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Trapezium-type $AB-$fractional Integral Inequalities Using Generalized Convex and Quasi $\phi-$Convex Functions

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Abstract: The Hermite-Hadamard inequality and an identity for $AB-$fractional integrals are demonstrated in this study using generalized $\phi-$convex functions. Some Hermite–Hadamard type inequalities are also established using the stated identity and the $AB-$fractional integral operator. A number of unique examples have been identified. The concept of generalized quasi $\phi-$convex functions is also introduced, as well as some fractional inequalities.

Keywords: Hermite-Hadamard inequality, Mittag-Leffler function, Hölder inequality, power mean inequality, AB-fractional integrals, convexity.

1 Introduction

The following inequalities are classic in the study of convex functions, and play a very important role in its development, due to its applications in different areas of mathematics such as optimization theory and mathematical economics. They are well known as Hermite-Hadamard inequality.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function and $\xi,\eta \in I$ with $\xi < \eta$. Then the following inequality holds:*

$$f\left(\frac{\xi+\eta}{2}\right) \leq \frac{1}{\eta-\xi}\int_{\xi}^{\eta} f(x)dx \leq \frac{f(\xi)+f(\eta)}{2}. \tag{1}$$

This inequality (1) is also known as trapezium inequality.

The left side of this inequality was demonstrated by Jaques Hadamard in 1893 [1] and the right side by Charles Hermite in 1883 [2]. The mean value of a continuous and convex function $f : [a,b] \to \mathbb{R}$ can be estimated using this basic inequality. Because of its wide application in Mathematical Analysis, the trapezium inequality is of great interest to researchers in this field.

In a variety of scientific fields, like as: Biology, Economics, Physics and optimization [3,4], among others, the convexity of functions plays an important role. In the last decades the concept of convexity has been generalized in several directions due to different problems and even to the theoretical development of research in pure mathematics, especially in the area of inequalities. Interested readers can see the references [5,6,7,8,9,10,11,12,13,14,15,16]. Recently some studies related to quantum mathematics and fractal sets have been development [17,18,19,20,21,22]. Even works that relate integral inequalities to fractional calculus have been published. [23,24,25,26,27].

M. Noor [28] introduced and studied one of the concepts associated with generalized convexity. Let's first establish the following notations: we denote with $\mathscr{K}$ a non-empty closed set in $\mathbb{R}^n$, and with $\mathscr{K}^\circ$ the interior of $\mathscr{K}$. For the inner

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product and norm on $\mathbb{R}^n$ we use $\langle .,. \rangle$ and $\|.\|$ respectively. Two continuous functions will also be considered, f and ϕ as real valued functions defined on $\mathscr{K}$.

Definition 1. *Let $u \in \mathscr{K}$. If there exists a function ϕ such that*

$$u + te^{i\phi}(v-u) \in \mathscr{K}$$

for all $u, v \in \mathscr{K}$ and $t \in [0,1]$ then $\mathscr{K}$ will be said a $\phi-$convex set.

Definition 2. *A function f defined on a ϕ-convex set $\mathscr{K}$ is said to be ϕ-convex, if*

$$f(\xi + te^{i\phi}(\eta - \xi)) \leq (1-t)f(\xi) + tf(\eta), \quad \forall\, \xi, \eta \in \mathscr{K}, \quad t \in [0,1].$$

If $(-f)$ is $\phi-$convex then the function f is said to be $\phi-$concave.

It is easy to observe that any convex function with $\phi = 0$ is $\phi-$ convex, but the converse is not true.

Using the following special function and their properties some researchers have presented more generalized concepts related to the aforementioned $\phi-$convex sets.

Definition 3.*[29]* *The special function*

$$\mathbf{E}_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}, \; \alpha \in C, \; Re(\alpha) > 0, \; z \in C, \tag{2}$$

where Γ is a Gamma function is called Mittag-Leffler function.

For $\alpha = 0,1,2,3,4$, we find the following expressions using the Mittag-Leffler function $\mathbf{E}_\alpha(z)$:

1. $\mathbf{E}_0(z) = \frac{1}{1-z}, \; |z| < 1$
2. $\mathbf{E}_1(z) = e^z$
3. $\mathbf{E}_1(iz) = e^{iz}$
4. $\mathbf{E}_2(z) = \cosh(\sqrt{z}), \; z \in C$
5. $\mathbf{E}_2(-z^2) = \cos z, \; z \in C$
6. $\mathbf{E}_3(z) = \frac{1}{2}\left[e^{z^{\frac{1}{3}}} + 2e^{-\frac{1}{2}z^{\frac{1}{3}}}\cos\left(\frac{\sqrt{3}}{2}z^{\frac{1}{3}}\right)\right]$
7. $\mathbf{E}_4(z) = \frac{1}{2}\left[\cos(z^{\frac{1}{4}}) + \cosh(z^{\frac{1}{4}})\right]$

Recently, in [30], Ali et al. have introduce some of the notions of generalized convexity.

Definition 4. *A non empty set $\mathscr{K}$ is said to be generalized ϕ-convex set if*

$$\xi + t\mathbf{E}_\alpha(\eta - \xi) \in \mathscr{K}, \quad \forall\, \xi, \eta \in \mathscr{K}, \quad t \in [0,1], \tag{3}$$

where $\alpha \in C$, $Re(\alpha) > 0$.

*Remark.*If $\alpha = 1$, then generalized ϕ-convex set reduces to the ϕ-convex set. The generalized ϕ-convex sets are non-convex.

Definition 5. *The function f is said to be generalized ϕ-convex, if*

$$f(\xi + t\mathbf{E}_\alpha(\eta - \xi)) \leq (1-t)f(\xi) + tf(\eta), \quad \forall\, \xi, \eta \in \mathscr{K}, \quad t \in [0,1]. \tag{4}$$

Remark. Obviously generalized ϕ-convex function is ϕ-convex function by replacing $\mathbf{E}_\alpha$ with $\mathbf{E}_1$, a special case of Mittag-Leffler function. The generalized ϕ-convex functions are non-convex.

In [31,32] the inequalities involving generalized fractional integral operators have been considered. This direction has gained the attention of many researches. Among the objectives proposed in this work is that of establishing a general formulation of inequalities of the Hermite–Hadamard type, so that the essential facts covered by the different fractional integrals become clearer and also produce new inequalities. We are interested in the following fractional integral operator.

Based on the Caputo and Riemann-Liouville definitions of fractional order derivatives, Atangana and Baleanu, in [33], created two new fractional derivatives. As the antiderivative of their operators, they asserted that their fractional derivative has a fractional integral. The kernel of the Atangana-Baleanu (AB) fractional order derivative, which uses the generalized Mittag-Leffler function, is known to be nonsingular and nonlocal, see [34,35,36,37].

Now, we recall the following definition related to the *AB*-fractional operator.

Definition 6. *The fractional AB-integral of the function $f \in H^*(\xi, q)$ is given by*

$$ {}^{AB}_{\xi}I^{\delta}_{t}f(t) = \frac{1-\delta}{\mathbb{B}(\delta)}f(t) + \frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}\int_{\xi}^{t}(t-u)^{\delta-1}f(u)du, \ \ t > \xi, \tag{5} $$

where $\xi < \eta$, $0 < \delta < 1$ and the normalization function $\mathbb{B}(\delta) > 0$ satisfies the property $\mathbb{B}(0) = \mathbb{B}(1) = 1$.

Similarly, the definition of the (5) opposite side is given by

$$ {}^{AB}_{\eta}I^{\delta}_{t}f(t) = \frac{1-\delta}{\mathbb{B}(\delta)}f(t) + \frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}\int_{t}^{\eta}(u-t)^{\delta-1}f(u)du, \ \ t < \eta. \tag{6} $$

Here, the Gamma function is represented as $\Gamma(\delta)$. Because the normalizing function $\mathbb{B}(\delta) > 0$ is positive, the fractional *AB*-integral of a positive function must also be positive. It's worth noting that when the order $\delta \to 1$ is used, the classical integral is recovered. When the fractional order $\delta \to 0$, the starting function is also restored.
Motivated by the connection between fractional calculus, convex analysis and optimization theory [38,39], and all the above literature cited, in this work we focus on establishing the Hermite-Hadamard inequalities for the $AB-$fractional integral operator, for which there are important applications. At the end, a brief conclusion is given.

2 Fractional integral inequalities of Hermite-Hadamard type

The Hermite—Hadamard inequalities can be expressed using generalized $\phi-$convex functions and $AB-$fractional integrals as follows.

Theorem 2. *Let $f : [\xi, \xi + \mathbf{E}_\alpha(\eta - \xi)] \to \mathbb{R}$ be a positive function for which ${}^{AB}_{\xi}I^{\delta}_{t}f(t) < \infty$. If f is generalized ϕ-convex function on $[\xi, \xi + \mathbf{E}_\alpha(\eta - \xi)]$, then we have the following double fractional integral inequality*

$$ \begin{aligned} &\frac{2[\mathbf{E}_\alpha(\eta-\xi)]^{\delta}}{\mathbb{B}(\delta)\Gamma(\delta)} f\left(\frac{2\xi + \mathbf{E}_\alpha(\eta-\xi)}{2}\right) + \frac{1-\delta}{\mathbb{B}(\delta)}[f(\xi) + f(\xi + \mathbf{E}_\alpha(\eta-\xi))] \\ &\qquad \le \left[{}^{AB}_{\xi}I^{\delta}_{\xi+\mathbf{E}_\alpha(\eta-\xi)} f(\xi + \mathbf{E}_\alpha(\eta-\xi)) + {}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^{\delta}_{\xi} f(\xi) \right] \\ &\qquad \le \frac{[\mathbf{E}_\alpha(\eta-\xi)]^{\delta}}{\mathbb{B}(\delta)\Gamma(\delta)}[f(\xi) + f(\eta)] + \frac{1-\delta}{\mathbb{B}(\delta)}[f(\xi) + f(\xi + \mathbf{E}_\alpha(\eta-\xi))]. \end{aligned} \tag{7} $$

with $0 < \delta < 1$ and $\alpha > 0$.

Proof. Since f is generalized ϕ-convex function, taking $x = \xi + (1-t)\mathbf{E}_\alpha(\eta - \xi)$, $y = \xi + t\mathbf{E}_\alpha(\eta - \xi)$, we get

$$ 2f\left(\frac{2\xi + \mathbf{E}_\alpha(\eta-\xi)}{2}\right) \le f(\xi + (1-t)\mathbf{E}_\alpha(\eta-\xi)) + f(\xi + t\mathbf{E}_\alpha(\eta-\xi)). \tag{8} $$

If we multiply both sides of the inequality (8) by the factor $\frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}t^{\delta-1}$ and then integrate with respect to t over $[0,1]$, then we obtain

$$ \begin{aligned} &\frac{2}{\mathbb{B}(\delta)\Gamma(\delta)} \ f\left(\frac{2\xi + \mathbf{E}_\alpha(\eta-\xi)}{2}\right) \\ &\quad \le \frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}\int_0^1 t^{\delta-1} f(\xi + (1-t)\mathbf{E}_\alpha(\eta-\xi))dt + \frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}\int_0^1 t^{\delta-1} f(\xi + t\mathbf{E}_\alpha(\eta-\xi))dt \\ &\quad = \frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)[\mathbf{E}_\alpha(\eta-\xi)]^{\delta}}\int_{\xi}^{\xi+\mathbf{E}_\alpha(\eta-\xi)} (\xi + \mathbf{E}_\alpha(\eta-\xi) - u)^{\delta-1} f(u)du \\ &\quad + \frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)[\mathbf{E}_\alpha(\eta-\xi)]^{\delta}}\int_{\xi}^{\xi+\mathbf{E}_\alpha(\eta-\xi)} (u-\xi)^{\delta-1} f(u)du. \end{aligned} $$

Therefore, it follows that

$$\frac{2[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)} f\left(\frac{2\xi+\mathbf{E}_\alpha(\eta-\xi)}{2}\right)+\frac{1-\delta}{\mathbb{B}(\delta)}[f(\xi)+f(\xi+\mathbf{E}_\alpha(\eta-\xi))]$$
$$\leq \frac{1-\delta}{\mathbb{B}(\delta)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))+\frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}\int_\xi^{\xi+\mathbf{E}_\alpha(\eta-\xi)}(\xi+\mathbf{E}_\alpha(\eta-\xi)-u)^{\delta-1}f(u)du$$
$$+\frac{1-\delta}{\mathbb{B}(\delta)}f(\xi)+\frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}\int_\xi^{\xi+\mathbf{E}_\alpha(\eta-\xi)}(u-\xi)^{\delta-1}f(u)du,$$

i.e.

$$\frac{2[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)} f\left(\frac{2\xi+\mathbf{E}_\alpha(\eta-\xi)}{2}\right)+\frac{1-\delta}{\mathbb{B}(\delta)}[f(\xi)+f(\xi+\mathbf{E}_\alpha(\eta-\xi))]$$
$$\leq \left[{}^{AB}_{\ \xi}I^\delta_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^\delta_\xi f(\xi)\right],$$

so the first inequality is proved.
Using the generalized $\phi-$convexity of f, we observe that

$$f(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))\leq tf(\xi)+(1-t)f(\eta)$$

and

$$f(\xi+t\mathbf{E}_\alpha(\eta-\xi))\leq(1-t)f(\xi)+tf(\eta).$$

Adding these inequalities, we get

$$f(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))+f(\xi+t\mathbf{E}_\alpha(\eta-\xi))\leq f(\xi)+f(\eta). \tag{9}$$

Now, in both sides of (9) we multiply by $\frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}t^{\delta-1}$ and integrate with respect to t over $[0,1]$ to obtain

$$\frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}\int_0^1 t^{\delta-1}f(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))dt+\frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}\int_0^1 t^{\delta-1}f(\xi+t\mathbf{E}_\alpha(\eta-\xi))dt$$
$$\leq\frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}[f(\xi)+f(\eta)]\int_0^1 t^{\delta-1}dt$$

i.e.

$$\left[{}^{AB}_{\ \xi}I^\delta_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^\delta_\xi f(\xi)\right]\leq\frac{[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)}[f(\xi)+f(\eta)]+\frac{1-\delta}{\mathbb{B}(\delta)}[f(\xi)+f(\xi+\mathbf{E}_\alpha(\eta-\xi))].$$

The proof has been completed.

Remark. Letting $\rho\to1$ in Theorem 2 we have

$$f\left(\frac{2\xi+\mathbf{E}_\alpha(\eta-\xi)}{2}\right)\leq\frac{1}{2\mathbf{E}_\alpha(\eta-\xi)}\int_\xi^{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(x)dx\leq\frac{f(\xi)+f(\eta)}{2}.$$

3 The *AB*-fractional integral inequalities and generalized ϕ-convex functions

The following Lemma is necessary to establish some new results related to the right side of the Hermite–Hadamard inequality with the $AB-$fractional integral operator and the use of generalized $\phi-$convex functions.

Lemma 1. *Let* $f:[\xi,\xi+\mathbf{E}_\alpha(\eta-\xi)]\to\mathbb{R}$ *be a differentiable mapping on* $(\xi,\xi+\mathbf{E}_\alpha(\eta-\xi))$ *for which* ${}^{AB}_{\ \xi}I^\delta_t f'(t)<\infty$. *Then the following equality for the AB-fractional integrals holds:*

$$\left(\frac{[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)}+\frac{1-\delta}{\mathbb{B}(\delta)}\right)[f(\xi)+f(\xi+\mathbf{E}_\alpha(\eta-\xi))]-\left[{}^{AB}_{\ \xi}I^\delta_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^\delta_\xi f(\xi)\right]$$
$$=\frac{[\mathbf{E}_\alpha(\eta-\xi)]^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)}\int_0^1\left[(1-t)^\delta-t^\delta\right]f'(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))dt, \tag{10}$$

with $0<\delta<1$ *and* $\alpha>0$.

Proof. Integrating by parts, we get

$$I_1 = \int_0^1 (1-t)^\delta \ f' \ (\xi + (1-t)\mathbf{E}_\alpha(\eta-\xi))\,dt$$

$$= -\frac{(1-t)^\delta}{\mathbf{E}_\alpha(\eta-\xi)} f(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))\Big|_0^1$$

$$-\frac{\delta}{\mathbf{E}_\alpha(\eta-\xi)}\int_0^1 (1-t)^{\delta-1} f(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))\,dt$$

$$= \frac{f(\xi+\mathbf{E}_\alpha(\eta-\xi))}{\mathbf{E}_\alpha(\eta-\xi)} - \frac{\delta}{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}\int_\xi^{\xi+\mathbf{E}_\alpha(\eta-\xi)} (x-\xi)^{\delta-1} f(x)dx.$$

Thus, multiplying both sides by $\frac{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)}$, it follows that

$$\frac{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)}\ I_1 \tag{11}$$

$$= \frac{[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)} f(\xi+\mathbf{E}_\alpha(\eta-\xi)) - \frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}\int_\xi^{\xi+\mathbf{E}_\alpha(\eta-\xi)} (x-\xi)^{\delta-1} f(x)dx$$

$$+\frac{(1-\delta)}{\mathbb{B}(\delta)} f(\xi) - \frac{(1-\delta)}{\mathbb{B}(\delta)} f(\xi)$$

$$= \frac{[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)} f(\xi+\mathbf{E}_\alpha(\eta-\xi)) + \frac{(1-\delta)}{\mathbb{B}(\delta)} f(\xi) - {}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^\delta_\xi f(\xi).$$

Similarly,

$$I_2 = \int_0^1 t^\delta \ f' \ (\xi + (1-t)\mathbf{E}_\alpha(\eta-\xi))\,dt$$

$$= -\frac{t^\delta}{\mathbf{E}_\alpha(\eta-\xi)} f(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))\Big|_0^1$$

$$-\frac{\delta}{\mathbf{E}_\alpha(\eta-\xi)}\int_0^1 t^{\delta-1} f(t\xi+(1-t)\xi+\mathbf{E}_\alpha(\eta-\xi))\,dt$$

$$= -\frac{f(\xi)}{\mathbf{E}_\alpha(\eta-\xi)} + \frac{\delta}{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}\int_\xi^{\xi+\mathbf{E}_\alpha(\eta-\xi)} (\xi+\mathbf{E}_\alpha(\eta-\xi)-x)^{\delta-1} f(x)dx.$$

If we multiply both sides by the factor$\frac{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)}$, it follows that

$$\frac{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)}\ I_2 \tag{12}$$

$$= -\frac{[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)} f(\xi) + \frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}\int_\xi^{\xi+\mathbf{E}_\alpha(\eta-\xi)} (\xi+\mathbf{E}_\alpha(\eta-\xi)-x)^{\delta-1} f(x)dx$$

$$+\frac{(1-\delta)}{\mathbb{B}(\delta)} f(\xi+\mathbf{E}_\alpha(\eta-\xi)) - \frac{(1-\delta)}{\mathbb{B}(\delta)} f(\xi+\mathbf{E}_\alpha(\eta-\xi))$$

$$= -\frac{[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)} f(\xi) - \frac{(1-\delta)}{\mathbb{B}(\delta)} f(\xi+\mathbf{E}_\alpha(\eta-\xi)) + {}^{AB}_{\xi}I^\delta_{\xi+\mathbf{E}_\alpha(\eta-\xi)} f(\xi+\mathbf{E}_\alpha(\eta-\xi)).$$

Now, using (11) and (12), we obtain

$$\frac{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)}[I_1-I_2]=\left(\frac{[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)}+\frac{1-\delta}{\mathbb{B}(\delta)}\right)[f(\xi)+f(\xi+\mathbf{E}_\alpha(\eta-\xi))] \\ -\left[{}^{AB}_{\ \xi}I^\delta_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^\delta_\xi f(\xi)\right].$$

The proof has been completed.

With an application of Lemma 1, we can obtain the following result.

Theorem 3. *Let $f:[\xi,\xi+\mathbf{E}_\alpha(\eta-\xi)]\to\mathbb{R}$ be a differentiable mapping on $(\xi,\xi+\mathbf{E}_\alpha(\eta-\xi))$ for which ${}^{AB}_{\ p}I^\delta_t f'(t)<\infty$. If $|f'|^r$ is generalized ϕ-convex function on $[\xi,\xi+\mathbf{E}_\alpha(\eta-\xi)]$ for $r\geq 1$, then we have*

$$\left|\left(\frac{[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)}+\frac{1-\delta}{\mathbb{B}(\delta)}\right)[f(\xi)+f(\xi+\mathbf{E}_\alpha(\eta-\xi))]-\left[{}^{AB}_{\ \xi}I^\delta_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^\delta_\xi f(\xi)\right]\right|$$

$$\leq\frac{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)}\times\begin{cases}\left(1-\frac{1}{2^\delta}\right)\left[\frac{|f'(\xi)|+|f'(\eta)|}{\delta+1}\right], & r=1\\ \left(\frac{2}{\delta s+1}\right)^{\frac{1}{s}}\left(1-\frac{1}{2^{\delta s}}\right)^{\frac{1}{s}}\left(\frac{|f'(\xi)|^r+|f'(\eta)|^r}{2}\right)^{\frac{1}{r}}, & r>1,\ \frac{1}{r}+\frac{1}{s}=1,\end{cases}\tag{13}$$

with $0<\delta<1$ and $\alpha>0$.

Proof. Firs, we suppose that $r=1$. Using Lemma 1 and the generalized ϕ-convexity of $|f'|$, we find

$$\left|\left(\frac{[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)}+\frac{1-\delta}{\mathbb{B}(\delta)}\right)[f(\xi)+f(\xi+\mathbf{E}_\alpha(\eta-\xi))]-\left[{}^{AB}_{\ \xi}I^\delta_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^\delta_\xi f(\xi)\right]\right|$$

$$\leq\frac{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)}\int_0^1\left|(1-t)^\delta-t^\delta\right|\left|f'(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))\right|dt$$
$$\leq\frac{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)}\left(|f'(\xi)|\int_0^1 t|(1-t)^\delta-t^\delta|dt+|f'(\eta)|\int_0^1(1-t)|(1-t)^\delta-t^\delta|dt\right)$$
$$=\frac{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)}[|f'(\xi)|+|f'(\eta)|]\left(1-\frac{1}{2^\delta}\right)\frac{1}{\delta+1}.$$

Here, it is noticeable seen that with simple integral calculation, we have

$$\int_0^1 t|(1-t)^\delta-t^\delta|dt=\int_0^1(1-t)|(1-t)^\delta-t^\delta|dt=\frac{1}{\delta+1}\left(1-\frac{1}{2^\delta}\right).\tag{14}$$

Second, we suppose that $r>1$. An application of Lemma 1, Hölder's inequality and the generalized ϕ-convexity of $|f'|^r$, we find

$$\left|\left(\frac{[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)}+\frac{1-\delta}{\mathbb{B}(\delta)}\right)[f(\xi)+f(\xi+\mathbf{E}_\alpha(\eta-\xi))]-\left[{}^{AB}_{\ \xi}I^\delta_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^\delta_\xi f(\xi)\right]\right|$$

$$\leq\frac{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)}\left(\int_0^1\left|(1-t)^\delta-t^\delta\right|^s dt\right)^{\frac{1}{s}}\left(\int_0^1\left|f'(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))\right|^r dt\right)^{\frac{1}{r}}$$
$$\leq\frac{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)}\left(\int_0^1\left|(1-t)^\delta-t^\delta\right|^s dt\right)^{\frac{1}{s}}\left(\frac{|f'(\xi)|^r+|f'(\eta)|^r}{2}\right)^{\frac{1}{r}}.$$

Using the inequality $(A-B)^s \le A^s - B^s$, with $A \ge B > 0$ and $s > 1$, we get

$$\begin{aligned}\int_0^1 \left|(1-t)^\delta - t^\delta\right|^s dt &= \int_0^{\frac{1}{2}} \left[(1-t)^\delta - t^\delta\right]^s dt + \int_{\frac{1}{2}}^1 \left[t^\delta - (1-t)^\delta\right]^s dt \\ &\le \int_0^{\frac{1}{2}} \left[(1-t)^{\delta s} - t^{\delta s}\right] dt + \int_{\frac{1}{2}}^1 \left[t^{\delta s} - (1-t)^{\delta s}\right] dt \\ &= \frac{2}{\delta s+1}\left(1-\frac{1}{2^{\delta s}}\right). \end{aligned} \tag{15}$$

The proof has been completed.

Corollary 1.
With the same conditions given in the Theorem 3 and if we choose $|f'| \le K$, we have

$$\left|\left(\frac{[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)} + \frac{1-\delta}{\mathbb{B}(\delta)}\right)[f(\xi)+f(\xi+\mathbf{E}_\alpha(\eta-\xi))] - \left[{}^{AB}_{\ \xi}I^\delta_{\xi+\mathbf{E}_\alpha(\eta-\xi)} f(\xi+\mathbf{E}_\alpha(\eta-\xi)) + {}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^\delta_\xi f(\xi)\right]\right|$$

$$\le \frac{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)} \times \begin{cases} \dfrac{2K}{\delta+1}\left(1-\dfrac{1}{2^\delta}\right), & r=1 \\ K\left(\dfrac{2}{\delta s+1}\right)^{\frac{1}{s}}\left(1-\dfrac{1}{2^{\delta s}}\right)^{\frac{1}{s}}, & r>1,\ \frac{1}{r}+\frac{1}{s}=1, \end{cases} \tag{16}$$

with $0<\delta<1$ *and* $\alpha>0$.

Theorem 4. *Let $f : [\xi, \xi+\mathbf{E}_\alpha(\eta-\xi)] \to \mathbb{R}$ be a differentiable mapping on $(\xi, \xi+\mathbf{E}_\alpha(\eta-\xi))$ for which ${}^{AB}_{\ \xi}I^\delta_t f'(t) < \infty$. If $|f'|^r$ is generalized ϕ-convex function on $[\xi, \xi+\mathbf{E}_\alpha(\eta-\xi)]$ for $r \ge 1$, then we have*

$$\left|\left(\frac{[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)} + \frac{1-\delta}{\mathbb{B}(\delta)}\right)[f(\xi)+f(\xi+\mathbf{E}_\alpha(\eta-\xi))] - \left[{}^{AB}_{\ \xi}I^\delta_{\xi+\mathbf{E}_\alpha(\eta-\xi)} f(\xi+\mathbf{E}_\alpha(\eta-\xi)) + {}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^\delta_\xi f(\xi)\right]\right|$$

$$\le 2\frac{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)} 2^{1-1/r}\left(\frac{1-2^{-\delta}}{\delta+1}\right)\left(|f'(\xi)|^r + |f'(\eta)|^r\right)^{\frac{1}{r}},$$

with $0<\delta<1$ *and* $\alpha>0$.

Proof. Let $r \ge 1$. An application of Lemma 1, the power mean inequality and the generalized ϕ-convexity of $|f'|^r$, we find

$$\left|\left(\frac{[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)} + \frac{1-\delta}{\mathbb{B}(\delta)}\right)[f(\xi)+f(\xi+\mathbf{E}_\alpha(\eta-\xi))] - \left[{}^{AB}_{\ \xi}I^\delta_{\xi+\mathbf{E}_\alpha(\eta-\xi)} f(\xi+\mathbf{E}_\alpha(\eta-\xi)) + {}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^\delta_\xi f(\xi)\right]\right|$$

$$\begin{aligned} &\le \frac{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)}\left(\int_0^1 \left|(1-t)^\delta - t^\delta\right| dt\right)^{1-\frac{1}{r}} \left(\int_0^1 \left|(1-t)^\delta - t^\delta\right| \left|f'(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))\right|^r dt\right)^{\frac{1}{r}} \\ &\le \frac{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)}\left(\int_0^1 \left|(1-t)^\delta - t^\delta\right| dt\right)^{1-\frac{1}{r}} \times \\ &\qquad \left(|f'(\xi)|^r \int_0^1 (1-t)\left|(1-t)^\delta - t^\delta\right| dt + |f'(\eta)|^r \int_0^1 t\left|(1-t)^\delta - t^\delta\right| dt\right)^{\frac{1}{r}}. \end{aligned} \tag{17}$$

Now we can observe that

$$\begin{aligned}\int_0^1 \left|(1-t)^\delta - t^\delta\right| dt &= \int_0^{1/2} ((1-t)^\delta - t^\delta) dt + \int_{1/2}^1 (t^\delta - (1-t)^\delta) dt \\ &= \frac{2(1-2^{-\delta})}{\delta+1}. \end{aligned} \tag{18}$$

Using this value and (14) (in proof Theorem 3), we can make the replacement in (17) to obtain

$$\left|\left(\frac{[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)}+\frac{1-\delta}{\mathbb{B}(\delta)}\right)[f(\xi)+f(\xi+\mathbf{E}_\alpha(\eta-\xi))]-\left[{}^{AB}_{\ \xi}I^{\delta}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^{\delta}_{\xi}f(\xi)\right]\right|$$

$$\leq\frac{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)}2^{1-1/r}\left(\frac{1-2^{-\delta}}{\delta+1}\right)\left(|f'(\xi)|^r+|f'(\eta)|^r\right)^{\frac{1}{r}}.$$

The desired result has been obtained.

Corollary 2. *With the same conditions given in Theorem 4, if we choose $|f'|\leq K$, we have*

$$\left|\left(\frac{[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)}+\frac{1-\delta}{\mathbb{B}(\delta)}\right)[f(\xi)+f(\xi+\mathbf{E}_\alpha(\eta-\xi))]-\left[{}^{AB}_{\ \xi}I^{\delta}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^{\delta}_{\xi}f(\xi)\right]\right|$$

$$\leq\frac{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)}K2^{1-1/r}\left(\frac{1-2^{-\delta}}{\delta+1}\right),$$

with $0<\delta<1$ and $\alpha>0$.

Theorem 5. *Let f,g be two real valued, non-negative and generalized ϕ-convex functions defined on $[\xi,\xi+\mathbf{E}_\alpha(\eta-\xi)]$, where $\mathbf{E}_\alpha(\eta-\xi)>0$. Then*

$$\left[{}^{AB}_{\ \xi}I^{\delta}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))g(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^{\delta}_{\xi}f(\xi)g(\xi)\right]$$

$$\leq\left(\frac{1-\delta}{\mathbb{B}(\delta)}+\frac{(\delta^2+\delta+2)\,[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)(\delta+1)(\delta+2)}\right)M(\xi,\eta)+\frac{2\delta[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)(\delta+1)(\delta+2)}N(\xi,\eta),$$

where $0<\delta<1$, $\alpha>0$, and

$$M(\xi,\eta)=f(\xi)g(\xi)+f(\eta)g(\eta),\quad N(\xi,\eta)=f(\xi)g(\eta)+f(\eta)g(\xi).$$

Proof. Using the generalized $\phi-$convexity of f and g we have

$$f(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))\leq tf(\xi)+(1-t)f(\eta) \tag{19}$$

and

$$g(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))\leq tg(\xi)+(1-t)g(\eta). \tag{20}$$

Based in (19) and (20), we get

$$f(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))g(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))$$

$$\leq t^2f(\xi)g(\xi)+(1-t)^2f(\eta)g(\eta)+t(1-t)[f(\xi)g(\eta)+f(\eta)g(\xi)].$$

Similarly,

$$f(\xi+t\mathbf{E}_\alpha(\eta-\xi))g(\xi+t\mathbf{E}_\alpha(\eta-\xi))$$

$$\leq(1-t)^2f(\xi)g(\xi)+t^2f(\eta)g(\eta)+t(1-t)[f(\xi)g(\eta)+f(\eta)g(\xi)].$$

By means of the sum of the two previous inequalities we obtain that

$$f(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))g(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))+f(\xi+t\mathbf{E}_\alpha(\eta-\xi))g(\xi+t\mathbf{E}_\alpha(\eta-\xi))$$

$$\leq(2t^2-2t+1)[f(\xi)g(\xi)+f(\eta)g(\eta)]+2t(1-t)[f(\xi)g(\eta)+f(\eta)g(\xi)].$$

If we multiply both sides of previous inequality by $\frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}t^{\delta-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, then we obtain

$$\begin{aligned}
&\frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}\int_0^1 t^{\delta-1}f(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))g(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))dt\\
&\qquad+\frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}\int_0^1 t^{\delta-1}f(\xi+t\mathbf{E}_\alpha(\eta-\xi))g(\xi+t\mathbf{E}_\alpha(\eta-\xi))dt\\
&\le\frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}\int_0^1 t^{\delta-1}(2t^2-2t+1)[f(\xi)g(\xi)+f(\eta)g(\eta)]dt\\
&\qquad+\frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}\int_0^1 t^{\delta-1}2t(1-t)[f(\xi)g(\eta)+f(\eta)g(\xi)]dt.
\end{aligned}$$

With an appropriate choice of variables and a simple integration, we get

$$\begin{aligned}
&\frac{1-\delta}{\mathbb{B}(\delta)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))g(\xi+\mathbf{E}_\alpha(\eta-\xi))\\
&\qquad+\frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}\int_\xi^{\xi+\mathbf{E}_\alpha(\eta-\xi)}(\xi+\mathbf{E}_\alpha(\eta-\xi)-u)^{\delta-1}f(u)g(u)du\\
&\qquad\qquad+\frac{1-\delta}{\mathbb{B}(\delta)}f(\xi)g(\xi)+\frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}\int_\xi^{\xi+\mathbf{E}_\alpha(\eta-\xi)}(u-\xi)^{\delta-1}f(u)g(u)du\\
&\le\left(\frac{1-\delta}{\mathbb{B}(\delta)}+\frac{\delta^2+\delta+2}{\mathbb{B}(\delta)\Gamma(\delta)(\delta+1)(\delta+2)}[\mathbf{E}_\alpha(\eta-\xi)]^\delta\right)[f(\xi)g(\xi)+f(\eta)g(\eta)]\\
&\qquad+\frac{2\delta[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)(\delta+1)(\delta+2)}[f(\xi)g(\eta)+f(\eta)g(\xi)].
\end{aligned}$$

As a result, the next expression follows

$$\begin{aligned}
&\left[{}^{AB}_{\ \xi}I^\delta_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))g(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{\quad AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^\delta_\xi f(\xi)g(\xi)\right]\\
&\le\left(\frac{1-\delta}{\mathbb{B}(\delta)}+\frac{\delta^2+\delta+2}{\mathbb{B}(\delta)\Gamma(\delta)(\delta+1)(\delta+2)}[\mathbf{E}_\alpha(\eta-\xi)]^\delta\right)M(\xi,\eta)+\frac{2\delta[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)(\delta+1)(\delta+2)}N(\xi,\eta).
\end{aligned}$$

The proof has been completed.

Corollary 3. *With the same conditions given in Theorem 5, if we choose $f=g$, we have*

$$\begin{aligned}
&\left[{}^{AB}_{\ \xi}I^\delta_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f^2(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{\quad AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^\delta_\xi f^2(\xi)\right]\\
&\le\left(\frac{1-\delta}{\mathbb{B}(\delta)}+\frac{(\delta^2+\delta+2)[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)(\delta+1)(\delta+2)}\right)M_1(\xi,\eta)+\frac{2\delta[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)(\delta+1)(\delta+2)}N_1(\xi,\eta),
\end{aligned}$$

where $0<\delta<1$, $\alpha>0$,

$$M_1(\xi,\eta)=f^2(\xi)+f^2(\eta),\quad N_1(\xi,\eta)=2f(\xi)f(\eta).$$

Corollary 4. *With the same conditions given in Theorem 5, if we choose $g:[\xi,\xi+\mathbf{E}_\alpha(\eta-\xi)]\to R$ as $g(t)\equiv 1$ for all $t\in[\xi,\xi+\mathbf{E}_\alpha(\eta-\xi)]$, we have*

$$\left[{}^{AB}_{\ \xi}I^\delta_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{\quad AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^\delta_\xi f(\xi)\right]\le\left(\frac{1-\delta}{\mathbb{B}(\delta)}+\frac{[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)}\right)[f(\xi)+f(\eta)]$$

which is the right hand side of (7).

Corollary 5. *With the same conditions given in Theorem 5, if we choose* $g:[\xi,\xi+\mathbf{E}_\alpha(\eta-\xi)]\to\mathbb{R}$ *is integrable and symmetric to* $\frac{2\xi+\mathbf{E}_\alpha(\eta-\xi)}{2}$, *then*

$$\left[{}^{AB}_{\xi}I^{\delta}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))g(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^{\delta}_{\xi}f(\xi)g(\xi)\right]$$
$$\leq\frac{1}{2}\left(\frac{1-\delta}{\mathbb{B}(\delta)}+\frac{[\mathbf{E}_\alpha(\eta-\xi)]^{\delta}}{\mathbb{B}(\delta)\Gamma(\delta)}\right)[f(\xi)+f(\eta)][g(\xi)+g(\eta)]. \tag{21}$$

Proof. Since g is symmetric to $(2\xi+\mathbf{E}_\alpha(\eta-\xi))/2$, we have $g(2\xi+\mathbf{E}_\alpha(\eta-\xi)-t)=g(t)$ for all $t\in[\xi,\xi+\mathbf{E}_\alpha(\eta-\xi)]$. Hence

$$\left[{}^{AB}_{\xi}I^{\delta}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))g(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^{\delta}_{\xi}f(\xi)g(\xi)\right]$$
$$\leq\left(\frac{1-\delta}{\mathbb{B}(\delta)}+\frac{[\mathbf{E}_\alpha(\eta-\xi)]^{\delta}}{\mathbb{B}(\delta)\Gamma(\delta)}\right)[f(\xi)+f(\eta)]g(\xi).$$

Similarly, we get

$$\left[{}^{AB}_{\xi}I^{\delta}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))g(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^{\delta}_{\xi}f(\xi)g(\xi)\right]$$
$$\leq\left(\frac{1-\delta}{\mathbb{B}(\delta)}+\frac{[\mathbf{E}_\alpha(\eta-\xi)]^{\delta}}{\mathbb{B}(\delta)\Gamma(\delta)}\right)[f(\xi)+f(\eta)]g(\eta).$$

By adding these inequalities, the proof has been completed.

Remark. If in Corollary 5, we let $\delta\to1$, then the inequality (21) becomes

$$\frac{1}{\mathbf{E}_\alpha(\eta-\xi)}\int_{\xi}^{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(u)g(u)du\leq\left(\frac{f(\xi)+f(\eta)}{2}\right)\left(\frac{g(\xi)+g(\eta)}{2}\right). \tag{22}$$

Theorem 6. *Let* f,g *be two real valued, non-negative and generalized* ϕ*-convex functions on* $[\xi,\xi+\mathbf{E}_\alpha(\eta-\xi)]$, *where* $\mathbf{E}_\alpha(\eta-\xi)>0$. *Then*

$$\frac{1-\delta}{\mathbb{B}(\delta)}M(\xi,\eta)+\frac{4[\mathbf{E}_\alpha(\eta-\xi)]^{\delta}}{\mathbb{B}(\delta)\Gamma(\delta)}f\left(\frac{2\xi+\mathbf{E}_\alpha(\eta-\xi)}{2}\right)g\left(\frac{2\xi+\mathbf{E}_\alpha(\eta-\xi)}{2}\right)$$
$$\leq\left[{}^{AB}_{\xi}I^{\delta}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))g(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^{\delta}_{\xi}f(\xi)g(\xi)\right]$$
$$+\frac{[\mathbf{E}_\alpha(\eta-\xi)]^{\delta}\left[2\delta M(\xi,\eta)+(\delta^2+\delta+2)N(\xi,\eta)\right]}{\mathbb{B}(\delta)\Gamma(\delta)(\delta+1)(\delta+2)},$$

where $0<\delta<1$, $\alpha>0$, *and* $M(\xi,\eta),N(\xi,\eta)$ *are defined as in Theorem 5.*

Proof. Using the generalized $\phi-$convexity of f and g we have

$$f\left(\frac{2\xi+\mathbf{E}_\alpha(\eta-\xi)}{2}\right)=f\left(\frac{\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi)+\xi+t\mathbf{E}_\alpha(\eta-\xi)}{2}\right)$$
$$\leq\frac{1}{2}f(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))+\frac{1}{2}f(\xi+t\mathbf{E}_\alpha(\eta-\xi)) \tag{23}$$

and

$$g\left(\frac{2\xi+\mathbf{E}_\alpha(\eta-\xi)}{2}\right)=g\left(\frac{\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi)+\xi+t\mathbf{E}_\alpha(\eta-\xi)}{2}\right)$$
$$\leq\frac{1}{2}g(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))+\frac{1}{2}g(\xi+t\mathbf{E}_\alpha(\eta-\xi)). \tag{24}$$

From (23) and (24), we get

$$f\left(\frac{2\xi+\mathbf{E}_\alpha(\eta-\xi)}{2}\right)g\left(\frac{2\xi+\mathbf{E}_\alpha(\eta-\xi)}{2}\right)$$

$$\leq \frac{1}{4}\left[f(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))+f(\xi+t\mathbf{E}_\alpha(\eta-\xi))\right]\left[g(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))+g(\xi+t\mathbf{E}_\alpha(\eta-\xi))\right]$$

$$=\frac{1}{4}\{f(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))g(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))+f(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))g(\xi+t\mathbf{E}_\alpha(\eta-\xi))$$

$$+f(\xi+t\mathbf{E}_\alpha(\eta-\xi))g(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))+f(\xi+t\mathbf{E}_\alpha(\eta-\xi))g(\xi+t\mathbf{E}_\alpha(\eta-\xi))\}$$

$$\leq \frac{1}{4}\{f(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))g(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))+f(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))g(\xi+t\mathbf{E}_\alpha(\eta-\xi))$$

$$+[(1-t)f(\xi)+tf(\eta)][tg(\xi)+(1-t)g(\eta)]+[(1-t)f(\xi)+tf(\eta)][(1-t)g(\xi)+tg(\eta)]\}$$

$$=\frac{1}{4}\{f(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))g(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))+f(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))g(\xi+t\mathbf{E}_\alpha(\eta-\xi))$$

$$+2(1-t)tM(\xi,\eta)+\left((1-t)^2+t^2\right)N(\xi,\eta)\}.$$

If we multiply both sides of previous inequality by $\frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}t^{\delta-1}$ and integrate with respect to t over $[0,1]$, we obtain

$$\frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}f\left(\frac{2\xi+\mathbf{E}_\alpha(\eta-\xi)}{2}\right)g\left(\frac{2\xi+\mathbf{E}_\alpha(\eta-\xi)}{2}\right)\int_0^1 t^{\delta-1}dt$$

$$\leq \frac{\delta}{4\mathbb{B}(\delta)\Gamma(\delta)}\int_0^1 t^{\delta-1}f(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))g(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))dt$$

$$+\frac{\delta}{4\mathbb{B}(\delta)\Gamma(\delta)}\int_0^1 t^{\delta-1}f(\xi+t\mathbf{E}_\alpha(\eta-\xi))g(\xi+t\mathbf{E}_\alpha(\eta-\xi))dt$$

$$+\frac{\delta M(\xi,\eta)}{2\mathbb{B}(\delta)\Gamma(\delta)}\int_0^1 t^{\delta}(1-t)dt+\frac{\delta N(\xi,\eta)}{4\mathbb{B}(\delta)\Gamma(\delta)}\int_0^1 t^{\delta-1}\left((1-t)^2+t^2\right)dt.$$

With a suitable choice of variables for a simple integral calculations, we get

$$\frac{1-\delta}{\mathbb{B}(\delta)}M(\xi,\eta)+\frac{4[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)}f\left(\frac{2\xi+\mathbf{E}_\alpha(\eta-\xi)}{2}\right)g\left(\frac{2\xi+\mathbf{E}_\alpha(\eta-\xi)}{2}\right)$$

$$\leq \frac{1-\delta}{\mathbb{B}(\delta)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))g(\xi+\mathbf{E}_\alpha(\eta-\xi))+\frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}\int_\xi^{\xi+\mathbf{E}_\alpha(\eta-\xi)}(\xi+\mathbf{E}_\alpha(\eta-\xi)-u)^{\delta-1}f(u)g(u)du$$

$$+\frac{1-\delta}{\mathbb{B}(\delta)}f(\xi)g(\xi)+\frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}\int_\xi^{\xi+\mathbf{E}_\alpha(\eta-\xi)}(u-\xi)^{\delta-1}f(u)g(u)du$$

$$+\frac{2\delta[\mathbf{E}_\alpha(\eta-\xi)]^\delta M(\xi,\eta)}{\mathbb{B}(\delta)\Gamma(\delta)(\delta+1)(\delta+2)}+\frac{\delta^2+\delta+2}{\mathbb{B}(\delta)\Gamma(\delta)(\delta+1)(\delta+2)}[\mathbf{E}_\alpha(\eta-\xi)]^\delta N(\xi,\eta).$$

Then we obtain the following the AB-fractional integral inequality:

$$\frac{1-\delta}{\mathbb{B}(\delta)}M(\xi,\eta)+\frac{4[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)}f\left(\frac{2\xi+\mathbf{E}_\alpha(\eta-\xi)}{2}\right)g\left(\frac{2\xi+\mathbf{E}_\alpha(\eta-\xi)}{2}\right)$$

$$\leq \left[{}^{AB}_{\xi}I^{\delta}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))g(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^{\delta}_{\xi}f(\xi)g(\xi)\right]$$

$$\frac{2\delta[\mathbf{E}_\alpha(\eta-\xi)]^\delta M(\xi,\eta)}{\mathbb{B}(\delta)\Gamma(\delta)(\delta+1)(\delta+2)}+\frac{\delta^2+\delta+2}{\mathbb{B}(\delta)\Gamma(\delta)(\delta+1)(\delta+2)}[\mathbf{E}_\alpha(\eta-\xi)]^\delta N(\xi,\eta).$$

The proof of this theorem is complete.

Corollary 6. *With the conditions given in Theorem 6, if we choose $f = g$, we have*

$$\frac{1-\delta}{\mathbb{B}(\delta)}M_1(\xi,\eta)+\frac{4[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)}f^2\left(\frac{2\xi+\mathbf{E}_\alpha(\eta-\xi)}{2}\right)$$
$$\leq\left[{}^{AB}_{\xi}I^{\delta}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f^2(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^{\delta}_{\xi}f^2(\xi)\right]$$
$$+\frac{[\mathbf{E}_\alpha(\eta-\xi)]^\delta\left[2\delta M_1(\xi,\eta)+\left(\delta^2+\delta+2\right)N_1(\xi,\eta)\right]}{\mathbb{B}(\delta)\Gamma(\delta)(\delta+1)(\delta+2)},$$

where $0<\delta<1$, $\alpha>0$, and $M_1(\xi,\eta)$, $N_1(\xi,\eta)$ are defined as in Corollary 3.

Corollary 7. *With the conditions given in Theorem 6, if we choose $g:[\xi,\xi+\mathbf{E}_\alpha(\eta-\xi)]\to R$ as $g(t)\equiv 1$ for all $t\in[\xi,\xi+\mathbf{E}_\alpha(\eta-\xi)]$, we have*

$$\frac{1-\delta}{\mathbb{B}(\delta)}[f(\xi)+f(\eta)]+\frac{4[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)}f\left(\frac{2\xi+\mathbf{E}_\alpha(\eta-\xi)}{2}\right)$$
$$\leq\left[{}^{AB}_{\xi}I^{\delta}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^{\delta}_{\xi}f(\xi)\right]+\frac{[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)}[f(\xi)+f(\eta)].$$

4 The *AB*-fractional inequalities for generalized quasi ϕ-convex functions

In this section we introduce the following definition.

Definition 7. *The function f is said to be generalized quasi ϕ-convex, if*

$$f(\xi+t\mathbf{E}_\alpha(\eta-\xi))\leq\max\{f(\xi),f(\eta)\},\quad\forall\,\xi,\eta\in\mathscr{K},\quad t\in[0,1].\tag{25}$$

We establish some inequalities of Hermite-Hadamard type using generalized quasi ϕ-convex functions.

Theorem 7. *Let $f:[\xi,\xi+\mathbf{E}_\alpha(\eta-\xi)]\to\mathbb{R}$ be a differentiable mapping on $(\xi,\xi+\mathbf{E}_\alpha(\eta-\xi))$ for which ${}^{AB}_{\xi}I^{\delta}_{t}f'(t)<\infty$. If $|f'|^r$ is generalized quasi ϕ-convex function on $[\xi,\xi+\mathbf{E}_\alpha(\eta-\xi)]$ for $r\geq 1$, then the following inequality for the AB-fractional integrals holds:*

$$\left|\left(\frac{[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)}+\frac{1-\delta}{\mathbb{B}(\delta)}\right)[f(\xi)+f(\xi+\mathbf{E}_\alpha(\eta-\xi))]-\left[{}^{AB}_{\xi}I^{\delta}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^{\delta}_{\xi}f(\xi)\right]\right|$$
$$\leq\frac{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)}\times\begin{cases}2\left(1-\frac{1}{2^\delta}\right)\left[\frac{\max\{|f'(\xi)|,|f'(\eta)|\}}{\delta+1}\right], & r=1\\ \left(\frac{2}{\delta s+1}\right)^{\frac{1}{s}}\left(1-\frac{1}{2^{\delta s}}\right)^{\frac{1}{s}}\max\{|f'(\xi)|,|f'(\eta)|\}, & r>1,\ \frac{1}{r}+\frac{1}{s}=1,\end{cases}\tag{26}$$

with $0<\delta<1$ and $\alpha>0$.

Proof. Let $r=1$. An application of Lemma 1 and the generalized quasi ϕ-convexity of $|f'|$, we find

$$\left|\left(\frac{[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)}+\frac{1-\delta}{\mathbb{B}(\delta)}\right)[f(\xi)+f(\xi+\mathbf{E}_\alpha(\eta-\xi))]-\left[{}^{AB}_{\xi}I^{\delta}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^{\delta}_{\xi}f(\xi)\right]\right|$$
$$\leq\frac{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)}\int_0^1\left|(1-t)^\delta-t^\delta\right|\left|f'(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))\right|dt$$
$$\leq\frac{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)}\max\{f'(\xi),f'(\eta)\}\int_0^1|(1-t)^\delta-t^\delta|dt$$
$$=\frac{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)}2\max\{f'(\xi),f'(\eta)\}\left(1-\frac{1}{2^\delta}\right)\frac{1}{\delta+1},$$

here it was used (18).

Now, let $r>1$. Making use of Lemma 1, Hölder's inequality and the generalized quasi ϕ-convexity of $|f'|^r$, we find

$$\left|\left(\frac{[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)}+\frac{1-\delta}{\mathbb{B}(\delta)}\right)[f(\xi)+f(\xi+\mathbf{E}_\alpha(\eta-\xi))]-\left[{}^{AB}_{\xi}I^{\delta}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^{\delta}_{\xi}f(\xi)\right]\right|$$

$$\leq \frac{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)}\left(\int_0^1\left|(1-t)^\delta-t^\delta\right|^s dt\right)^{\frac{1}{s}}\left(\int_0^1\left|f'(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))\right|^r dt\right)^{\frac{1}{r}}$$

$$\leq \frac{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)}\left(\int_0^1\left|(1-t)^\delta-t^\delta\right|^s dt\right)^{\frac{1}{s}}\left(\max\{|f'(\xi)|,|f'(\eta)|\}\right).$$

As in the proof of Theorem 3 we use (15) to complete the proof.

Corollary 8. *With the notations in Theorem 7, if we choose $|f'|\leq K$, we have*

$$\left|\left(\frac{[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)}+\frac{1-\delta}{\mathbb{B}(\delta)}\right)[f(\xi)+f(\xi+\mathbf{E}_\alpha(\eta-\xi))]-\left[{}^{AB}_{\xi}I^{\delta}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^{\delta}_{\xi}f(\xi)\right]\right|$$

$$\leq \frac{(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)}\times\begin{cases}\frac{2K}{\delta+1}\left(1-\frac{1}{2^\delta}\right), & r=1\\ K\left(\frac{2}{\delta s+1}\right)^{\frac{1}{s}}\left(1-\frac{1}{2^{\delta s}}\right)^{\frac{1}{s}}, & r>1,\ \frac{1}{r}+\frac{1}{s}=1,\end{cases}\tag{27}$$

with $0<\delta<1$ and $\alpha>0$.

Theorem 8. *Let $f:[\xi,\xi+\mathbf{E}_\alpha(\eta-\xi)]\to\mathbb{R}$ be a differentiable mapping on $(\xi,\xi+\mathbf{E}_\alpha(\eta-\xi))$ for which ${}^{AB}_{\xi}I^{\delta}_{t}f'(t)<\infty$. If $|f'|^r$ is generalized quasi ϕ-convex function on $[\xi,\xi+\mathbf{E}_\alpha(\eta-\xi)]$ for $r\geq 1$, then the following inequality for the AB-fractional integrals holds:*

$$\left|\left(\frac{[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)}+\frac{1-\delta}{\mathbb{B}(\delta)}\right)[f(\xi)+f(\xi+\mathbf{E}_\alpha(\eta-\xi))]-\left[{}^{AB}_{\xi}I^{\delta}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^{\delta}_{\xi}f(\xi)\right]\right|$$

$$\leq \frac{2(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)}\left(\frac{1-2^{-\delta}}{\delta+1}\right)\max\{|f'(\xi)|,|f'(\eta)|\},$$

with $0<\delta<1$ and $\alpha>0$.

Proof. As in the proof of Theorem 4, the expression in (17) will have the following form

$$\left(\int_0^1\left|(1-t)^\delta-t^\delta\right|\left|f'(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))\right|^r dt\right)^{\frac{1}{r}}\leq\max\{|f'(\xi)|,|f'(\eta)|\}\left(\int_0^1\left|(1-t)^\delta-t^\delta\right|dt\right)^{\frac{1}{r}}$$
$$=\max\{|f'(\xi)|,|f'(\eta)|\}\left(\frac{2(1-2^{-\delta})}{\delta+1}\right)^{1/r},$$

so, by a corresponding replacement we attain the desired result.

Corollary 9. *With the conditions given in Theorem 8, if we choose $|f'|\leq K$, we have*

$$\left|\left(\frac{[\mathbf{E}_\alpha(\eta-\xi)]^\delta}{\mathbb{B}(\delta)\Gamma(\delta)}+\frac{1-\delta}{\mathbb{B}(\delta)}\right)[f(\xi)+f(\xi+\mathbf{E}_\alpha(\eta-\xi))]-\left[{}^{AB}_{\xi}I^{\delta}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^{\delta}_{\xi}f(\xi)\right]\right|$$

$$\leq\frac{2K(\mathbf{E}_\alpha(\eta-\xi))^{\delta+1}}{\mathbb{B}(\delta)\Gamma(\delta)}\left(\frac{1-2^{-\delta}}{\delta+1}\right),$$

with $0<\delta<1$ and $\alpha>0$.

Theorem 9. *Let f,g be two real valued, non-negative and generalized quasi ϕ-convex functions on $[\xi,\xi+\mathbf{E}_\alpha(\eta-\xi)]$, where $\mathbf{E}_\alpha(\eta-\xi)>0$. Then*

$$\left[{}^{AB}_{\ \xi}I^{\delta}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))g(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^{\delta}_{\xi}f(\xi)g(\xi)\right]$$
$$\leq\frac{1-\delta}{\mathbb{B}(\delta)}N(\xi,\eta)+\frac{[\mathbf{E}_\alpha(\eta-\xi)]^{\delta}}{\mathbb{B}(\delta)\Gamma(\delta)}2\max\{f(\xi),f(\eta)\}\max\{g(\xi),g(\eta)\}.$$

where

$$N(\xi,\eta)=f(\xi+\mathbf{E}_\alpha(\eta-\xi))g(\xi+\mathbf{E}_\alpha(\eta-\xi))+f(\xi)g(\xi)$$

Proof. Since f and g are generalized ϕ-convex functions on $[\xi,\xi+\mathbf{E}_\alpha(\eta-\xi)]$, then

$$f(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))\leq\max\{f(\xi),f(\eta)\}\tag{28}$$

and

$$g(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))\leq\max\{g(\xi),g(\eta)\}.\tag{29}$$

From (28) and (29), we get

$$f(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))g(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))\leq\max\{f(\xi),f(\eta)\}\max\{g(\xi),g(\eta)\}.$$

Similarly,

$$f(\xi+t\mathbf{E}_\alpha(\eta-\xi))g(\xi+t\mathbf{E}_\alpha(\eta-\xi))\leq\max\{f(\xi),f(\eta)\}\max\{g(\xi),g(\eta)\}.$$

Adding the above two inequalities, it follows that

$$f(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))g(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))+f(\xi+t\mathbf{E}_\alpha(\eta-\xi))g(\xi+t\mathbf{E}_\alpha(\eta-\xi))$$
$$\leq 2\max\{f(\xi),f(\eta)\}\max\{g(\xi),g(\eta)\}.$$

If we multiply both sides of the previous inequality by $\frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}t^{\delta-1}$ and integrate with respect to t over $[0,1]$, we obtain

$$\frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}\int_0^1 t^{\delta-1}f(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))g(\xi+(1-t)\mathbf{E}_\alpha(\eta-\xi))dt$$
$$+\frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}\int_0^1 t^{\delta-1}f(\xi+t\mathbf{E}_\alpha(\eta-\xi))g(\xi+t\mathbf{E}_\alpha(\eta-\xi))dt$$
$$\leq\frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}2\max\{f(\xi),f(\eta)\}\max\{g(\xi),g(\eta)\}\int_0^1 t^{\delta-1}dt$$

With a suitable choose of variables for a simple integral calculations, we get

$$\frac{1-\delta}{\mathbb{B}(\delta)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))g(\xi+\mathbf{E}_\alpha(\eta-\xi))$$
$$+\frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}\int_\xi^{\xi+\mathbf{E}_\alpha(\eta-\xi)}(\xi+\mathbf{E}_\alpha(\eta-\xi)-u)^{\delta-1}f(u)g(u)du$$
$$+\frac{1-\delta}{\mathbb{B}(\delta)}f(\xi)g(\xi)+\frac{\delta}{\mathbb{B}(\delta)\Gamma(\delta)}\int_\xi^{\xi+\mathbf{E}_\alpha(\eta-\xi)}(u-\xi)^{\delta-1}f(u)g(u)du$$
$$\leq\frac{1-\delta}{\mathbb{B}(\delta)}N(\xi,\eta)+\frac{[E_\alpha(\eta-\xi)]^{\delta}}{\mathbb{B}(\delta)\Gamma(\delta)}2\max\{f(\xi),f(\eta)\}\max\{g(\xi),g(\eta)\},$$

where

$$N(\xi,\eta)=f(\xi+\mathbf{E}_\alpha(\eta-\xi))g(\xi+\mathbf{E}_\alpha(\eta-\xi))+f(\xi)g(\xi)$$

Then we obtain the following the *AB*-fractional integral inequality:

$$\left[{}^{AB}_{\ \xi}I^{\delta}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi))g(\xi+\mathbf{E}_\alpha(\eta-\xi))+{}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^{\delta}_{\xi}f(\xi)g(\xi)\right]$$
$$\leq\frac{1-\delta}{\mathbb{B}(\delta)}N(\xi,\eta)+\frac{[\mathbf{E}_\alpha(\eta-\xi)]^{\delta}}{\mathbb{B}(\delta)\Gamma(\delta)}2\max\{f(\xi),f(\eta)\}\max\{g(\xi),g(\eta)\}.$$

The proof has been completed.

Remark. If in Theorem 9 we have $f = g$ then

$$\left[{}^{AB}_{\xi}I^{\delta}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f^2(\xi+\mathbf{E}_\alpha(\eta-\xi)) + {}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^{\delta}_{\xi}f^2(\xi)\right] \leq \frac{1-\delta}{\mathbb{B}(\delta)}\left(f^2\left(\xi+\mathbf{E}_\alpha(\eta-\xi)\right)+f^2(\xi)\right) + \frac{2\left[\mathbf{E}_\alpha(\eta-\xi)\right]^\delta\left(\max\{f(\xi),f(\eta)\}\right)^2}{\mathbb{B}(\delta)\Gamma(\delta)}$$

and if $g \equiv 1$ then

$$\left[{}^{AB}_{\xi}I^{\delta}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}f(\xi+\mathbf{E}_\alpha(\eta-\xi)) + {}^{AB}_{\xi+\mathbf{E}_\alpha(\eta-\xi)}I^{\delta}_{\xi}f(\xi)\right] \leq \frac{1-\delta}{\mathbb{B}(\delta)}\left(f\left(\xi+\mathbf{E}_\alpha(\eta-\xi)\right)+f(\xi)\right) + \frac{2\left[\mathbf{E}_\alpha(\eta-\xi)\right]^\delta\max\{f(\xi),f(\eta)\}}{\mathbb{B}(\delta)\Gamma(\delta)}.$$

5 Conclusion

The so-called generalized ϕ-convex functions can be applied to obtain several results in convex analysis as well as related optimization theory and may stimulate further research in different areas of pure and applied sciences. We have established the Hermite-Hadamard type inequality for generalized $\phi-$convex functions using the $AB-$fractional integral operator and others related to the right side inequality of the aforementioned. The concept of generalized quasi $\phi-$convex functions was introduced and some fractional integral inequalities were also established for the type of functions under study.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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