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Generalized Record Values from Distributions Having Power Hazard Function and Characterization

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Abstract: This paper presents exact expressions as well as some recurrence relations for single and product moments for the generalized upper record values ($k-$th upper record values) , when the parent population follows the distribution having power hazard function, abbreviated as (DPHF). Further, various of its deductions and particular cases are discussed. At the end, the characterization results based on recurrence relations and conditional moment are established and some computational work also carried out.

Keywords: Generalized record values, $k-$th record values, single and product moments, recurrence relations, conditional expectation and characterization.

1 Introduction

Record values and associated statistics are of great importance in several real life problems involving weather, economic studies, sports and so on. The prediction of a future record value is an interesting problem in many real life applications. For example, the predicted value of the amount of next record level of water that a dam will storage from rain and control or discharge is helpful for future planning purposes, predicted intensity of the next strongest earthquake is essential for disaster management planning, prediction of next level of new record in athletic events is helpful for subjecting the prospective athletes to rigorous training, practice and so on.

Chandler[1]introduced the study of record values and documented many of the basic properties of records. For more details and applications of record values, readers may be refer to Glick [2] , Ahsanullah [3] and Arnold et al. [4] .
Applications of $k-$th record values can be found in the literature, for instance, see the examples cited in Kamps [5] and Danielak and Raqab [6] in reliability theory.
Record values can be viewed as order statistics from a sample whose size is determined by the values and the order of occurrence of the observations.
Let $\{X_n\,,\, n \geq 1\}$ be a sequence of *iid* random variables with distribution function $(df)F(x)$ and probability density function $(pdf)f(x)$. The $j-$th order statistic of a sample $X_1, X_2, \ldots, X_n$ is denoted by $X_{j:n}$. For a fixed positive integer k, we define the sequence $\{U_n^{(k)},\, n \geq 1\}$ of $k-$th upper record times of $\{X_n\,,\, n \geq 1\}$ as follows:

$$U_1^{(k)} = 1$$
$$U_{n+1}^{(k)} = min\{j > U_n^{(k)} : X_{j:j+k-1} > X_{U_n^{(k)}:U_n^{(k)}+k-1}\}.$$

The sequence $\{Y_n^{(k)},\, n \geq 1\}$, where $Y_n^{(k)} = X_{U_n^{(k)}}$ is called the sequence of $k-$th upper record values or generalized upper record values of $\{X_n,\, n \geq 1\}$. Note that for $k = 1$, we write $Y_n^{(1)} = X_{U_n},\, n \geq 1$, which are the upper record values of $\{X_n,\, n \geq 1\}$ as defined in (Ahsanullah[3]). Moreover, we see that $Y_0^{(k)} = 0$ and $Y_1^{(k)} = \min(X_1, X_2, \ldots, X_n) = X_{1:k}$. The

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pdf of $Y_n^{(k)}$ and the joint pdf of $Y_m^{(k)}$ and $Y_n^{(k)}$ are given by (Dziubdziela and Kopociński [7] and, Grudzień [8])

$$f_{Y_n^{(k)}}(x) = \frac{k^n}{(n-1)!}[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}f(x), \quad n \geq 1, \tag{1}$$

and

$$f_{Y_m^{(k)},Y_n^{(k)}}(x,y) = \frac{k^n}{(m-1)!(n-m-1)!}[-\ln \bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}$$

$$\times[\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1}[\bar{F}(y)]^{k-1}f(y), \quad x < y, 1 \leq m < n, n \geq 2, \tag{2}$$

where $\bar{F}(x) = 1 - F(x)$.

Several authors have been extensively studied on the topic of generalized upper record values for different distributions . For a detailed survey one may refer to Grudzieǹ and Szynal [9], Pawlas and Szynal [10,11,12], Minimol and Thomas [13,14], Khan and Khan [15] and Khan, et al.[16] respectively. In this paper, we focus the moments properties of generalized upper record values, when parent population follows DPHF.

The power hazard function has been defined by Mugdadi [17],

$$h(x) = \alpha x^{\beta}, \qquad x > 0, \ \alpha > 0, \ \beta > -1 \tag{3}$$

Corresponding to this hazard function, its cumulative distribution function *(cdf)* is given by,

$$F(x) = 1 - exp\left(-\frac{\alpha}{\beta+1}x^{\beta+1}\right), \qquad x > 0, \ \alpha > 0, \ \beta > -1 \tag{4}$$

with the corresponding *pdf*

$$f(x) = \alpha x^{\beta} exp\left(-\frac{\alpha}{\beta+1}x^{\beta+1}\right), \qquad x > 0, \ \alpha > 0, \ \beta > -1 \tag{5}$$

The distribution with density defined in (5) is known as DPHF . For $-1 < \beta < 0$ DPHF has a decreasing hazard function and $\beta > 0$, DPHF has an increasing hazard function. For more details, properties of DPHF, see Ismail [18].
It is clear that some well- known life time distributions as Weibull, Rayleigh and exponential are special cases of DPHF. Such that,
If $\beta = \alpha - 1$ then DPHF reduces to Weibull $(\alpha, 1)$.
If $\alpha = \frac{1}{\theta^2}, \beta = 1$ then DPHF reduces to Rayleigh (θ) .
If $\beta = 0$ then DPHF reduces to exponential distribution with mean $\frac{1}{\alpha}$.
Note that for DPHF defined in (5),

$$f(x) = \alpha x^{\beta} \bar{F}(x) \tag{6}$$

The relation in (6) will be used to derive some simple recurrence relations for the moments of $k-th$ upper record values from DPHF.
It appears from literature that no attention has been paid on the characterization of DPHF based on $k-th$ upper record values.

2 Relations for Single Moments

Here we derive the exact expressions and recurrence relations for single moments of generalized upper record values from DPHF. First we describe the exact expression for single moments of generalized upper record values in the following theorem.
Theorem 2.1. For the distribution given in (4). Fix a positive integer $k \geq 1$, for $n \geq 1$, $n \geq k$ and $j = 0, 1, \ldots$

$$E(Y_n^{(k)})^j = \left(\frac{\beta+1}{\alpha k}\right)^{j/(\beta+1)} \times \frac{\Gamma[n + j/(\beta+1)]}{(n-1)!}. \tag{7}$$

Proof. From (1), we have

$$E(Y_n^{(k)})^j = \frac{k^n}{(n-1)!}\int_0^{\infty} x^j[-\ln \bar{F}(x)]^{n-1}[\bar{F}(x)]^{k-1}f(x)\,dx. \tag{8}$$

Setting $t = -\ln \bar{F}(x)$ in (8), we find that

$$E(Y_n^{(k)})^j = \left(\frac{\beta+1}{\alpha k}\right)^{j/(\beta+1)} \frac{1}{(n-1)!} \int_0^\infty t^{n+j/(\beta+1)-1} e^{-tk} dt \tag{9}$$

which upon simplification yields (7).

Table 2.1. First four moments of upper record values

n	$\alpha=1, \beta=1, k=1$				$\alpha=1, \beta=2, k=1$			
	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$
1	2.506628	2.000000	2.506628	4.000	3.863698	2.816679	3.00000	3.863698
2	1.414214	2.000000	5.656854	24.000	1.952976	1.857472	3.00000	6.50992
3	0.6266571	2.000000	16.44975	240.000	0.7211248	1.040042	3.00000	12.98025
4	0.2357023	2.000000	56.56854	3360.000	0.2146499	0.521607	3.00000	28.93798
5	0.0783321	2.000000	220.5294	60480.000	0.0542493	0.240783	3.00000	70.13568
n	$\alpha=1, \beta=3, k=1$				$\alpha=2, \beta=4, k=1$			
	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$
1	5.127387	3.544908	3.466002	4.00000	6.334123	4.222608	3.911405	4.21905
2	2.506628	2.000000	2.506628	4.00000	3.060461	2.21629	2.411596	3.23800
3	0.866500	0.886226	1.602308	4.00000	1.027341	0.873937	1.223153	2.25074
4	0.235702	0.333333	0.942809	4.00000	0.267720	0.283490	0.543765	1.46403
5	0.053410	0.110778	0.521254	4.00000	0.057488	0.079318	0.218877	0.905974

Corollary 2.1. The exact expression for single moments of upper record values from DPHF has the form

$$E(Y_n^{(1)})^j = E(X_{U_n}^j) = \left(\frac{\beta+1}{\alpha}\right)^{j/(\beta+1)} \times \frac{\Gamma[n+j/(\beta+1)]}{(n-1)!}.$$

The following theorem gives the recurrence relations for single moments of generalized upper record values from DPHF.
Theorem 2.2. For the distribution given in (4). Fix a positive integer $k \geq 1$, for $n \geq 1$, $n \geq k$ and $j = 0, 1, \ldots$

$$E(Y_n^{(k)})^{j+\beta+1} = E(Y_{n-1}^{(k)})^{j+\beta+1} + \frac{(j+\beta+1)}{\alpha k} E(Y_n^{(k)})^j. \tag{10}$$

Proof. From (1) and (6), for $n \geq 1$ and $j = 0, 1, \ldots$, we have

$$E(Y_n^{(k)})^j = \frac{\alpha k^n}{(n-1)!\,\theta^\beta} \int_0^\infty x^{j+\beta} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^k dx. \tag{11}$$

Integrating by parts, taking $x^{j+\beta}$ as the part to be integrated and simplifying the resulting expression, we get the relation given in (10).

Remark 2.1.
i) Putting $\beta = 0$ in (10), we deduce the recurrence relation for single moments of $k-$ th upper record values from exponential distribution as established by Pawlas and Szynal [10].
ii) Setting $\beta = \alpha - 1$ in (10), we get the recurrence relation for single moments of $k-$ th upper record values from Weibull distribution, which verify the results obtained by Pawlas and Szynal [12].
iii) Putting $\alpha = 1/\theta^2$ and $\beta = 1$ in (10), the recurrence relation for single moments of $k-$ th upper record values is deduced for Rayleigh distribution.
Corollary 2.2. The recurrence relation for single moments of upper record values from DPHF has the following form

$$E[X_{U_n}^{j+\beta+1}] = E[X_{U_{n-1}}^{j+\beta+1}] + \frac{(j+\beta+1)}{\alpha} E[X_{U_n}^j]$$

3 Relations for Product Moments

This section contains the explicit expressions and recurrence relations for product moments of generalized upper record values from DPHF. We shall first establish the explicit expression for the product moments of generalized upper record values in the following theorem.

Theorem 3.1. For the distribution given in (4). Fix a positive integer $k \geq 1$, for $1 \leq m \leq n-2$ and $i, j = 0, 1, \ldots$

$$E[(Y_m^{(k)})^i (Y_n^{(k)})^j] = \left(\frac{\beta+1}{\alpha}\right)^{(i+j)/(\beta+1)} \frac{k^n}{(m-1)!(n-m-1)!} \sum_{u=0}^{n-m-1} \frac{(-1)^{n-m-u-1}}{[n+\{i/(\beta+1)\}-u-1]}$$

$$\times \binom{n-m-1}{u} \frac{\Gamma[n+\{(i+j)/(\beta+1)\}]}{k^{n+\{(i+j)/(\beta+1)\}}} \quad (12)$$

Proof. From (2), we have

$$E[(Y_m^{(k)})^i (Y_n^{(k)})^j] = \frac{k^n}{(m-1)!(n-m-1)!} \int_0^{\infty} \int_0^{y} x^i y^j [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)}$$

$$\times [\ln \overline{F}(x) - \ln \overline{F}(y)]^{n-m-1} [\bar{F}(y)]^{k-1} f(y)\, dx\, dy. \quad (13)$$

On expanding $[\ln \overline{F}(x) - \ln \overline{F}(y)]^{n-m-1}$ binomially in (13), we get

$$E[(Y_m^{(k)})^i (Y_n^{(k)})^j] = \frac{k^n}{(m-1)!(n-m-1)!} \sum_{u=0}^{n-m-1} (-1)^{n-m-u-1} \binom{n-m-1}{u}$$

$$\times \int_0^{\infty} y^j [-\ln \bar{F}(y)]^u [\bar{F}(y)]^{k-1} f(y) I(y)\, dy, \quad (14)$$

where,

$$I(y) = \int_0^y x^i [-\ln \bar{F}(x)]^{n-u-2} \frac{f(x)}{\bar{F}(x)} dx. \quad (15)$$

By setting $t = -\ln \overline{F}(x)$ in (15), we obtain

$$I(y) = \left(\frac{\beta+1}{\alpha}\right)^{i/(\beta+1)} \int_0^{-\ln \bar{F}(y)} t^{n+\{i/(\beta+1)\}-u-2} dt$$

$$= \left(\frac{\beta+1}{\alpha}\right)^{i/(\beta+1)} \frac{[-\ln F(y)]^{n+i/(\beta+1)-u-1}}{[n+\{i/(\beta+1)\}-u-1]}.$$

On substituting the above expression of $I(y)$ in (14), we find that

$$E[(Y_m^{(k)})^i (Y_n^{(k)})^j] = \left(\frac{\beta+1}{\alpha}\right)^{(i+j)/(\beta+1)} \frac{k^n}{(m-1)!(n-m-1)!} \sum_{u=0}^{n-m-1} \frac{(-1)^{n-m-u-1}}{[n+\{i/(\beta+1)\}-u-1]} \binom{n-m-1}{u}$$

$$\times \int_0^{\infty} y^j [-\ln F(y)]^{n+\{i/(\beta+1)\}-1} [\bar{F}(y)]^{k-1} f(y)\, dy. \quad (16)$$

Again by setting $z = -\ln \overline{F}(y)$ in (16) and simplifying the resulting expression, we get

$$E[(Y_m^{(k)})^i (Y_n^{(k)})^j] = A \int_0^{\infty} z^{[n-(i+j)(\beta+1)]-1} e^{-kz} dz,$$

where

$$A = \left(\frac{\beta+1}{\alpha}\right)^{(i+j)/(\beta+1)} \frac{k^n}{(m-1)!(n-m-1)!} \sum_{u=0}^{n-m-1} \frac{(-1)^{n-m-u-1}}{[n+\{i/(\beta+1)\}-u-1]} \binom{n-m-1}{u}$$

and hence the result.

Identity 3.1. For $1 \le m < n$

$$\sum_{u=0}^{n-m-1} \frac{(-1)^{n-m-u-1}}{(n-u-1)} \binom{n-m-1}{u} = \frac{(m-1)!(n-m-1)!}{(n-1)!}. \tag{17}$$

Proof. Putting $i = j = 0$ in (12), we get the required result.
Remark 3.1. At $i = 0$ in (12), we have

$$[E(Y_n^{(k)})^j = \left(\frac{\beta+1}{\alpha}\right)^{j/(\beta+1)} \frac{k^n}{(m-1)!(n-m-1)!} \sum_{u=0}^{n-m-1} \frac{(-1)^{n-m-u-1}}{(n-u-1)} \binom{n-m-1}{u} \frac{\Gamma[n+j/(\beta+1)]}{k^{n+j/(\beta+1)}} \tag{18}$$

Making use of (17) in (18), we find that

$$E(Y_n^{(k)})^j = \left(\frac{\beta+1}{\alpha k}\right)^{j/(\beta+1)} \times \frac{\Gamma[n+j/(\beta+1)]}{(n-1)!},$$

which is the exact expression for single moments from DPHF as obtained in (7).
Corollary 3.1. The exact expression for product moments of upper record values from DPHF has the form

$$E[(Y_m^{(1)})^i (Y_n^{(1)})^j] = E(X_{U_m}^i X_{U_n}^j) = \left(\frac{\beta+1}{\alpha}\right)^{(i+j)/(\beta+1)} \frac{1}{(m-1)!(n-m-1)!} \sum_{u=0}^{n-m-1} \frac{(-1)^{n-m-u-1}}{[n+\{i/(\beta+1)\}-u-1]}$$
$$\times \binom{n-m-1}{u} \Gamma[n+\{(i+j)/(\beta+1)\}].$$

The following theorem gives the recurrence relations for product moments of generalized upper record values.
Theorem 3.2. For the distribution given in (4) and $m \ge 1$, $m \ge k$ and $i,\ j = 0, 1, \dots$

$$E[(Y_m^{(k)})^i (Y_{m+1}^{(k)})^{j+\beta+1}] = E(Y_m^{(k)})^{i+j+\beta+1} + \frac{(j+\beta+1)}{\alpha k} E[(Y_m^{(k)})^i (Y_{m+1}^{(k)})^j] \tag{19}$$

and for $1 \le m \le n-2$, $i,\ j = 0, 1, \dots$

$$E[(Y_m^{(k)})^i (Y_n^{(k)})^{j+\beta+1}] = E[(Y_m^{(k)})^i (Y_{n-1}^{(k)})^{j+\beta+1}] + \frac{(j+\beta+1)}{\alpha k} E[(Y_m^{(k)})^i (Y_n^{(k)})^j]. \tag{20}$$

Proof. From (2) and (5), we have

$$E[(Y_m^{(k)})^i (Y_n^{(k)})^j] = \frac{\alpha k^n}{(m-1)!\,(n-m-1)!} \int_0^\infty \int_x^\infty x^i y^{j+\beta-1} [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)}$$
$$\times [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} [\bar{F}(y)]^k dy\, dx. \tag{21}$$

(21) can be solved in view of Khan *et al.*[16] by noting that

$$E[(Y_m^{(k)})^i (Y_n^{(k)})^j] - E[(Y_m^{(k)})^i (Y_{n-1}^{(k)})^j] = \frac{j k^{n-1}}{(m-1)!(n-m-1)!} \int_0^\infty \int_x^\infty x^i y^{j-1}$$
$$\times [-\ln \bar{F}(x)]^{m-1} \frac{f(x)}{\bar{F}(x)} [\ln \bar{F}(x) - \ln \bar{F}(y)]^{n-m-1} [\bar{F}(y)]^k dy dx.$$

Proceeding in a similar manner for the case $n = m+1$, the recurrence relation given in (19) can easily be established. One can also note that Theorem 2.2 can be deduced from Theorem 3.2 by putting $i = 0$.

Remark 3.2.
i)Putting $\beta = 0$ in (20), we deduce the recurrence relation for product moments of $k-th$ upper record values from exponential distribution as established by Pawlas and Szynal [10].
ii) Setting $\beta = \alpha - 1$ in (20), we get the recurrence relation for product moments of $k-th$ upper record values from Weibull distribution, which verify the results obtained by Pawlas and Szynal [12].
iii) Putting $\alpha = \frac{1}{\theta^2}$ and$\beta = 1$ in (20), the recurrence relation for product moments of $k-th$ upper record values is deduced for Rayleigh distribution.
Corollary 3.2. The recurrence relations for product moments of upper record values from DPHF has the following form,

$$E(X_{U_m}^i X_{U_n}^{j+\beta}) = E(X_{U_m}^i X_{U_{n-1}}^{j+\beta}) + \frac{(j+\beta+1)}{\alpha k} E(X_{U_m}^i X_{U_n}^j).$$

4 Characterizations

Theorem 4.1. Fix a positive integer $k \geq 1$ and let j be a non-negative integer, a necessary and sufficient condition for a random variable X to be distributed with $f(x)$ given by (4) is that

$$E(Y_n^{(k)})^{j+\beta+1} = E(Y_{n-1}^{(k)})^{j+\beta+1} + \frac{(j+\beta+1)}{\alpha k} E(Y_n^{(k)})^j. \tag{22}$$

for $n = 1, 2, \ldots$ and $n \geq k$.
Proof. The necessary part follows immediately from (10). On the other hand if the recurrence relation in (22) is satisfied, then on using (1), we have

$$\frac{k^n}{(n-1)!} \int_0^\infty x^{j+\beta+1} [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x)\, dx$$
$$= \frac{k^{n-1}}{(n-2)!} \int_0^\infty x^{j+\beta+1} [-\ln \bar{F}(x)]^{n-2} [\bar{F}(x)]^{k-1} f(x) dx$$
$$+ \frac{(j+\beta+1)\, k^{n-1}}{\alpha\, (n-1)!} \int_0^\infty x^j [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} f(x) dx. \tag{23}$$

Integrating the left hand side in (23) by parts and simplifying the resulting expression, we find that

$$\frac{(j+\beta+1)\, k^{n-1}}{(n-1)!} \int_0^\infty x^j [-\ln \bar{F}(x)]^{n-1} [\bar{F}(x)]^{k-1} \left\{ \frac{1}{\alpha} f(x) - x^\beta \bar{F}(x) \right\} dx = 0. \tag{24}$$

Now applying a generalization of the Müntz-Szász Theorem (see for example Hwang and Lin [19]) to (24), we obtain,

$$f(x) = \alpha x^\beta \bar{F}(x),$$

which proves that $f(x)$ has the form as in (5).
Corollary 4.1. Under the assumptions of Theorem 4.1 with $j = 0$, we obtain the following equations

$$E(Y_n^{(k)})^{\beta+1} = E(Y_{n-1}^{(k)})^{\beta+1} + \frac{\beta+1}{\alpha k}, n = 1, 2, \ldots$$

Putting $n = 1$ in (22), we get the result in terms of moments of minimal order statistics. Now Theorem 4.1 can be used in a characterization of DPHF in terms of moments of minimal order statistics. Putting $n = 1$ in (22), we get

$$E[X_{1:k}^j] = \frac{\alpha k}{(j+\beta+1)} E[X_{1:k}^{j+\beta+1}].$$

Remark 4.1. If $k = 1$ we obtain the following characterizing results for upper record values of DPHF,

$$E[X_{U_n}^{\beta+1}] = E[X_{U_{n-1}}^{\beta+1}] + \frac{\beta+1}{\alpha}, n = 1, 2, \ldots.$$

The following theorem characterizes DPHF through the conditional expectation.
Theorem 4.2. Let X be a non-negative random variable having an absolutely continuous df $F(x)$ with $F(0) = 0$ and $0 \leq F(x) \leq 1$ for all $x > 0$, then

$$E[\xi(Y_n^{(k)}) | (Y_l^{(k)}) = x] = \exp(-\frac{\alpha}{\beta+1} x^{\beta+1}) \left(\frac{k}{k+1}\right)^{n-l}, l = m, m+1 \tag{25}$$

if and only if

$$\bar{F}(x) = \exp(-\frac{\alpha}{\beta+1} x^{\beta+1}), \quad 0 < x < \infty, \alpha > 0, \beta > -1$$

where,

$$\xi(y) = \exp(-\frac{\alpha}{\beta+1} y^{\beta+1}).$$

Proof. From (2) and (1), we have

$$E[\xi(Y_n^{(k)})|(Y_m^{(k)})=x]=\frac{k^{n-m}}{(n-m-1)!}\int_x^\infty \exp(-\frac{\alpha}{\beta+1}y^{\beta+1})\,[\ln\bar{F}(x)-\ln\bar{F}(y)]^{n-m-1}$$

$$\times\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{k-1}\frac{f(y)}{\bar{F}(x)}dy. \tag{26}$$

By setting $u=\frac{\bar{F}(y)}{\bar{F}(x)}=\frac{\exp(-\frac{\alpha}{\beta+1}y^{\beta+1})}{\exp(-\frac{\alpha}{\beta+1}x^{\beta+1})}$ from (4) in (26), we have

$$E[\xi(Y_n^{(k)})|(Y_m^{(k)})=x]=\frac{k^{n-m}}{(n-m-1)!}\exp(-\frac{\alpha}{\beta+1}x^{\beta+1})\int_0^1(-\ln u)^{n-m-1}u^k du. \tag{27}$$

We have Gradshteyn and Ryzhik ([20], p-551)

$$\int_0^1(-\ln x)^{\mu-1}x^{\upsilon-1}dx=\frac{\Gamma\mu}{\upsilon^\mu},\mu>0,\upsilon>0. \tag{28}$$

On using (28) in (27), we have the result given in (25).
To prove the sufficiency part, we have

$$\frac{k^{n-m}}{(n-m-1)!}\int_x^\infty \exp(-\frac{\alpha}{\beta+1}x^{\beta+1})\,[\ln F(x)-\ln F(y)]^{n-m-1}$$

$$\times[\bar{F}(y)]^{k-1}f(y)dy=[\bar{F}(x)]^k g_{n|m}(x), \tag{29}$$

where

$$g_{n|m}(x)=\exp(-\frac{\alpha}{\beta+1}x^{\beta+1})\left(\frac{k}{k+1}\right)^{n-m}.$$

Differentiating (29) both sides with respect to x, we get

$$-\frac{k^{n-m}f(x)}{\bar{F}(x)(n-m-2)!}\int_x^\infty \exp(-\frac{\alpha}{\beta+1}x^{\beta+1})[\ln F(x)-\ln F(y)]^{n-m-2}$$

$$\times[\bar{F}(y)]^{k-1}f(y)\,dy=g'_{n|m}(x)[\bar{F}(x)]^k-k\,g_{n|m}(x)[\bar{F}(x)]^{k-1}f(x)$$

or,

$$-k\,g_{n|m+1}(x)[\bar{F}(x)]^{k-1}f(x)=g'_{n|m}(x)[\bar{F}(x)]^k-k\,g_{n|m}(x)[\bar{F}(x)]^{k-1}f(x).$$

Therefore,

$$\frac{f(x)}{\bar{F}(x)}=-\frac{g'_{n|m}(x)}{k[g_{n|m+1}(x)-g_{n|m}(x)]} \qquad \text{Khan } et\ al., [21]$$

$$=\alpha x^\beta, \tag{30}$$

where

$$g'_{n|m}(x)=-\alpha x^\beta\exp(-\frac{\alpha}{\beta+1}x^{\beta+1})\left(\frac{k}{k+1}\right)^{n-m},$$

$$g_{n|m+1}(x)-g_{n|m}(x)=\frac{1}{k}\exp(-\frac{\alpha}{\beta+1}x^{\beta+1})\left(\frac{k}{k+1}\right)^{n-m}.$$

Integrating both sides of (30) with respect to x between $(0,y)$, the sufficiency part is proved.

5 Conclusion

In this paper, we have established the characterization results through the generalized upper record values when a sample is available from a DPHF. It has been seen that some well- known life time distributions namely Weibull, Rayleigh and exponential are the special cases of a DPHF. Further, the results provided in this paper are useful in the field of ordered random variables.

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