and the MSEs are reported in Tables 3 and 4, while the results of ACL and CP for all parameters are reported in Tables 5 and 6.

Table (3). AVG and MSE of ML and Bayes estimates for the parameters with $(\alpha,\gamma,\beta)=(0.1,1.5,2.5)$

			MLE			*Boot*$_p$			*MCMC*		
T	(n,m)	CS	α	γ	β	α	γ	β	α	γ	β
0.8	$(30,10)$	A	0.0916	1.7027	2.5969	0.1233	1.8556	2.8740	0.0937	1.7042	2.5939
			(0.0062)	(0.1909)	(0.3297)	(0.0078)	(0.1770)	(0.3111)	(0.0065)	(0.1915)	(0.3320)
		B	0.0992	1.6435	2.6790	0.1310	1.7445	2.9200	0.1010	1.6440	2.6774
			(0.0066)	(0.1406)	(0.3691)	(0.0080)	(0.1276)	(0.3342)	(0.0068)	(0.1409)	(0.3697)
		C	0.0965	1.6297	2.6998	0.1568	1.6511	2.7930	0.0985	1.6297	2.6987
			(0.0055)	(0.1192)	(0.4497)	(0.0078)	(0.0827)	(0.2201)	(0.0057)	(0.1194)	(0.4500)
	$(50,10)$	A	0.0933	1.7105	2.5668	0.1256	1.8655	2.8401	0.0954	1.7108	2.5660
			(0.0067)	(0.1967)	(0.3037)	(0.0086)	(0.1858)	(0.2987)	(0.0070)	(0.1970)	(0.3047)
		B	0.0951	1.6604	2.6514	0.1283	1.7495	2.9090	0.0972	1.6607	2.6514
			(0.0051)	(0.1346)	(0.3012)	(0.0064)	(0.1197)	(0.3155)	(0.0054)	(0.1347)	(0.3014)
		C	0.1037	1.6096	2.6854	0.1655	1.6185	2.7857	0.1060	1.6097	2.6851
			(0.0055)	(0.1050)	(0.4599)	(0.0090)	(0.0761)	(0.2326)	(0.0057)	(0.1054)	(0.4597)
	$(60,15)$	A	0.0993	1.6742	2.5863	0.1262	1.8517	2.8950	0.1008	1.6750	2.5854
			(0.0087)	(0.1810)	(0.2877)	(0.0096)	(0.1756)	(0.3467)	(0.0081)	(0.1814)	(0.2878)
		B	0.1077	1.6159	2.6313	0.1298	1.7302	2.9692	0.1095	1.6159	2.6313
			(0.0082)	(0.1281)	(0.2661)	(0.0090)	(0.1180)	(0.3969)	(0.0085)	(0.1283)	(0.2664)
		C	0.1126	1.5687	2.7048	0.1762	1.5643	2.8051	0.1144	1.5690	2.7046
			(0.0078)	(0.0901)	(0.3932)	(0.0122)	(0.0605)	(0.2435)	(0.0081)	(0.0903)	(0.3933)
	$(60,20)$	A	0.0981	1.6961	2.5411	0.1240	1.8576	2.903	0.0992	1.6962	2.5412
			(0.0087)	(0.1926)	(0.2570)	(0.0099)	(0.1815)	(0.3536)	(0.0089)	(0.1927)	(0.2573)
		B	0.1051	1.6182	2.6115	0.1171	1.7496	3.0026	0.1064	1.6185	2.6112
			(0.0079)	(0.1260)	(0.2278)	(0.0079)	(0.1284)	(0.4246)	(0.0082)	(0.1263)	(0.2280)
		C	0.1142	1.5719	2.6682	0.1782	1.5644	2.7809	0.1156	1.5720	2.6679
			(0.0096)	(0.0916)	(0.3318)	(0.0136)	(0.0631)	(0.2382)	(0.0099)	(0.0917)	(0.3314)

Table (4). AVG and MSE of ML and Bayes estimates for the parameters with $(\alpha,\gamma,\beta)=(0.1,1.5,2.5)$

			MLE			*Boot*$_p$			*MCMC*		
T	(n,m)	CS	α	γ	β	α	γ	β	α	γ	β
0.9	$(30,10)$	A	0.0939	1.6893	2.5352	0.1325	1.7710	2.7311	0.0959	1.6895	2.5348
			(0.0056)	(0.1901)	(0.3503)	(0.0068)	(0.1313)	(0.2520)	(0.0059)	(0.1906)	(0.3508)
		B	0.0989	1.6440	2.6774	0.1428	1.7041	2.8396	0.1011	1.6440	2.6775
			(0.0064)	(0.1398)	(0.3760)	(0.0080)	(0.1083)	(0.2767)	(0.0067)	(0.1396)	(0.3762)
		C	0.0952	1.6355	2.7345	0.1549	1.6547	2.8039	0.0974	1.6347	2.7345
			(0.0051)	(0.1216)	(0.4653)	(0.0075)	(0.0848)	(0.2261)	(0.0053)	(0.1214)	(0.4647)
	$(50,10)$	A	0.1018	1.6622	2.5583	0.1398	1.7575	2.7399	0.1041	1.6631	2.5571
			(0.0073)	(0.1853)	(0.3175)	(0.0087)	(0.1306)	(0.2410)	(0.0077)	(0.1861)	(0.3182)
		B	0.0992	1.6488	2.6462	0.1440	1.6992	2.8277	0.1014	1.6488	2.6456
			(0.0056)	(0.1401)	(0.3260)	(0.0076)	(0.1028)	(0.2685)	(0.0059)	(0.1403)	(0.3265)
		C	0.1081	1.5806	2.7445	0.1695	1.5952	2.8204	0.1104	1.5801	2.7445
			(0.0062)	(0.0860)	(0.4751)	(0.0103)	(0.0657)	(0.2393)	(0.0065)	(0.0861)	(0.4747)
	$(60,15)$	A	0.1201	1.6166	2.4985	0.1524	1.7196	2.7211	0.1220	1.6165	2.4980
			(0.0108)	(0.1746)	(0.2768)	(0.0119)	(0.1165)	(0.2610)	(0.0113)	(0.1745)	(0.2769)
		B	0.1097	1.6063	2.5839	0.1494	1.6620	2.8034	0.1115	1.6066	2.5825
			(0.0084)	(0.1167)	(0.2375)	(0.0099)	(0.0851)	(0.2566)	(0.0087)	(0.1169)	(0.2382)
		C	0.1120	1.5790	2.7051	0.1744	1.5751	2.8056	0.1137	1.5794	2.7044
			(0.0080)	(0.0943)	(0.3899)	(0.0122)	(0.0650)	(0.2466)	(0.0083)	(0.0942)	(0.3894)
	$(60,20)$	A	0.1214	1.5962	2.4845	0.1473	1.7032	2.7562	0.1229	15970	2.4829
			(0.0114)	(0.1571)	(0.2242)	(0.0111)	(0.1057)	(0.2700)	(0.0118)	(0.1572)	(0.2251)
		B	0.1185	1.5871	2.5599	0.1523	1.6474	2.8047	0.1198	1.5874	2.5596
			(0.0106)	(0.1224)	(0.2300)	(0.0114)	(0.0864)	(0.2781)	(0.0110)	(0.1225)	(0.2303)
		C	0.1240	1.5476	2.6743	0.1858	1.5397	2.7836	0.1255	1.5476	2.6742
			(0.0120)	(0.1025)	(0.3333)	(0.0163)	(0.0678)	(0.2366)	(0.0124)	(0.1024)	(0.3327)

Table (5). Comparisons of ACL and CP of 95% CIs for the parameters with $(\alpha,\gamma,\beta) = (0.1, 1.5, 2.5)$

			MLE			$Boot_p$			MCMC		
T	(n,m)	CS	α	γ	β	α	γ	β	α	γ	β
0.8	(30,10)	A	2.4925	12.4468	11.0308	0.6517	1.7030	2.0218	0.1136	0.0183	0.0189
			(0.9462)	(0.9541)	(0.9300)	(0.9628)	(0.9425)	(0.9401)	(0.9412)	(0.9501)	(0.9408)
		B	2.5753	11.6176	9.7114	0.6505	1.5485	2.0135	0.1226	0.0172	0.0182
			(0.9425)	(0.9315)	(0.9233)	(0.9442)	(0.9351)	(0.9523)	(0.9422)	(0.9378)	(0.9287)
		C	3.5009	15.3351	10.4122	0.7709	1.6560	2.2261	0.1196	0.0234	0.0198
			(0.9252)	(0.9465)	(0.9104)	(0.9574)	(0.9458)	(0.9115)	(0.9224)	(0.9460)	(0.9325)
	(50,10)	A	2.0406	9.7570	7.4489	0.6480	1.7105	1.9890	0.1158	0.0151	0.0140
			(0.9465)	(0.9507)	(0.9501)	(0.9458)	(0.9600)	(0.9625)	(0.9524)	(0.9500)	(0.9201)
		B	2.0482	9.2230	6.6203	0.6421	1.5159	1.9360	0.1180	0.0156	0.0144
			(0.9431)	(0.9452)	(0.9635)	(0.9462)	(0.9302)	(0.9135)	(0.9201)	(0.9120)	(0.9410)
		C	3.3664	14.6134	8.2852	0.7659	1.5572	2.2225	0.1285	0.0240	0.0180
			(0.9250)	(0.9461)	(0.9601)	(0.9204)	(0.9225)	(0.9462)	(0.9465)	(0.9431)	(0.9250)
	(60,15)	A	1.8686	8.1270	6.4119	0.6203	1.6392	1.8782	0.1006	0.0128	0.0137
			(0.9352)	(0.9240)	(0.9642)	(0.9102)	(0.9452)	(0.9625)	(0.9548)	(0.9421)	(0.9456)
		B	1.8583	7.2812	5.3908	0.5873	1.3941	1.7902	0.1091	0.0118	0.0115
			(0.9302)	(0.9258)	(0.9425)	(0.9421)	(0.9245)	(0.9611)	(0.9324)	(0.9421)	(0.9541)
		C	2.8662	11.2472	6.8116	0.7682	1.4573	2.0819	0.1141	0.0181	0.0144
			(0.9625)	(0.9325)	(0.9121)	(0.9611)	(0.9325)	(0.9625)	(0.9521)	(0.9544)	(0.9425)
	(60,20)	A	1.5166	6.6671	4.9896	0.6189	1.6173	1.8279	0.0858	0.0107	0.0107
			(0.9201)	(0.9452)	(0.9201)	(0.9214)	(0.9421)	(0.9452)	(0.9201)	(0.9511)	(0.9584)
		B	1.7230	7.0832	5.3442	0.5272	1.3393	1.7337	0.0925	0.0115	0.0114
			(0.9312)	(0.9254)	(0.9401)	(0.9542)	(0.9625)	(0.9542)	(0.9461)	(0.9025)	(0.9102)
		C	2.4635	9.5514	6.0873	0.7655	1.4705	1.9814	0.1001	0.0152	0.0129
			(0.9421)	(0.9485)	(0.9120)	(0.9314)	(0.9435)	(0.9121)	(0.9524)	(0.9485)	(0.9640)

Table (6). Comparisons of ACL and CP of 95% CIs for the parameters with $(\alpha,\gamma,\beta) = (0.1, 1.5, 2.5)$

			MLE			$Boot_p$			MCMC		
T	(n,m)	CS	α	γ	β	α	γ	β	α	γ	β
0.9	(30,10)	A	2.1999	10.6267	8.7352	0.7006	1.7278	2.0699	0.1162	0.0173	0.0169
			(0.9447)	(0.9304)	(0.9542)	(0.9325)	(0.9115)	(0.9356)	(0.9423)	(0.9354)	(0.9425)
		B	2.3617	10.0343	7.6759	0.7087	1.6112	2.0446	0.1226	0.0163	0.0163
			(0.9625)	(0.9402)	(0.9410)	(0.9628)	(0.9420)	(0.9241)	(0.9245)	(0.9425)	(0.9314)
		C	3.0092	13.4749	8.8810	0.7614	1.6546	2.2006	0.1179	0.0219	0.0187
			(0.9431)	(0.9411)	(0.9208)	(0.9470)	(0.9421)	(0.9462)	(0.9144)	(0.9256)	(0.9324)
	(50,10)	A	2.1652	9.5724	7.1258	0.6986	1.7096	2.0249	0.1261	0.0155	0.0154
			(0.9621)	(0.9654)	(0.9482)	(0.9472)	(0.9425)	(0.9431)	(0.9255)	(0.9425)	(0.9414)
		B	2.2524	9.7740	7.6133	0.7027	1.5583	1.9585	0.1234	0.0159	0.0157
			(0.9402)	(0.9547)	(0.9622)	(0.9142)	(0.9632)	(0.9325)	(0.9147)	(0.9425)	(0.9365)
		C	3.5613	14.4537	8.4505	0.7667	1.5250	2.2161	0.1341	0.0234	0.0180
			(0.9465)	(0.9408)	(0.9485)	(0.9465)	(0.9503)	(0.9452)	(0.9402)	(0.9142)	(0.9425)
	(60,15)	A	1.9534	7.2655	5.4047	0.7025	1.6427	1.9314	0.1217	0.0114	0.0116
			(0.9425)	(0.9285)	(0.9430)	(0.9257)	(0.9645)	(0.9400)	(0.9425)	(0.9647)	(0.9103)
		B	2.1180	9.2248	7.1143	0.6966	1.4983	1.8610	0.1110	0.0146	0.0148
			(0.9617)	(0.9408)	(0.9574)	(0.9257)	(0.9447)	(0.9421)	(0.9521)	(0.9257)	(0.9450)
		C	2.9082	11.5716	6.9645	0.7658	1.4635	2.0759	0.1135	0.0185	0.0147
			(0.9146)	(0.9406)	(0.9130)	(0.9348)	(0.9201)	(0.9420)	(0.9214)	(0.9414)	(0.9420)
	(60,20)	A	2.3386	9.5359	8.0624	0.6839	1.5936	1.8959	0.1067	0.0136	0.0145
			(0.9214)	(0.9254)	(0.9104)	(0.9364)	(0.9136)	(0.9205)	(0.9432)	(0.9205)	(0.9421)
		B	1.7841	6.7589	5.1026	0.6732	1.4505	1.8020	0.1038	0.0108	0.0114
			(0.9105)	(0.9417)	(0.9154)	(0.9441)	(0.9105)	(0.9403)	(0.9104)	(0.9151)	(0.9452)
		C	2.5725	9.2196	5.9628	0.7647	1.4288	1.9788	0.1091	0.0144	0.0128
			(0.9465)	(0.9374)	(0.9420)	(0.9330)	(0.9214)	(0.9642)	(0.9425)	(0.9645)	(0.9254)

8 Conclusion

In this paper, we have discussed the classical and Bayesian estimation for the unknown parameters of WED based on A-II-PRO-C. The asymptotic normality of ML and parametric bootstrap methods have been used to construct the CIs for the unknown parameters of WED. The MCMC is used to compute the approximate Bayes estimates and corresponding credible intervals using Metropolis – Hastings algorithm with Gibbs sampling. A simulated data set is presented to show how the MCMC methods work based on A-II-PRO-C. Monte Carlo simulation study has been used to compare the performance of the proposed methods for different sample size (n, m) and different censoring scheme R. From Tables (3-6), it is possible to conclude that:

1. As sample size n increases, the MSEs decrease and Bayes estimates have the smallest MSEs among all other estimates proposed.
2. Percentile bootstrap method performs better than ML method in the sense of having smaller MSEs for α, γ and β.
3. The MSEs which is obtained at time $T = 0.8$ is smaller than MSEs which are obtained at time $T = 0.9$ for most estimators.
4. The average length for all estimates at time $T = 0.9$ is smaller than at time $T = 0.8$.

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