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Parameter Estimation for the Power Generalized Weibull Distribution Based on One- and Two-Stage Ranked Set Sampling Designs

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Abstract: Parameter estimation Based on Double Ranked Set Sampling (DRSS) designs was recently developed by Sabry et al., (2019) and shows high efficiency and precision of the likelihood estimators when applied to the two-parameter Weibull distribution. In this paper the likelihood function of the General Double Ranked Set Sampling (GDRSS) design discussed by Taconeli & Cabral (2019) is derived and the two double ranked set sampling designs DRSS and GDRSS are compared along with the usual Ranked Set Sampling (RSS) and Extreme Ranked Set Sampling (ERSS) designs for the estimation of the parameters of the Power Generalized Weibull (PGW) distribution which is an extension of the two parameter Weibull distribution. An intensive simulation has been made to compare the one- and the two- stages designs. The results show that likelihood estimation based on DRSS and GDRSS designs provide more efficient estimators than the usual RSS and ERSS designs. Moreover, the GDRSS is slightly more efficient that the DRSS designs in the case of estimating the PGW distribution.

Keywords: Simple Random Sampling, Ranked Set Sampling, Extreme Ranked Set Sampling, Double Ranked Set Sampling, Parameter Estimation, Maximum Likelihood Estimation.

1 Introduction

The Weibull distribution proposed by Waloddi Weibull (1951) is a very popular lifetime distribution that has been extensively used over the past decades for modeling data in reliability, engineering and biological studies. Its simplified density and cumulative distributions made it attractive for many authors to consider it in their different applications. It can be approximately symmetric or skewed either positively or negatively. However, the Weibull distribution may not provide a reasonable parametric fit in cases where the hazard rates are bathtub or unimodal shapes. To solve this problem, many authors started to add more flexibility to it by developing new generalizations of the Weibull distribution by adding new parameters. On the other hand, when increasing the number of parameters, the forms of the survival and hazard functions may turn into complicated functions and the estimation problems may be challenging. [see for example Bebbington et al. (2007), Mudholkar and Srivastava (1993), Ghitany et al. (2005), Wahed et al. (2009), Cordeiro et al. (2010), Silva et al. (2010), Risti´c and Balakrishnan (2012)]. In this paper the power generalized Weibull (PGW) distribution which is another generalization of the Weibull distribution proposed by Bagdonovačius and Nikulin in (2002). The hazard function of PGW distribution can be constant, monotone (increasing and decreasing), bathtub shaped and upside down bathtub shaped. This distribution is often used for constructing accelerated failures rate models. They also used the chi-square goodness-of-fit test to illustrate that the PGW fits the randomly-censored survival time's data for patients at arm A of the head-and-neck cancer clinical trial. Nikulin and Highlight (2009) have studied some of the statistical properties of PGW distribution. The cumulative density function (cdf), probability density function (pdf) and quantile function of the power generalized Weibull (PGW(λ, α, β)) distribution are

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$$F(x;\lambda,\alpha,\beta) = 1 - e^{1-(1+\lambda x^{\beta})^{\alpha}} \quad (1)$$

$$f(x;\lambda,\alpha,\beta) = \lambda\alpha\beta x^{\beta-1}\left(1+\lambda x^{\beta}\right)^{\alpha-1}\left(exp\left[1-\left(1+\lambda x^{\beta}\right)^{\alpha}\right]\right) \quad (2)$$

and $x_q = \left[\frac{([1-\ln(1-u)]^{\frac{1}{\alpha}})-1}{\lambda}\right]^{\frac{1}{\beta}}, \quad u = F(x)$.

respectively, where $\beta > 0$ and $\alpha > 0$ are two-shape parameters and $\lambda > 0$ is a scale parameter and $x > 0,\ 0 < u < 1$.

Ranked Set Sampling (RSS) was proposed by McIntyre (1952) to improve the estimation of the population mean while sampling units are constituted in the RSS method, due to not doing certain measurements, the possibility of doing error in ranking increases. In order to overcome this problem, various modifications of RSS have been suggested such as; Samawi et al., (1996) introduced Extreme Ranked Set Sampling (ERSS), Muttlak, (1997) proposed median ranked set sampling (MRSS), the two-stage RSS designs Double Ranked Set Sampling (DRSS), as developed by Al-Saleh and AlKadiri, (2000), Muttlak, (2003) introduced Percentile Ranked Set Sampling (PRSS), L Ranked Set Sampling (LRSS) was proposed by Al-Nasser (2007) and Neoteric Ranked Set Sampling, (NRSS) was introduced by Zamanzade and Al-Omari (2016). Besides these studies, several authors have considered the estimation of the parameters of well-known distributions using RSS or modifications of it. For example, estimation of $P(X < Y)$ using Some Modifications of RSS for Weibull Distribution was proposed by Akgül and Senoglu (2017) , The estimation of unknown parameters of exponential, extreme-value, logistic, Weibull and Pareto distributions was studied by Lam et al. (1994), and maximum likelihood estimators of the parameters of a modified Weibull distribution using extreme ranked set sampling was introduced by Al-Omari and Al-Hadrami (2011), Bhoj (1997), AbuDayyeh et al. (2004), Helu et al. (2010) and Abu-Dayyeh et al. (2013).and a k-stage RSS design uses n^{k+1} sample units from the target population to produce a sample of size n after k stages of ranking developed by Taconeli & Cabral (2019).

The rest of the paper is organized as follows: In Section 2, theRSS and ERSS designs as examples of one-stage RSS designs are described briefly. In Section 3, we describe briefly the DRSS and GDRSS designs as examples of two-stage RSS designs. In Section 4, the maximum likelihood estimation of the PGW distribution parameters are considered when samples are from the one-stage or two-stages RSS designs. Section 5 is devoted for the simulation for the numerical solutions of the likelihood equations in each case. Finally, in Section 6, some concluding remarks are discussed.

2 The One-Stage RSS Designs

In this paper RSS and ERSS are used as an example of the one-stage ranked set samples. The following steps are employed to obtain a RSS of size m:

Step 1: Randomly select m^2 units from the population; these units are randomly allocated into m sets, each of size m.

Step 2: The m units of each set are ranked either visually or by any inexpensive method with respect to the variable of interest.

Step3: From the first set of m units, the smallest ranked unit is measured; from the second set of m units the second smallest ranked unit is measured. The process continues until the m^{th} smallest unit (largest) is measured from the last set.

Step 4: The procedure can be repeated n times if needed to increase the sample size to nm units.

The Likelihood function corresponding to RSS scheme is as follows:

$$L(x;\theta) = \prod_{j=1}^{r}\prod_{i=1}^{m} C_i\, f\left(x_{(i)j};\theta\right)\left[F\left(x_{(i)j};\theta\right)\right]^{i-1}\left[1-F\left(x_{(i)j};\theta\right)\right]^{n-i}. \quad (3)$$

It should be noted that the error in ranking reduces the efficiency of the method. Samawi et al. (1996) proposed extreme ranked set sampling as a useful modification of RSS. It requires identifying the extreme units only, as opposed to all ranks as in the usual RSS. The method gives an unbiased estimate of the population mean in the case of symmetric distributions and it is more efficient than simple random sampling (SRS).

ERSS introduced by Al-Omari and Ali-Hadrami (2011) method can be described as follows:

Step 1: Select m random samples each of size m units from the target population.

Step 2: Rank the units within each sample with respect to a variable of interest by visual inspection or any other inexpensive method.

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Step 3: For actual measurement, if the sample size m is even, from the first $m/2$ sets select the lowest ranked unit of each set and from the other $m/2$ m sets select the largest ranked unit. If the sample size is odd, from the first $(m-1)/2$ sets select the lowest ranked unit, from the other $(m-1)/2$ sets select the largest ranked unit, and from the remaining set select the median ranked unit.

Step 4: The procedure can be repeated r times if needed to increase the sample size to r ×m units.

$$\begin{bmatrix} x_{(11)}^{(1)} & x_{(12)}^{(1)} & \cdots & x_{(1m)}^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ x_{(\frac{m}{2}1)}^{(1)} & x_{(\frac{m}{2}2)}^{(1)} & \cdots & x_{(\frac{m}{2}m)}^{(1)} \\ x_{((\frac{m}{2}+1)1)}^{(1)} & x_{((\frac{m}{2}+1)2)}^{(1)} & \cdots & x_{((\frac{m}{2}+1)m)}^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ x_{(m1)}^{(1)} & x_{(m2)}^{(1)} & \cdots & x_{(mm)}^{(1)} \end{bmatrix} \xrightarrow{for\ m\ even\ and\ r\ times} \begin{bmatrix} \boxed{x_{(11)}^{(1)}} & x_{(12)}^{(1)} & \cdots & x_{(1m)}^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ \boxed{x_{(\frac{m}{2}1)}^{(1)}} & x_{(\frac{m}{2}2)}^{(1)} & \cdots & x_{(\frac{m}{2}m)}^{(1)} \\ x_{((\frac{m}{2}+1)1)}^{(1)} & x_{((\frac{m}{2}+1)2)}^{(1)} & \cdots & \boxed{x_{((\frac{m}{2}+1)m)}^{(1)}} \\ \vdots & \vdots & \vdots & \vdots \\ x_{(m1)}^{(1)} & x_{(m2)}^{(1)} & \cdots & \boxed{x_{(mm)}^{(1)}} \end{bmatrix}$$

Fig. 1: ERSS design in case of even sample size.

The Likelihood function corresponding to ERSS scheme for even set sizes ($m = 2p$) and with r cycles is given as follows:

$$L(\theta) = h\left[\prod_{j=1}^{r}\prod_{i=1}^{p} m\, f\left(x_{(1)i,j};\theta\right)\ \left[F\left(x_{(1)i,j};\theta\right)\right]^{m-1}\right].$$

$$\times\left[\prod_{j=1}^{r}\prod_{i=p+1}^{m} m\, f\left(x_{(m)i,j};\theta\right)\ \left[1-F\left(x_{(m)i,j};\theta\right)\right]^{m-1}\right] \tag{4}$$

where h is a constant and $m = 2q+1$, The Likelihood function corresponding to ERSS scheme for m is odd and with r cycles is given as follows:

$$L(\theta) = K\left[\prod_{j=1}^{r}\prod_{i=1}^{q} m\, f\left(x_{(1)i,j};\theta\right)\ \left[F\left(x_{(1)i,j};\theta\right)\right]^{m-1}\right].$$

$$\times\left[\prod_{j=1}^{r}\prod_{i=q+1}^{m-1} m\, f\left(x_{(m)i,j};\theta\right)\ \left[1-F\left(x_{(m)i,j};\theta\right)\right]^{m-1}\right].$$

$$\times\left[f\left(x_{\left(\frac{m+1}{2}\right),j};\theta\right)\left(F\left(x_{\left(\frac{m+1}{2}\right),j};\theta\right)\left(1-F\left(x_{\left(\frac{m+1}{2}\right),j};\theta\right)\right)\right)^{\frac{m-1}{2}}\right]. \tag{5}$$

where $q = (m-1)/2$ and K is a constant and the variable $X_{(k)i,j}$ denotes the k^{th} ranked unit ($k = 1$ or m or $(m+1)/2$) of the i^{th} sample at the j^{th} cycle.

3 The two-Stages RSS Designs

As mentioned before the DRSS and General Double Ranked Set Sampling (GDRSS) are used as an example of the one-stage ranked set samples. The following steps are employed to obtain a DRSS of size m.

DRSS is a two-stage design was proposed by Al-Saleh and Al-Kadiri (2000).

Step 1: Select m^3 elements from the target population and divide these elements randomly into n sets (of sizem^2).

Step 2: Select a sample of size m in each set using RSS method.

Step 3: Apply the RSS procedure again to elements selected in step 2 to obtain a DRSS of size m.

Step 4: The cycle may be repeated m times.

$$\begin{bmatrix} x_{(11)}^{(1)} & \cdots & x_{(1m)}^{(1)} \\ \vdots & \ddots & \vdots \\ x_{(m1)}^{(1)} & \cdots & x_{(mm)}^{(1)} \end{bmatrix}, \begin{bmatrix} x_{(11)}^{(2)} & \cdots & x_{(1m)}^{(2)} \\ \vdots & \ddots & \vdots \\ x_{(m1)}^{(2)} & \cdots & x_{(mm)}^{(2)} \end{bmatrix}, \begin{bmatrix} x_{(11)}^{(3)} & \cdots & x_{(1m)}^{(3)} \\ \vdots & \ddots & \vdots \\ x_{(m1)}^{(3)} & \cdots & x_{(mm)}^{(3)} \end{bmatrix} and \begin{bmatrix} x_{(11)}^{(m)} & \cdots & x_{(1m)}^{(m)} \\ \vdots & \ddots & \vdots \\ x_{(m1)}^{(m)} & \cdots & x_{(mm)}^{(m)} \end{bmatrix}$$

Fig. 2: DRSS design in case of even sample size.

So, we have four judgment ranked sets of size m each:

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$X_{1,j} = min\left\{\left\{x_{(11)}^{(j)}, x_{(22)}^{(j)}, \ldots . x_{(mm)}^{(j)}\right\}, j = 1,2, \ldots, r\right\}$, and

$X_{m,k} = max\left\{\left\{x_{(11)}^{(k)}, x_{(22)}^{(k)}, \ldots . x_{(mm)}^{(k)}\right\}, k = r+1, r+2, \ldots, m\right\}$.

The likelihood function for DRSS that is proposed by Sabry et al. (2019) is given as:

Case I: $\boldsymbol{m}$ even ($\boldsymbol{m = 2r}$)

$$L(\theta) = \left[\prod_{j=1}^{r} m f_{1:m}(x_{1,j})[1 - F_{1:m}(x_{1,j})]^{m-1}\right]\left[\prod_{k=r+1}^{m} m\, f_{m:m}(x_{m,k})\, [F_{m:m}(x_{m,k})]^{m-1}\right] \tag{6}$$

where $f_{1:m}(x_j) = m\, f(x_{1,j})[1 - F(x_{1,j})]^{m-1}$, $F_{1:m}(x_{1,j}) = 1 - [1 - F(x_{1,j})]^{m}$, $f_{m:m}(x_k) = mf(x_{m,k})\, [F(x_{m,k})]^{m-1}$ and $F_{m:m}(x_{m,k}) = [F(x_{m,k})]^{m}$

Case II: $\boldsymbol{m}$ odd ($\boldsymbol{m = 2r+1}$)

$$L(\theta) = \left[\prod_{j=1}^{r} m f_{1:m}(x_{1,j})[1 - F_{1:m}(x_{1,j})]^{m-1}\right]\left[\prod_{k=r+2}^{m} m f_{m:m}(x_{m,k})\, [F_{m:m}(x_{m,k})]^{m-1}\right]$$
$$\times\left[\frac{(2r+1)!}{(r)!\,(r)!} f_{r+1:m}(x_{(r+1),(r+1)})\left(F_{r+1:m}(x_{(r+1),(r+1)})\left(1 - F_{r+1:m}(x_{(r+1),(r+1)})\right)\right)^{r}\right]. \tag{7}$$

where $F_{r+1:m}(x_{(r+1),(r+1)}) = \sum_{t=r+1}^{m}\binom{m}{t}\left(F(x_{(r+1),(r+1)})\right)^{t}\left(1 - F(x_{(r+1),(r+1)})\right)^{m-t}$.

GDRSS is a two-stage design discussed by Taconeli & Cabral (2019) in which the first stage is defined by RSS scheme, followed by a second stage in which the RSS scheme must be performed

Step 1: Select m^3 elements from the target population and divide these elements randomly into n sets (of size m^2).

Step 2: Select a sample of size m in each set using RSS method.

Step 3: Ranked the units from the first stage into m sets, each of size m and apply the RSS procedure to obtain a DRSS of size m

Step 4: The cycle may be repeated m times.

Assume the elements are $x_{11}^{(1)}, x_{12}^{(1)}, \ldots, x_{mm}^{(1)}, x_{11}^{(2)}, x_{12}^{(2)}, \ldots, x_{mm}^{(2)}, \ldots, x_{11}^{(m)}, x_{12}^{(m)}, \ldots, x_{mm}^{(m)}$

After ranking the elements of each set (visually), we obtain

$$Stage\ 1: \begin{bmatrix} x_{(11)}^{(1)} & \cdots & x_{(1m)}^{(1)} \\ \vdots & \ddots & \vdots \\ x_{(m1)}^{(1)} & \cdots & x_{(mm)}^{(1)} \end{bmatrix}, \begin{bmatrix} x_{(11)}^{(2)} & \cdots & x_{(1m)}^{(2)} \\ \vdots & \ddots & \vdots \\ x_{(m1)}^{(2)} & \cdots & x_{(mm)}^{(2)} \end{bmatrix}, \begin{bmatrix} x_{(11)}^{(3)} & \cdots & x_{(1m)}^{(3)} \\ \vdots & \ddots & \vdots \\ x_{(m1)}^{(3)} & \cdots & x_{(mm)}^{(3)} \end{bmatrix} and \begin{bmatrix} x_{(11)}^{(m)} & \cdots & x_{(1m)}^{(m)} \\ \vdots & \ddots & \vdots \\ x_{(m1)}^{(m)} & \cdots & x_{(mm)}^{(m)} \end{bmatrix}$$

The RSS sample is

$$\left\{\underbrace{x_{(11)}^{(1)}}_{z_1^{(1)}}, \underbrace{x_{(22)}^{(1)}}_{z_2^{(1)}}, \ldots, \underbrace{x_{(mm)}^{(1)}}_{z_m^{(1)}}, \underbrace{x_{(11)}^{(2)}}_{z_1^{(2)}}, x_{(22)}^{(2)}, \ldots, \underbrace{x_{(mm)}^{(2)}}_{z_m^{(2)}}, \ldots, \underbrace{x_{(11)}^{(m)}}_{z_1^{(m)}}, x_{(22)}^{(m)}, \ldots, \underbrace{x_{(mm)}^{(m)}}_{z_m^{(m)}}\right\}$$

$$Stage\ 2: \begin{bmatrix} z_{(1)}^{(1)} & \cdots & z_{(m)}^{(1)} \\ \vdots & \ddots & \vdots \\ z_{(1)}^{(m)} & \cdots & x_{(m)}^{(m)} \end{bmatrix}.$$

Fig. 3: GDRSS design.

and we apply the RSS procedure to obtain a DRSS of size m, then $z_{(1)}^{(1)}, z_{(2)}^{(2)}, z_{(3)}^{(3)}, \ldots, z_{(m)}^{(m)}$ is GDRSS .

Since GDRSS is applied on existing order statistics variables, it can be easily shown that the likelihood function of a random sample selected by the GDRSS is given as

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$L(\theta) = \prod_{j=1}^{m} \frac{m!}{(j-1)!(m-j)!} f_{(j;m)}\left(z_{(j)}; \boldsymbol{\theta}\right) \left[F_{(j;m)}\left(z_{(j)}; \boldsymbol{\theta}\right)\right]^{j-1} [1 - F_{(j;m)}\left(z_{(j)}; \boldsymbol{\theta}\right)]^{m-j}$. (8)

where $f_{(j;m)}\left(z_{(j)}; \theta\right) = \frac{m!}{(j-1)!(m-j)!} f\left(z_j; \boldsymbol{\theta}\right)[F\left(z_j; \boldsymbol{\theta}\right)]^{j-1}[1 - F\left(z_j; \boldsymbol{\theta}\right)]^{m-i}$ and

$F_z\left(x_{(j)}; \theta\right) = \sum_{t=i}^{m} \binom{m}{t} [F\left(z_j; \boldsymbol{\theta}\right)]^t [1 - F\left(z_j; \boldsymbol{\theta}\right)]^{m-t}$.

4 Maximum Likelihood Estimation

This section is devoted to the MLE for the unknown parameters of PGW distribution based on RSS, ERSS, DRSS and GDRSS designs.

4.1 Estimation Based on RSS

Let $\{X_i^j, i = 1,2,..,n; j = 1,2,..,r\}$ be a ranked set sample with cdf and pdf given in Eqs. (1) and (2), where m is the set size, r is the number of cycles and $n = m\,r$. According to Eq. (3) The Likelihood function of the RSS sample for PGW distribution is given by,

$$L(x;\theta) = \prod_{j=1}^{r} \prod_{i=1}^{m} C_i \left(\lambda\alpha\beta x_{i,j}^{\beta-1}\left(1 + \lambda x_{i,j}^{\beta}\right)^{\alpha-1} \left(\exp\left[1 - \left(1 + \lambda x_{i,j}^{\beta}\right)^{\alpha}\right]\right)\right).$$

$$\times \left(1 - e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}}\right)^{i-1} \left(e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}}\right)^{n-i}$$

where $C_i = \frac{m!}{(i-1)!(m-i)!}$ and $\theta = (\lambda, \beta, \alpha)$. The log likelihood function can be derived directly as follows ($\ell(\theta) = log\, L(\theta)$)

$$\ell(\theta) \propto m\,r \log\lambda + m\,r\log\beta + m\,r\log\alpha + (\beta - 1) \sum_{j=1}^{r} \sum_{i=1}^{m} \log x_{(i)j},$$

$$+(\alpha - 1) \sum_{j=1}^{r} \sum_{i=1}^{m} log\left(1 + \lambda x_{i,j}^{\beta}\right) + \sum_{j=1}^{r} \sum_{i=1}^{m} \left[1 - \left(1 + \lambda x_{i,j}^{\beta}\right)^{\alpha}\right].$$

$$+ \sum_{j=1}^{r} \sum_{i=1}^{m} (i-1)\log\left[1 - e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}}\right] + \sum_{j=1}^{r} \sum_{i=1}^{m} (m-i)\left[1 - \left(1 + \lambda x_{i,j}^{\beta}\right)^{\alpha}\right],$$

The 1st derivatives becomes

$$\frac{\partial\ell}{\partial\lambda} = \frac{m\,r}{\lambda} + (\alpha - 1)\sum_{j=1}^{r}\sum_{i=1}^{m} \frac{x_{i,j}^{\beta}}{\left(1 + \lambda x_{i,j}^{\beta}\right)} - \sum_{j=1}^{r}\sum_{i=1}^{m} \alpha\left[\left(1 + \lambda x_{i,j}^{\beta}\right)^{\alpha-1}\right] x_{i,j}^{\beta}$$

$$+ \sum_{j=1}^{r} \sum_{i=1}^{m} (i-1) \frac{e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}} \alpha\left(1+\lambda x_{i,j}^{\beta}\right)^{\alpha-1} x_{i,j}^{\beta}}{1-e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}}} - \sum_{j=1}^{r} \sum_{i=1}^{n} (m-i)\alpha\left(1 + \lambda x_{i,j}^{\beta}\right)^{\alpha-1} x_{i,j}.,$$

$$\frac{\partial\ell}{\partial\beta} = \frac{m\,r}{\beta} + \sum_{j=1}^{r}\sum_{i=1}^{m} \log x_{(i)j} + (\alpha - 1)\sum_{j=1}^{r}\sum_{i=1}^{m} \frac{\lambda x_{i,j}^{\beta} \log x_{i,j}}{\left(1 + \lambda x_{i,j}^{\beta}\right)} - \sum_{j=1}^{r}\sum_{i=1}^{m} \alpha\left[\left(1 + \lambda x_{i,j}^{\beta}\right)^{\alpha-1}\right] \lambda x_{i,j}^{\beta} \log x_{i,j}$$

$$+ \sum_{j=1}^{r} \sum_{i=1}^{m} (i-1) \frac{e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}} \alpha\left(1+\lambda x_{i,j}^{\beta}\right)^{\alpha-1} \lambda x_{i,j}^{\beta} \log x_{i,j}}{1-e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}}} - \sum_{j=1}^{r} \sum_{i=1}^{m} (n-i)\alpha\left(1 + \lambda x_{i,j}^{\beta}\right)^{\alpha-1} \lambda x_{i,j}^{\beta} \log x_{i,j}.,$$

and

$$\frac{\partial\ell}{\partial\alpha} = \frac{m\,r}{\alpha} + \sum_{j=1}^{r}\sum_{i=1}^{m} \log\left(1 + \lambda x_{i,j}^{\beta}\right) - \sum_{j=1}^{r}\sum_{i=1}^{m} \left[\left(1 + \lambda x_{i,j}^{\beta}\right)^{\alpha}\right] \log\left(1 + \lambda x_{i,j}^{\beta}\right)$$

$$+ \sum_{j=1}^{r} \sum_{i=1}^{m} (i-1) \frac{e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}} \left(1+\lambda x^{\beta}\right)^{\alpha} \log\left(1+\lambda x_{i,j}^{\beta}\right)}{1-e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}}} - \sum_{j=1}^{r} \sum_{i=1}^{m} (m-i)\left[\left(1 + \lambda x_{i,j}^{\beta}\right)^{\alpha}\right] \log\left(1 + \lambda x_{i,j}^{\beta}\right).$$

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4.2 Estimation Based on ERSS

According to Eq. (4) The Likelihood function for even set sizes ($m = 2p$) and with r cycles based on ERSS is given by

$$L(\theta) = h\left[\prod_{j=1}^{r}\prod_{i=1}^{p} m\ \lambda\alpha\beta x_{(1)i,j}^{\beta-1}\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha-1}\left(exp\ \left[1-\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha}\right]\right)\right]$$

$$\times\left[\prod_{j=1}^{r}\prod_{i=p+1}^{m} m\ \lambda\alpha\beta x_{(m)i,j}^{\beta-1}\left(1+\lambda x_{(m)i,j}^{\beta}\right)^{\alpha-1}\left(exp\ \left[1-\left(1+\lambda x_{(m)i,j}^{\beta}\right)^{\alpha}\right]\right)^{m}\right],$$

The log likelihood function is given as:

$$\ell(\theta) = rm\log m + rm\log\lambda + rm\log\alpha + rm\log\beta + (\beta-1)\sum_{j=1}^{r}\sum_{i=1}^{p}\log x_{(1)i,j}$$
$$+(\beta-1)\sum_{j=1}^{r}\sum_{i=p+1}^{m}\log x_{(m)i,j} + (\alpha-1)\sum_{j=1}^{r}\sum_{i=1}^{p}\log\left(1+\lambda x_{(1)i,j}^{\beta}\right)$$
$$+\sum_{j=1}^{r}\sum_{i=1}^{p}\left[1-\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha}\right] + (\alpha-1)\sum_{j=1}^{r}\sum_{i=p+1}^{m}\log\left(1+\lambda x_{(m)i,j}^{\beta}\right)$$
$$+\textstyle\sum_{j=1}^{r}\sum_{i=1}^{p}(m-1)\log\left[1-e^{1-\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha}}\right] + \sum_{j=1}^{r}\sum_{i=p+1}^{m} m\left(1-\left(1+\lambda x_{(m)i,j}^{\beta}\right)^{\alpha}\right),$$

and the first derivatives of the $\ell(\theta)$ are given by

$$\frac{\partial\ell}{\partial\lambda} = \frac{rm}{\lambda} + (\alpha-1)\sum_{j=1}^{r}\sum_{i=1}^{p}\frac{x_{(1)i,j}^{\beta}}{\left(1+\lambda x_{(1)i,j}^{\beta}\right)} + (\alpha-1)\sum_{j=1}^{r}\sum_{i=p+1}^{m}\frac{x_{(m)i,j}^{\beta}}{\left(1+\lambda x_{(m)i,j}^{\beta}\right)}$$
$$-\alpha\sum_{j=1}^{r}\sum_{i=1}^{p}\left[\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha-1}\right]x_{(1)i,j}^{\beta} + \sum_{j=1}^{r}\sum_{i=1}^{p}(m-1)\frac{e^{1-\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha}}\alpha\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha-1}x_{(1)i,j}^{\beta}}{\left(\left[1-e^{1-\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha}}\right]\right)}$$
$$-\textstyle\sum_{j=1}^{r}\sum_{i=p+1}^{m} m\,\alpha\left(1+\lambda x_{(m)i,j}^{\beta}\right)^{\alpha-1}x_{(m)i,j}^{\beta},$$

$$\frac{\partial\ell}{\partial\beta} = \frac{rm}{\beta} + \sum_{j=1}^{r}\sum_{i=1}^{p}\log x_{(1)i,j} + \sum_{j=1}^{r}\sum_{i=p+1}^{m}\log x_{(m)i,j} + (\alpha-1)\sum_{j=1}^{r}\sum_{i=1}^{p}\frac{\lambda x_{(1)i,j}^{\beta}\log x_{(1)i,j}}{\left(1+\lambda x_{(1)i,j}^{\beta}\right)}$$
$$-\alpha\sum_{j=1}^{r}\sum_{i=1}^{p}\left[\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha-1}\right]\lambda x_{(1)i,j}^{\beta}\log x_{(1)i,j} + (\alpha-1)\sum_{j=1}^{r}\sum_{i=p+1}^{m}\frac{\lambda x_{(m)i,j}^{\beta}\log x_{(m)i,j}}{\left(1+\lambda x_{(m)i,j}^{\beta}\right)}$$
$$+\sum_{j=1}^{r}\sum_{i=1}^{p}(m-1)\frac{e^{1-\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha}}\alpha\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha-1}\lambda x_{(1)i,j}^{\beta}\log x_{(1)i,j}}{\left(\left[1-e^{1-\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha}}\right]\right)}$$
$$-\textstyle\sum_{j=1}^{r}\sum_{i=p+1}^{m} m\alpha\left(1+\lambda x_{(m)i,j}^{\beta}\right)^{\alpha-1}\lambda x_{(m)i,j}^{\beta}\log x_{(m)i,j},$$

and

$$\frac{\partial\ell}{\partial\alpha} = \frac{rm}{\alpha} + \sum_{j=1}^{r}\sum_{i=1}^{p}\log\left(1+\lambda x_{(1)i,j}^{\beta}\right) - \sum_{j=1}^{r}\sum_{i=1}^{p}\left[\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha}\right]\log\left(1+\lambda x_{(1)i,j}^{\beta}\right) + \sum_{j=1}^{r}\sum_{i=p+1}^{m}\log\left(1+\lambda x x_{(m)i,j}^{\beta}{}^{\beta}\right)$$
$$+\sum_{j=1}^{r}\sum_{i=1}^{p}(m-1)\frac{e^{1-\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha}}\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha}\log\left(1+\lambda x_{(1)i,j}^{\beta}\right)}{\left(\left[1-e^{1-\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha}}\right]\right)}$$
$$-\sum_{j=1}^{r}\sum_{i=p+1}^{m} m\left(1+\lambda x_{(m)i,j}^{\beta}\right)^{\alpha}\log\left(1+\lambda x_{(m)i,j}^{\beta}\right).$$

When m is odd ($m = 2q + 1$) and based on ERSS, according to Eq. (5) the likelihood function L is given by

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$$L(\theta)=K\left[\prod_{j=1}^{r}\prod_{i=1}^{q} m\ \lambda\alpha\beta x_{(1)i,j}^{\beta-1}\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha-1}\left(\exp\left[1-\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha}\right]\right)\times\left[1-e^{1-\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha}}\right]^{m-1}\right]$$

$$\times\left[\prod_{j=1}^{r}\prod_{i=q+1}^{m-1} m\ \lambda\alpha\beta x_{(m)i,j}^{\beta-1}\left(1+\lambda x_{(m)i,j}^{\beta}\right)^{\alpha-1}\left(\exp\left[1-\left(1+\lambda x_{(m)i,j}^{\beta}\right)^{\alpha}\right]\right)^{m}\right]$$

$$\times\left[\left(\lambda\alpha\beta x_{\left(\frac{m+1}{2}\right)j}^{\beta-1}\left(1+\lambda x_{\left(\frac{m+1}{2}\right)j}^{\beta}\right)^{\alpha-1}\left(\exp\left[1-\left(1+\lambda x_{\left(\frac{m+1}{2}\right)j}^{\beta}\right)^{\alpha}\right]\right)\right)\times\left[1-e^{1-\left(1+\lambda x_{((m+1)/2)j}^{\beta}\right)^{\alpha}}\right]^{\frac{m-1}{2}}\left(\exp\left[1-\left(1+\lambda x_{((m+1)/2)j}^{\beta}\right)^{\alpha}\right]\right)^{\frac{m-1}{2}}\right].$$

Taking the natural logarithm of $L(\theta)$ gets,

$$\ell(\theta)=rm\log m+rm\log\lambda+rm\log\alpha+rm\log\beta+(\beta-1)\sum_{j=1}^{r}\sum_{i=1}^{q}\log x_{(1)i,j}$$

$$+(\beta-1)\sum_{i=1}^{r}\sum_{j=q+1}^{m-1}\log x_{(m)i,j}+(\alpha-1)\sum_{j=1}^{r}\sum_{i=1}^{q}\log\left(1+\lambda x_{(1)i,j}^{\beta}\right)+\sum_{j=1}^{r}\sum_{i=1}^{q}\left[1-\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha}\right]$$

$$+(\alpha-1)\sum_{j=1}^{r}\sum_{i=p+1}^{m-1}\log\left(1+\lambda x_{(m)i,j}^{\beta}\right)+(m-1)\sum_{j=1}^{r}\sum_{i=1}^{q}\log\left[1-e^{1-\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha}}\right]$$

$$+m\sum_{j=1}^{r}\sum_{i=q+1}^{m-1}\log\left(1-\left(1+\lambda x_{(m)i,j}^{\beta}\right)^{\alpha}\right)+(\beta-1)\log x_{\left(\frac{m+1}{2}\right)j}+(\alpha-1)\log\left(1+\lambda x_{\left(\frac{m+1}{2}\right)j}^{\beta}\right)$$

$$+\left(\frac{m+1}{2}\right)\left[1-\left(1+\lambda x_{((m+1)/2)j}^{\beta}\right)^{\alpha}\right]+\frac{m-1}{2}\log\left(1-e^{1-\left(1+\lambda x_{((m+1)/2)j}^{\beta}\right)^{\alpha}}\right),$$

and the first derivatives are given by

$$\frac{\partial\ell}{\partial\beta}=\frac{rm}{\beta}+\sum_{j=1}^{r}\sum_{i=1}^{q}\log x_{(1)i,j}+\sum_{j=1}^{r}\sum_{i=q+1}^{m-1}\log x_{(m)i,j}+(\alpha-1)\sum_{j=1}^{r}\sum_{i=1}^{q}\frac{\lambda x_{(1)i,j}^{\beta}\log x_{(1)i,j}}{\left(1+\lambda x_{(1)i,j}^{\beta}\right)}$$

$$-\alpha\sum_{j=1}^{r}\sum_{i=1}^{q}\left[\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha-1}\right]\lambda x_{(1)i,j}^{\beta}\log x_{(1)i,j}+(\alpha-1)\sum_{j=1}^{r}\sum_{i=q+1}^{m-1}\frac{\lambda x_{(m)i,j}^{\beta}x_{(m)i,j}}{\left(1+\lambda x_{(m)i,j}^{\beta}\right)}$$

$$+(m-1)\sum_{j=1}^{r}\sum_{i=1}^{q}\frac{e^{1-\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha}}\alpha\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha-1}\lambda x_{(1)i,j}^{\beta}\log x_{(1)i,j}}{\left(\left[1-e^{1-\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha}}\right]\right)}$$

$$-m\sum_{j=1}^{r}\sum_{i=q+1}^{m}\alpha\left(1+\lambda x_{(m)i,j}^{\beta}\right)^{\alpha-1}\lambda x_{(m)i,j}^{\beta}\log x_{(m)i,j}+\log x_{\left(\frac{m+1}{2}\right)j}$$

$$+(\alpha-1)\frac{\lambda x_{\left(\frac{m+1}{2}\right)j}^{\beta}x_{\left(\frac{m+1}{2}\right)j}}{\left(1+\lambda x_{\left(\frac{m+1}{2}\right)j}^{\beta}\right)}+\left(\frac{m+1}{2}\right)\alpha\left[\left(1+\lambda x_{\left(\frac{m+1}{2}\right)j}^{\beta}\right)^{\alpha-1}\right]\lambda x_{\left(\frac{m+1}{2}\right)j}^{\beta}\log x_{\left(\frac{m+1}{2}\right)j}$$

$$+\left(\frac{m-1}{2}\right)\frac{e^{1-\left(1+\lambda x_{((m+1)/2)j}^{\beta}\right)^{\alpha}}\alpha\left(1+\lambda x_{((m+1)/2)j}^{\beta}\right)^{\alpha-1}\lambda x_{((m+1)/2)j}^{\beta}\log x_{((m+1)/2)j}}{\left(\left[1-e^{1-\left(1+\lambda x_{((m+1)/2)j}^{\beta}\right)^{\alpha}}\right]\right)},$$

$$\frac{\partial\ell}{\partial\lambda}=\frac{rm}{\lambda}+(\alpha-1)\sum_{j=1}^{r}\sum_{j=1}^{q}\frac{x_{(1)i,j}^{\beta}}{\left(1+\lambda x_{(1)i,j}^{\beta}\right)}+(m-1)\sum_{j=1}^{r}\sum_{i=1}^{q}\frac{e^{1-\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha}}\alpha\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha-1}x_{(1)i,j}^{\beta}}{\left(\left[1-e^{1-\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha}}\right]\right)}$$

$$-m\sum_{j=1}^{r}\sum_{i=q+1}^{m-1}\alpha\left(1+\lambda x_{(m)i,j}^{\beta}\right)^{\alpha-1}x_{(m)i,j}^{\beta}+(\alpha-1)\frac{x_{\left(\frac{m+1}{2}\right)j}^{\beta}}{\left(1+\lambda x_{\left(\frac{m+1}{2}\right)j}^{\beta}\right)}-\left(\frac{m+1}{2}\right)\alpha\left[\left(1+\lambda x_{\left(\frac{m+1}{2}\right)j}^{\beta}\right)^{\alpha-1}\right]x_{\left(\frac{m+1}{2}\right)j}^{\beta},$$

and

$$\frac{\partial \ell}{\partial \alpha}=\frac{rm}{\alpha}+\sum_{j=1}^{r}\sum_{i=1}^{q}\log\left(1+\lambda x_{(1)i,j}^{\beta}\right)-\sum_{j=1}^{r}\sum_{i=1}^{q}\left[\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha}\right]\log\left(1+\lambda x_{(1)i,j}^{\beta}\right)$$

$$+\sum_{j=1}^{r}\sum_{i=q+1}^{m-1}\log\left(1+\lambda x_{(m)i,j}^{\beta}\right)+(m-1)\sum_{j=1}^{r}\sum_{i=1}^{q}\frac{e^{1-\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha}}\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha}\log\left(1+\lambda x_{(1)i,j}^{\beta}\right)}{\left(\left[1-e^{1-\left(1+\lambda x_{(1)i,j}^{\beta}\right)^{\alpha}}\right]\right)}$$

$$-m\sum_{j=1}^{r}\sum_{i=q+1}^{m-1}\left(1+\lambda x_{(m)i,j}^{\beta}\right)^{\alpha}\log\left(1+\lambda x_{(m)i,j}^{\beta}\right)+\log\left(1+\lambda x_{\left(\frac{m+1}{2}\right)j}^{\beta}\right)$$

$$-(r+1)\left[\left(1+\lambda x_{\left(\frac{m+1}{2}\right)j}^{\beta}\right)^{\alpha}\right]\log\left(1+\lambda x_{\left(\frac{m+1}{2}\right)j}^{\beta}\right)$$

$$+r\frac{e^{1-\left(1+\lambda x_{((m+1)/2)j}^{\beta}\right)^{\alpha}}\left(1+\lambda x_{((m+1)/2)j}^{\beta}\right)^{\alpha}\log\left(1+\lambda x_{((m+1)/2)j}^{\beta}\right)}{\left(\left[1-e^{1-\left(1+\lambda x_{((m+1)/2)j}^{\beta}\right)^{\alpha}}\right]\right)}.$$

4.3 Estimation Based on DRSS

According to Eq. (6) the likelihood function for the DRSS design is derived as follows.

Case I: m even ($m=2r$)

$$L(\theta)=\left[\prod_{j=1}^{r}m\left(m\,\lambda\alpha\beta x_{1,j}^{\beta-1}\left(1+\lambda x_{1,j}^{\beta}\right)^{\alpha-1}e^{1-\left(1+\lambda x_{1,j}^{\beta}\right)^{\alpha}}\left(e^{1-\left(1+\lambda x_{1,j}^{\beta}\right)^{\alpha}}\right)^{m-1}\right)\left(e^{1-\left(1+\lambda x_{1,j}^{\beta}\right)^{\alpha}}\right)^{m(m-1)}\right]$$
$$\times\left[\prod_{k=r+1}^{m}m\left(m\,\lambda\alpha\beta x_{m,k}^{\beta-1}\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha-1}e^{1-\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha}}\right)\left(1-e^{1-\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha}}\right)^{m-1}\times\left[\left(1-e^{1-\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha}}\right)^{m}\right]^{(m-1)}\right],$$

Then, the associated log-likelihood function is obtained as

$$\ell=c+m\log\lambda+m\log\alpha+m\log\beta+(\beta-1)\sum_{j=1}^{r}\log x_{1,j}+(\beta-1)\sum_{k=r+1}^{m}\log x_{m,k}.$$
$$+(\alpha-1)\sum_{j=1}^{r}\log\left(1+\lambda x_{1,j}^{\beta}\right)+(\alpha-1)\sum_{k=r+1}^{m}\log\left(1+\lambda x_{m,k}^{\beta}\right)$$
$$+m^{2}\sum_{j=1}^{r}\left(1-\left(1+\lambda x_{1,j}^{\beta}\right)^{\alpha}\right)+\sum_{k=r+1}^{m}\left(1-\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha}\right)$$
$$+(m^{2}-1)\sum_{k=r+1}^{m}\log\left(1-e^{1-\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha}}\right),$$

and the first derivatives are given by

$$\frac{\partial \ell}{\partial \lambda}=\frac{m}{\lambda}+(\alpha-1)\sum_{j=1}^{r}\frac{x_{1,j}^{\beta}}{\left(1+\lambda x_{1,j}^{\beta}\right)}+(\alpha-1)\sum_{k=r+1}^{m}\frac{x_{m,k}^{\beta}}{\left(1+\lambda x_{m,k}^{\beta}\right)}$$
$$-m^{2}\alpha\sum_{j=1}^{r}\left(\left(1+\lambda x_{1,j}^{\beta}\right)^{\alpha-1}\right)x_{1,j}^{\beta}-\alpha\sum_{k=r+1}^{m}\left(\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha-1}\right)x_{m,k}^{\beta}$$
$$+(m^{2}-1)\sum_{k=r+1}^{m}\frac{e^{1-\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha}}\alpha\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha-1}x_{m,k}^{\beta}}{\left(1-e^{1-\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha}}\right)},$$

$$\frac{\partial \ell}{\partial \beta}=\frac{m}{\beta}+\sum_{j=1}^{r}\log x_{1,j}+\sum_{k=r+1}^{m}\log x_{m,k}+(\alpha-1)\sum_{j=1}^{r}\frac{\lambda x_{1,j}^{\beta}\log x_{1,j}}{\left(1+\lambda x_{1,j}^{\beta}\right)}+(\alpha-1)\sum_{k=r+1}^{m}\frac{\lambda x_{m,k}^{\beta}\log x_{m,k}}{\left(1+\lambda x_{m,k}^{\beta}\right)}$$
$$-m^{2}\alpha\sum_{j=1}^{r}\left(\left(1+\lambda x_{1,j}^{\beta}\right)^{\alpha-1}\right)\lambda x_{1,j}^{\beta}-\alpha\sum_{k=r+1}^{m}\left(\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha-1}\right)\lambda x_{m,k}^{\beta}\log x_{m,k}$$
$$+(m^{2}-1)\sum_{k=r+1}^{m}\frac{e^{1-\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha}}\alpha\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha-1}\lambda x_{m,k}^{\beta}\log x_{m,k}}{\left(1-e^{1-\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha}}\right)},$$

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and

$$\frac{\partial \ell}{\partial \alpha} = \frac{m}{\alpha} + \sum_{j=1}^{r} \log\left(1 + \lambda x_{1,j}^{\beta}\right) + \sum_{k=r+1}^{m} \log\left(1 + \lambda x_{m,k}^{\beta}\right) - m^2 \sum_{j=1}^{r} \left(1 + \lambda x_{1,j}^{\beta}\right)^{\alpha} \log\left(1 + \lambda x_{1,j}^{\beta}\right)$$

$$- \sum_{k=r+1}^{m} \left(1 + \lambda x_{m,k}^{\beta}\right)^{\alpha-1} \log\left(1 + \lambda x_{m,k}^{\beta}\right)$$

$$+(m^2 - 1) \sum_{k=r+1}^{m} \frac{e^{1-\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha}} \left(1 + \lambda x_{m,k}^{\beta}\right)^{\alpha} \log\left(1 + \lambda x_{m,k}^{\beta}\right)}{\left(1 - e^{1-\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha}}\right)}.$$

Case II: m odd ($m = 2r + 1$)

According to Eq.(7) the likelihood function of the PGW distribution for odd set size is given by

$$L(\theta) = \left[\prod_{j=1}^{r} \left(m^{2m}\, \lambda\alpha\beta x_{1,j}^{\beta-1} \left(1 + \lambda x_{1,j}^{\beta}\right)^{\alpha-1} \left(e^{1-\left(1+\lambda x_{1,j}^{\beta}\right)^{\alpha}}\right)^{m^2}\right)\right]$$

$$\times \left[\prod_{k=r+2}^{m} \left(m^2\, \lambda\alpha\beta x_{m,k}^{\beta-1} \left(1 + \lambda x_{m,k}^{\beta}\right)^{\alpha-1} e^{1-\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha}}\right) \left(1 - e^{1-\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha}}\right)^{m^2-1}\right]$$

$$\times \left[\frac{(2r+1)!}{(r)!(r)!} \left(\frac{m!}{r!r!} \left(\lambda\alpha\beta x_{(r+1),(r+1)}^{\beta-1} \left(1 + \lambda x_{(r+1),(r+1)}^{\beta}\right)^{\alpha-1}\right)\right)\right.$$

$$\times \left(\left(1 - e^{1-\left(1+\lambda x_{(r+1),(r+1)}^{\beta}\right)^{\alpha}}\right)^{r}\right) \times \left(F_{r+1:m}\left(x_{(r+1),(r+1)}\right)\right)^{r} \left(1 - F_{r+1:m}\left(x_{(r+1),(r+1)}\right)\right)^{r},$$

Then, the associated log-likelihood function is obtained as

$$\ell = c + m \log \lambda + m \log \alpha + m \log \beta + (\beta - 1) \sum_{j=1}^{r} \log x_{1,j} + (\beta - 1) \sum_{k=r+2}^{m} \log x_{m,k}$$

$$+(\alpha - 1) \sum_{j=1}^{r} \log\left(1 + \lambda x_{1,j}^{\beta}\right) + (\alpha - 1) \sum_{k=r+2}^{m} \log\left(1 + \lambda x_{m,k}^{\beta}\right)$$

$$+m^2 \sum_{j=1}^{r} \left(1 - \left(1 + \lambda x_{1,j}^{\beta}\right)^{\alpha}\right) + \sum_{k=r+2}^{m} \left(1 - \left(1 + \lambda x_{m,k}^{\beta}\right)^{\alpha}\right)$$

$$+(m^2 - 1) \sum_{k=r+1}^{m} \log\left(1 - e^{1-\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha}}\right) + (\beta - 1) \log x_{(r+1),(r+1)}$$

$$+(\alpha - 1) \log\left(1 + \lambda x_{(r+1),(r+1)}^{\beta}\right) + (r + 1)\left[1 - \left(1 + \lambda x_{(r+1),(r+1)}^{\beta}\right)^{\alpha}\right]$$

$$+r \log\left(1 - e^{1-\left(1+\lambda x_{(r+1),(r+1)}^{\beta}\right)^{\alpha}}\right) + r \log F_{r+1:m}\left(x_{(r+1),(r+1)}\right)$$

$$+r\, log\, \left(1 - F_{r+1:m}\left(x_{(r+1),(r+1)}\right)\right),$$

and the first derivatives are given by

$$\frac{\partial \ell}{\partial \lambda} = \frac{m}{\lambda} + (\alpha - 1) \sum_{j=1}^{r} \frac{x_{1,j}^{\beta}}{\left(1+\lambda x_{1,j}^{\beta}\right)} + (\alpha - 1) \sum_{k=r+2}^{m} \frac{x_{m,k}^{\beta}}{\left(1+\lambda x_{m,k}^{\beta}\right)} - m^2 \alpha \sum_{j=1}^{r} \left(\left(1 + \lambda x_{1,j}^{\beta}\right)^{\alpha-1}\right) x_{1,j}^{\beta}.$$

$$-\alpha \sum_{k=r+2}^{m} \left(\left(1 + \lambda x_{m,k}^{\beta}\right)^{\alpha-1}\right) x_{m,k}^{\beta} + (m^2 - 1) \sum_{k=r+2}^{m} \frac{e^{1-\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha}} \alpha\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha-1} x_{m,k}^{\beta}}{\left(1-e^{1-\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha}}\right)}$$

$$+ \log x_{(r+1),(r+1)} + (\alpha - 1) \sum_{k=r+2}^{m} \frac{x_{(r+1),(r+1)}^{\beta}}{\left(1+\lambda x_{(r+1),(r+1)}^{\beta}\right)} - (\mathrm{r} + 1)\alpha \left(\left(1 + \lambda x_{(r+1),(r+1)}^{\beta}\right)^{\alpha-1}\right) x_{(r+1),(r+1)}^{\beta}$$

$$+r \frac{e^{1-\left(1+\lambda x_{(r+1),(r+1)}^{\beta}\right)^{\alpha}} \alpha\left(1+\lambda x_{(r+1),(r+1)}^{\beta}\right)^{\alpha-1} x_{(r+1),(r+1)}^{\beta}}{\left(1-e^{1-\left(1+\lambda x_{(r+1),(r+1)}^{\beta}\right)^{\alpha}}\right)} + F_{\lambda}\left(\frac{1-2F_{r+1:m}\left(x_{(r+1),(r+1)}\right)}{F_{r+1:m}\left(x_{(r+1),(r+1)}\right)\left(1-F_{r+1:m}\left(x_{(r+1),(r+1)}\right)\right)}\right),$$

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$$\frac{\partial \ell}{\partial \beta}=\frac{m}{\beta}+\sum_{j=1}^{r}\log x_{1,j}+\sum_{k=r+2}^{m}\log x_{m,k}+(\alpha-1)\sum_{j=1}^{r}\frac{\lambda x_{1,j}^{\beta}\log x_{1,j}}{\left(1+\lambda x_{1,j}^{\beta}\right)}$$

$$+(\alpha-1)\sum_{k=r+2}^{m}\frac{\lambda x_{m,k}^{\beta}\log x_{m,k}}{\left(1+\lambda x_{m,k}^{\beta}\right)}-m^{2}\alpha\sum_{j=1}^{r}\left(\left(1+\lambda x_{1,j}^{\beta}\right)^{\alpha-1}\right)\lambda x_{1,j}^{\beta}\log x_{1,j}-\alpha\sum_{k=r+2}^{m}\left(\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha-1}\right)\lambda x_{m,k}^{\beta}\log x_{m,k}$$

$$+(m^{2}-1)\sum_{k=r+1}^{m}\frac{e^{1-\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha}}\alpha\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha-1}\lambda x_{m,k}^{\beta}\log x_{m,k}}{\left(1-e^{1-\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha}}\right)}+\log x_{(r+1),(r+1)}$$

$$+(\alpha-1)\frac{\lambda x_{(r+1),(r+1)}^{\beta}\log x_{(r+1),(r+1)}^{\beta}}{\left(1+\lambda x_{(r+1),(r+1)}^{\beta}\right)}-(r+1)\,\alpha\left(\left(1+\lambda x_{(r+1),(r+1)}^{\beta}\right)^{\alpha-1}\right)\lambda x_{(r+1),(r+1)}^{\beta}\log x_{(r+1),(r+1)}$$

$$+r\frac{e^{1-\left(1+\lambda x_{(r+1),(r+1)}^{\beta}\right)^{\alpha}}\alpha\left(1+\lambda x_{(r+1),(r+1)}^{\beta}\right)^{\alpha-1}\lambda x_{(r+1),(r+1)}^{\beta}\log x_{(r+1),(r+1)}}{\left(1-e^{1-\left(1+\lambda x_{(r+1),(r+1)}^{\beta}\right)^{\alpha}}\right)}+F_{\beta}\left(\frac{1-2F_{r+1:m}\left(x_{(r+1),(r+1)}\right)}{F_{r+1:m}\left(x_{(r+1),(r+1)}\right)\left(1-F_{r+1:m}\left(x_{(r+1),(r+1)}\right)\right)}\right),$$

$$\frac{\partial \ell}{\partial \alpha}=\frac{m}{\alpha}+\sum_{j=1}^{r}\log\left(1+\lambda x_{1,j}^{\beta}\right)+\sum_{k=r+2}^{m}\log\left(1+\lambda x_{m,k}^{\beta}\right)-m^{2}\sum_{j=1}^{r}\left(1+\lambda x_{1,j}^{\beta}\right)^{\alpha}\log\left(1+\lambda x_{1,j}^{\beta}\right)$$

$$-\sum_{k=r+2}^{m}\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha}\log\left(1+\lambda x_{m,k}^{\beta}\right)+(m^{2}-1)\sum_{k=r+1}^{m}\frac{e^{1-\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha}}\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha}\log\left(1+\lambda x_{m,k}^{\beta}\right)}{\left(1-e^{1-\left(1+\lambda x_{m,k}^{\beta}\right)^{\alpha}}\right)}$$

$$+\log\left(1+\lambda x_{(r+1),(r+1)}^{\beta}\right)-(r+1)\left(1+\lambda x_{(r+1),(r+1)}^{\beta}\right)^{\alpha}\log\left(1+\lambda x_{(r+1),(r+1)}^{\beta}\right)$$

$$+r\frac{e^{1-\left(1+\lambda x_{(r+1),(r+1)}^{\beta}\right)^{\alpha}}\left(1+\lambda x_{(r+1),(r+1)}^{\beta}\right)^{\alpha}\log\left(1+\lambda x_{(r+1),(r+1)}^{\beta}\right)}{\left(1-e^{1-\left(1+\lambda x_{(r+1),(r+1)}^{\beta}\right)^{\alpha}}\right)}+F_{\alpha}\left(\frac{1-2F_{r+1:m}\left(x_{(r+1),(r+1)}\right)}{F_{r+1:m}\left(x_{(r+1),(r+1)}\right)\left(1-F_{r+1:m}\left(x_{(r+1),(r+1)}\right)\right)}\right).$$

Where $F_{\lambda}=\partial F_{r+1:m}\left(x_{(r+1),(r+1)}\right)/\partial\lambda$, $F_{\beta}=\partial F_{r+1:m}\left(x_{(r+1),(r+1)}\right)/\partial\beta$ and $F_{\alpha}=\partial F_{r+1:m}\left(x_{(r+1),(r+1)}\right)/\partial\alpha$.

4.4 Estimation Based on GDRSS

According to Eq. (7) the likelihood function of the PGW distribution for GDRSS design is given by

$$L(\theta)=\prod_{j=1}^{m}\left(\frac{m!}{(j-1)!\,(m-j)!}\right)\frac{m!}{(j-1)!\,(m-j)!}\left(\lambda\alpha\beta x^{\beta-1}\left(1+\lambda x^{\beta}\right)^{\alpha-1}\left(\exp\left[1-\left(1+\lambda x^{\beta}\right)^{\alpha}\right]\right)\right)$$

$$\times\left[1-e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}}\right]^{i-1}\left[e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}}\right]^{m-i}\times\left[\sum_{t=j}^{n}\binom{n}{t}\left[1-e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}}\right]^{t}\left[e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}}\right]^{n-t}\right]^{j-1}$$

$$\times\left[1-\sum_{t=j}^{n}\binom{n}{t}\left[1-e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}}\right]^{t}\left[e^{1-\left(1+\lambda x^{\beta}\right)^{\alpha}}\right]^{m-t}\right]^{m-j}.$$

the associated log-likelihood function is as follows

$$\ell=2m\log c+m\log\lambda+m\log\alpha+m\log\beta+(\beta-1)\sum_{j=1}^{m}\log x_{j}+(\alpha-1)\sum_{j=1}^{m}\log\left(1+\lambda x_{j}^{\beta}\right)$$

$$+\sum_{j=1}^{m}(j-1)\log\left(1-e^{1-\left(1+\lambda x_{j}^{\beta}\right)^{\alpha}}\right)+\sum_{j=1}^{m}(m-j+1)\left(1-\left(1+\lambda x_{j}^{\beta}\right)^{\alpha}\right)$$

$$+\sum_{j=1}^{m}(j-1)\log\left(\sum_{t=j}^{m}\binom{m}{t}\left[1-e^{1-\left(1+\lambda x_{j}^{\beta}\right)^{\alpha}}\right]^{t}\left[e^{1-\left(1+\lambda x_{j}^{\beta}\right)^{\alpha}}\right]^{m-t}\right)$$

$$+\sum_{j=1}^{m}(m-j+1)\log\left(1-\sum_{t=j}^{m}\binom{m}{t}\left[1-e^{1-\left(1+\lambda x_{j}^{\beta}\right)^{\alpha}}\right]^{t}\left[e^{1-\left(1+\lambda x_{j}^{\beta}\right)^{\alpha}}\right]^{m-t}\right).$$

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and the first derivatives are given by

$$\frac{\partial \ell}{\partial \lambda}=\frac{m}{\lambda}+(\alpha-1)\sum_{j=1}^{m}\frac{x_j^{\beta}}{\left(1+\lambda x_j^{\beta}\right)}+\sum_{j=1}^{m}(j-1)\frac{e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\alpha\left(1+\lambda x_j^{\beta}\right)^{\alpha-1}x_j^{\beta}}{1-e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}}$$

$$-\sum_{j=1}^{m}(n-j+1)\left(\alpha\left(1+\lambda x_j^{\beta}\right)^{\alpha-1}x_j^{\beta}\right)+\sum_{j=1}^{m}(j-1)\frac{\partial\alpha\left(\sum_{t=j}^{m}\binom{m}{t}\left[1-e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\right]^{t}\left[e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\right]^{m-t}\right)}{\left(\sum_{t=j}^{m}\binom{m}{t}\left[1-e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\right]^{t}\left[e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\right]^{m-t}\right)}$$

$$+\sum_{j=1}^{m}(m-j+1)\frac{\partial\alpha\left(1-\sum_{t=j}^{m}\binom{m}{t}\left[1-e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\right]^{t}\left[e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\right]^{m-t}\right)}{\left(1-\sum_{t=j}^{m}\binom{m}{t}\left[1-e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\right]^{t}\left[e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\right]^{m-t}\right)},$$

$$\frac{\partial \ell}{\partial \beta}=\frac{m}{\beta}+\sum_{j=1}^{m}\log x_j+(\alpha-1)\sum_{j=1}^{m}\frac{\lambda x_j^{\beta}\log x_j^{\beta}}{\left(1+\lambda x_j^{\beta}\right)}$$

$$+\sum_{j=1}^{m}(j-1)\frac{e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\alpha\left(1+\lambda x_j^{\beta}\right)^{\alpha-1}\lambda x_j^{\beta}\log x_j^{\beta}}{1-e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}}-\sum_{j=1}^{m}(m-j+1)\left(\alpha\left(1+\lambda x_j^{\beta}\right)^{\alpha-1}\lambda x_j^{\beta}\log x_j^{\beta}\right).$$

$$+\sum_{j=1}^{m}(j-1)\frac{\partial\beta\left(\sum_{t=j}^{m}\binom{m}{t}\left[1-e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\right]^{t}\left[e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\right]^{m-t}\right)}{\left(\sum_{t=j}^{m}\binom{m}{t}\left[1-e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\right]^{t}\left[e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\right]^{m-t}\right)}$$

$$+\sum_{j=1}^{m}(m-j+1)\frac{\partial\beta\left(1-\sum_{t=j}^{m}\binom{m}{t}\left[1-e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\right]^{t}\left[e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\right]^{m-t}\right)}{\left(1-\sum_{t=j}^{m}\binom{m}{t}\left[1-e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\right]^{t}\left[e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\right]^{m-t}\right)},$$

and

$$\frac{\partial \ell}{\partial \alpha}=\frac{m}{\alpha}+\sum_{j=1}^{m}\log\left(1+\lambda x_j^{\beta}\right)+\sum_{j=1}^{m}(j-1)\frac{e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\left(1+\lambda x_j^{\beta}\right)^{\alpha}\log\left(1+\lambda x_j^{\beta}\right)}{1-e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}}$$

$$-\sum_{j=1}^{m}(m-j+1)\left(\left(1+\lambda x_j^{\beta}\right)^{\alpha}\log\left(1+\lambda x_j^{\beta}\right)\right)$$

$$+\sum_{j=1}^{m}(j-1)\frac{\partial\alpha\left(\sum_{t=j}^{m}\binom{m}{t}\left[1-e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\right]^{t}\left[e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\right]^{m-t}\right)}{\left(\sum_{t=j}^{m}\binom{m}{t}\left[1-e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\right]^{t}\left[e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\right]^{m-t}\right)}$$

$$+\sum_{j=1}^{n}(n-j+1)\frac{\partial\alpha\left(1-\sum_{t=j}^{m}\binom{m}{t}\left[1-e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\right]^{t}\left[e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\right]^{m-t}\right)}{\left(1-\sum_{t=j}^{m}\binom{m}{t}\left[1-e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\right]^{t}\left[e^{1-\left(1+\lambda x_j^{\beta}\right)^{\alpha}}\right]^{m-t}\right)}.$$

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5 Simulation Study

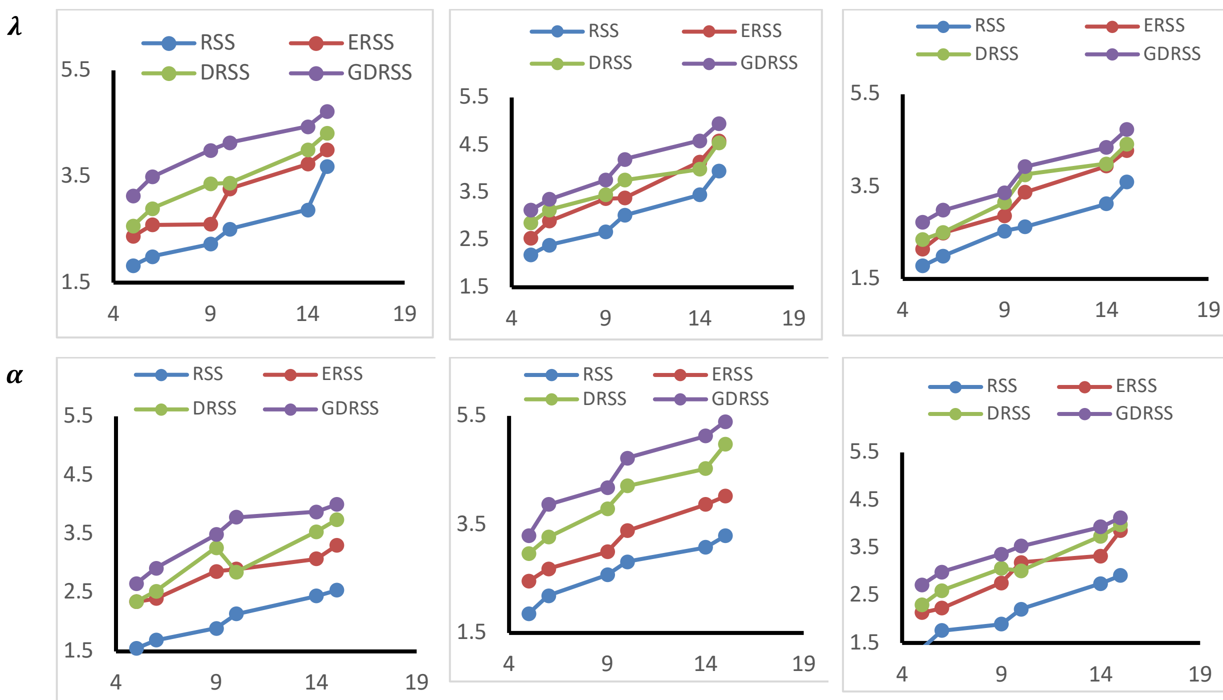
Sample units generated by the proposed sampling designs only become order statistics when the ranking process is done without any error (perfect ranking). Because of this, the RSS-Based designs produce sample units that are neither independent nor identically distributed which makes it difficult to analytically derive some of the properties of their respective estimators [see Taconeli & Cabral (2019)]. Therefore, an extensive simulation study is conducted to evaluate the derived likelihood function estimator's performance and compare their performance with other **RSS**-based designs estimators' performance. The Monte Carlo simulation is made for the **PGW** distribution with different parameter values to ensure a wide range of shapes of the **PGW**, namely PGW(1,1.3,1.5), **PGW**(0.4, 0.7, 4) and PGW(0.06,0.025,5). Figure 1 shows the density function for the PGW distribution for the initial parameter values used in the simulation. The simulation is made for samples of sizes $\mathbf{5, 6, 9, 10, 14}$ and $\mathbf{15}$ and 50,000 replications. Let $\hat{\boldsymbol{\theta}}_{\boldsymbol{k}}$ be the $\boldsymbol{k}$th sample estimator generated by a particular RSS-based sampling design $\boldsymbol{k} = \mathbf{1, 2, \ldots, 50,000}$. The comparison are made using two criteria: the Relative Bias (**RB**) and Mean Square Errors (**MSE**), which are calculated as follows:

$$\boldsymbol{RB} = \sum_{i=1}^{50,000} \frac{\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}}{\boldsymbol{\theta}}; \quad \boldsymbol{MSE}(\hat{\boldsymbol{\theta}}) = \frac{\mathbf{1}}{\mathbf{50,000}} \sum_{k=1}^{50,000} (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}).$$

The Relative Efficiency (**RE**) to **SRS** estimators is calculated for each **RSS**-based design, by

$$\boldsymbol{RE}(\hat{\boldsymbol{\theta}}) = \frac{\boldsymbol{MSE}_{\boldsymbol{SRS}}(\hat{\boldsymbol{\theta}})}{\boldsymbol{MSE}_{\boldsymbol{SRS}}(\hat{\boldsymbol{\theta}})}.$$

All simulations are performed using routines developed by the authors in the R environment for statistical computing. Simulation results are shown in Tables 1 and 2. Also figure 4 shows the performance of the different RSS designs for the different parameters.



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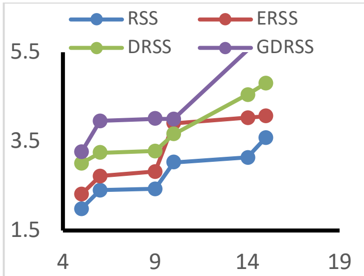
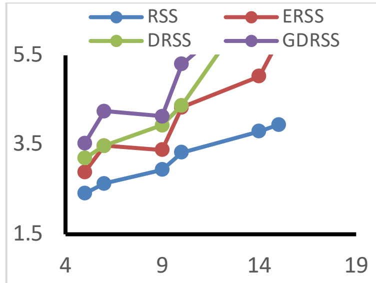
β



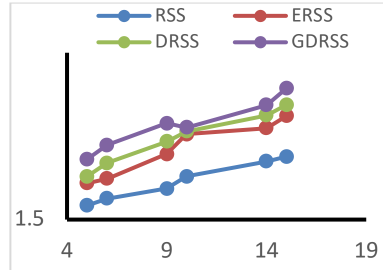
a

PGW(1,1.3,1.5)



b

PGW(0.4,0.7,4)



c

PGW(0.06,0.25,5)

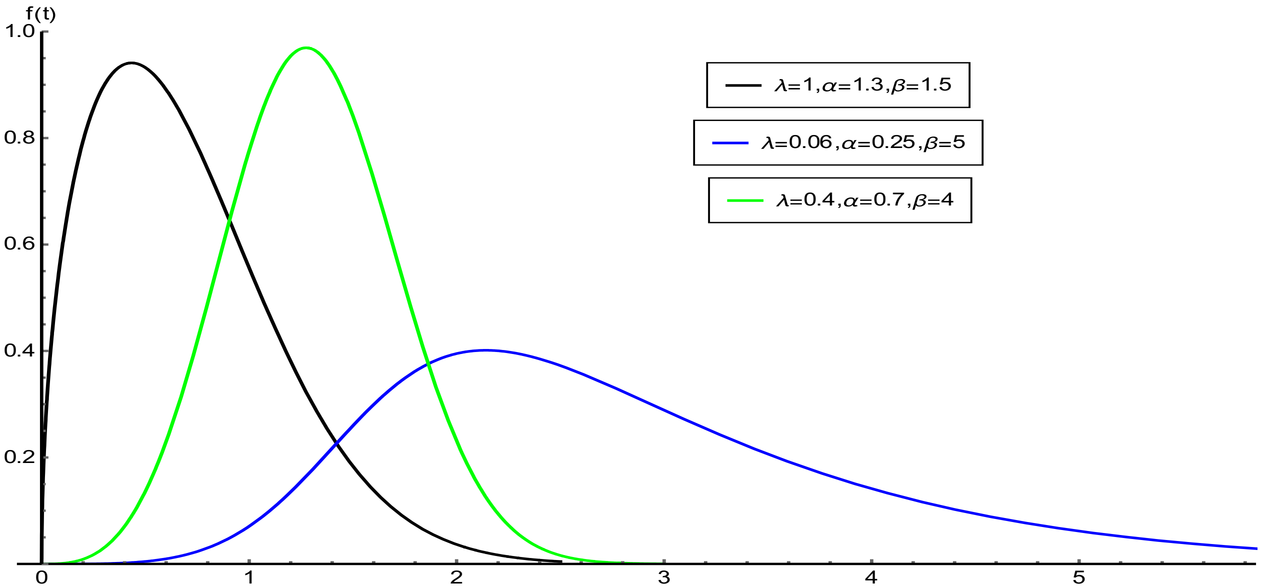


Fig. 4: The density function of the PGW distribution for different parameter values.

From figures and tables it can be noticed that:

1. As the sample size increases the relative bias decreases for λ, α, and β.
2. As λ and α decreases and the sample sizes increase, the performance of the estimators of λ, α, and β for different designs become higher.
3. GDRSS Design provides more efficient estimator than ERSS and RSS estimator and is slightly higher than the DRSS design for all the distribution parameters.
4. The efficiency of both ERSS and DRSS for some sample sizes are nearly close but the overall performance of DRSS is higher than the ERSS design.
5. Regarding the distribution shape, as the distribution becomes almost symmetric the RE is always higher than the RE for the asymmetric shapes of the distribution.

6 Conclusions

In this paper, we have derived the likelihood function for the **GDRSS** design and compare it with the RSS, ERSS and DRSS designs. Moreover, the **MLE** for **PGW** distribution based on **SRS**, **RSS**, **ERSS**, **DRSS** and **GDRSS** has been done.

An intensive numerical comparison between the **SRS** and different **RSS** deigns is done and it is shown than that the **GDRSS** is more efficient for all values for the scale parameter and the two-shape parameters of the PGW distribution. we have found that the likelihood estimation based on GDRSS proposed by Taconeli & Cabral (2019) provides slightly more efficient estimators than the likelihood estimation based on the DRSS designs proposed by (Sabry et al., 2019).

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Table 1: Relative efficiency for RSS-based estimators compared to SRS based estimators under perfect ranking.

PGW(λ,α,β)	m	RSS			ERSS			DRSS			GDRSS		
		λ	α	β	λ	α	β	λ	α	β	λ	α	β
PGW(1,1.3,1.5)	5	1.823	2.370	2.570	3.136	1.550	2.344	2.344	2.652	1.990	2.312	3.010	3.258
	6	1.992	2.590	2.896	3.496	1.693	2.398	2.525	2.915	2.403	2.723	3.242	3.960
	9	2.226	2.601	3.360	3.992	1.892	2.860	3.266	3.488	2.430	2.823	3.279	4.006
	10	2.514	3.268	3.382	4.138	2.137	2.899	2.852	3.782	3.032	3.905	3.662	4.001
	14	2.873	3.735	3.999	4.440	2.442	3.075	3.539	3.875	3.136	4.026	4.544	5.550
	15	3.687	3.998	4.315	4.732	2.539	3.301	3.740	3.999	3.585	4.062	4.802	5.682
PGW(0.4,0.7,4)	5	2.188	2.538	2.854	3.125	1.859	2.452	2.963	3.296	2.412	2.894	3.198	3.535
	6	2.390	2.897	3.132	3.353	2.186	2.680	3.265	3.865	2.635	3.479	3.482	4.249
	9	2.671	3.371	3.451	3.762	2.570	2.994	3.789	4.179	2.945	3.387	3.939	4.136
	10	3.017	3.382	3.758	4.200	2.813	3.382	4.207	4.721	3.326	4.340	4.384	5.301
	14	3.448	4.145	3.992	4.592	3.076	3.865	4.526	5.126	3.801	5.028	6.855	6.789
	15	3.958	4.584	4.545	4.945	3.287	4.018	4.973	5.397	3.952	5.837	6.981	7.130
PGW(0.06,0.025,5)	5	1.788	2.154	2.354	2.735	1.385	2.134	2.302	2.721	1.838	2.371	2.533	2.943
	6	1.997	2.497	2.513	2.989	1.759	2.231	2.599	2.990	2.008	2.478	2.858	3.287
	9	2.539	2.871	3.157	3.366	1.892	2.763	3.066	3.366	2.244	3.068	3.373	3.806
	10	2.631	3.382	3.758	3.937	2.212	3.190	3.016	3.532	2.534	3.543	3.615	3.705
	14	3.128	3.945	3.996	4.351	2.744	3.327	3.739	3.939	2.896	3.696	3.998	4.241
	15	3.599	4.284	4.418	4.738	2.915	3.864	3.984	4.131	3.011	3.983	4.251	4.647

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Table 2: Relative bias for RSS-based estimators under perfect ranking.

PGW(λ,α,β)	m	RSS			ERSS			DRSS			GDRSS		
		λ	α	β	λ	α	β	λ	α	β	λ	α	β
PGW(1,1.3,1.5)	5	0.0125	0.0101	0.0081	0.0095	0.0077	0.0062	0.0093	0.0075	0.0060	0.0082	0.0066	0.0053
	6	0.0097	0.0078	0.0063	0.0074	0.0059	0.0047	0.0072	0.0058	0.0046	0.0063	0.0051	0.0041
	9	0.0052	0.0042	0.0034	0.0040	0.0032	0.0025	0.0039	0.0031	0.0025	0.0034	0.0027	0.0022
	10	0.0026	0.0021	0.0017	0.0020	0.0016	0.0013	0.0019	0.0015	0.0012	0.0017	0.0014	0.0011
	14	0.0008	0.0006	0.0005	0.0006	0.0005	0.0004	0.0006	0.0005	0.0004	0.0005	0.0004	0.0003
	15	0.0005	0.0004	0.0003	0.0004	0.0003	0.0002	0.0004	0.0003	0.0002	0.0003	0.0003	0.0002
PGW(0.4,0.7,4)	5	0.0100	0.0081	0.0065	0.0008	0.0007	0.0005	0.0007	0.0006	0.0005	0.0006	0.0005	0.0004
	6	0.0078	0.0062	0.0050	0.0006	0.0005	0.0004	0.0006	0.0004	0.0004	0.0005	0.0004	0.0003
	9	0.0042	0.0033	0.0027	0.0003	0.0003	0.0002	0.0003	0.0002	0.0002	0.0003	0.0002	0.0002
	10	0.0021	0.0017	0.0013	0.0002	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
	14	0.0006	0.0005	0.0004	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	15	0.0004	0.0003	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
PGW(0.06,0.025,5)	5	0.0108	0.0087	0.0070	0.0086	0.0070	0.0056	0.0010	0.0008	0.0007	0.0009	0.0007	0.0006
	6	0.0084	0.0067	0.0054	0.0067	0.0054	0.0043	0.0008	0.0007	0.0005	0.0007	0.0006	0.0005
	9	0.0045	0.0036	0.0029	0.0036	0.0029	0.0023	0.0004	0.0003	0.0003	0.0004	0.0003	0.0002
	10	0.0022	0.0018	0.0014	0.0018	0.0014	0.0012	0.0002	0.0002	0.0001	0.0002	0.0002	0.0001
	14	0.0007	0.0006	0.0004	0.0006	0.0004	0.0004	0.0001	0.0001	0.0000	0.0001	0.0000	0.0000
	15	0.0004	0.0003	0.0003	0.0003	0.0003	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

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References

[1] Abu-Dayyeh, W. A., Al-Subh, S. A. and Muttlak, H. A. (2004). Logistic parameters estimation using simple random sampling and ranked set sampling. Appl. Math. Comput., **150**, 543-554, 2004.

[2] Abu-Dayyeh, W. A., Assrhani, A. and Ibrahim, K. (2013). Estimation of the shape and scale parameters of Pareto distribution using ranked set sampling. Stat. Pap., **54(1)**, 207-225, 2013.

[3] Akgül, F. G. and Şenoğlu, B. (2017). Estimation of P(X<Y) Using Ranked Set Sampling for Weibull Distribution. QTQM, **14(3)**, 296-309, 2017.

[4] Al-Nasser, A. D. (2007). L ranked set sampling: A generalization procedure for robust visual sampling. Commun. Stat. Simulat., **36**, 33-43, 2007.

[5] Al-Omari, A. I. and Al-Hadhrami, S., (2011). On Maximum Likelihood Estimators of the Parameters of a Modifed Weibull Distribution Using Extreme Ranked Set Sampling. Journal of Modern Applied Statistical Methods, **10(2)**, 607-617, 2011

[6] Al-Saleh, M. F. and Al-Kadiri, M. A. (2000). Double ranked set sampling. Stat. Prob. Lett., **48(2)**, 205-212, 2000.

[7] Bagdonavicius, V., and M. Nikulin. 2001. Accelerated life models: modeling and statistical analysis: Chapman and Hall/CRC, Boca Raton, 2001.

[8] Bebbington, M., Lai, C.D. and Zitikis, R. (2007). A flexible weibull extension. Reliability.

[9] Bhoj, D. S. (1997). Estimation of parameters of the Extreme Value distribution using ranked set sampling. Commun. Stat. Theory., **26(3)**, 653-667, 1997.

[10] Cordeiro, G.M., Edwin, M.M. Ortega and Nadarajah, S., (2010). The Kumaraswamy Weibull distribution with application to failure data. J. Franklin Inst., 347 , 1399-1429. Engineering and System Safety, **92**, 719-726, 2010.

[11] Ghitany M.E., Al-Hussaini E.K. and Al-Jarallah R.A., (2005). Marshall-Olkin extended Weibull distribution and its application to censored data. J. Appl. Stat. **32**, 1025-1034, 2005.

[12] Helu, A., Abu-Salih, M. and Alkam, O. (2010). Bayes estimation of Weibull distribution parameters using ranked set sampling. Commun. Stat. Theory., **39**, 2533-2551, 2010.

[13] Lam, K., Sinha, B. K. and Wu, Z. (1994). Estimation of parameters in a two parameter exponential distribution using ranked set sampling. Ann. Inst. Statist. Math. **46(4)**, 723-736, 243-266, 1994.

[14] McIntyre, G. A. (1952). A method for unbiased selective sampling, using ranked sets. Aust. J. Agr. Res., **3**, 385–390, 1952.

[15] Mudholkar, G.S. and Srivastava, D.K., (1993). Exponentiated Weibull family for analyzing bathtub failure-real data, IEEE Transaction Reliability., **42**, 299-302, 1993.

[16] Muttlak, H. A. (1997). Median ranked set sampling. J. Appl. Stat. Sci., **6(4)**, 245- 255, 1997.

[17] Muttlak, H. A. (2003). Modified ranked set sampling methods. Pak. J. Stat., **19(3)**, 315-323, 2003.

[18] Nikulin, N and Haghighi, F., (2007). A chi-squared test for the genralized power Weibull family for the head-and-neck cancer censored data. Journal of Mathematical Sciences., **142 (3)**, 2204-2204, 2007.

[19] Nikulin, N and Haghighi, F., (2009). On the power generalized Weibull family: model for cancer censored data. Metron., **67(1)**, 75-86, 2009.

[20] Risti´c, M. M. and Balakrishnan, N. (2012). The gamma-exponentiated exponential distribution. J. Statist. Comput., Simul., **82** , 1191-1206, 2012.

[21] Sabry, M. A., Shaaban, M., Muhammed, H. Z. and Nabih, A. (2019). Parameter Estimation based on Double Ranked Set Samples with Applications to Weibull Distribution. Journal of Modern Applied Statistical Methods. 19(1), Under publication, 2019.

[22] Samawi, H. M., Ahmed, M. S. and Abu-Dayyeh, W. A. (1996). Estimating the population mean using extreme ranked set sampling. The Biometrical J., **38(5)**, 577-586, 1996.

[23] Silva, G.O., Edwin, M. M. Ortega and Cordeiro, G. M., (2010). The beta modified Weibull distribution. Lifetime Data Anal., **16** , 409-430, 2010.

[24] Taconeli, C. A., and Cabral, A. S., (2019). New two-stage sampling designs based on neoteric ranked set sampling, Journal of Statistical Computation and Simulation, 89:2, 232-248. [25] B. Gates, Why word is used, *Microsoft Lett.*, **69 (5)**, 53-60, 1998, 2019.

[26] Wahed, Abdus S., Luong, The Minh and Jeong, Jong-Hyeon. (2009). A new generalization of Weibull distribution with application to a breast cancer data set. Stat Med., **28**, 2077-2094, 2009.

[27] Weibull, W. 1951. Wide applicability. Journal of applied mechanics., **103(730)**, 293-297, 1951.

[28] Zamanzade, E., and Al-Omari, A.I. (2016). New ranked set sampling for estimating the population mean and variance. Hacettepe Journal of Mathematics and Statistics, **45(6)**, 1891-1905, 2016.

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