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A New Compound Lifetime Distribution: Ishita Power Series Distribution with Properties and Applications

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Abstract: In this paper, we explore a new family of models for lifetime data called Ishita Power Series family of distributions by compounding a lifetime distribution called Ishita distribution and Power Series distribution. The proposed model has special cases of several lifetime distributions that are very flexible to fit different types of data sets. It is pertinent to mention that the probability density function and hazard rate can take up different forms such as increasing, decreasing and upside down bathtub shapes which are shown through graphs. Some mathematical properties like; moments, moment generating function and order statistics of the new class of distribution have been discussed. The model parameters are estimated through MLE technique. Finally, the potentiality of the proposed model has been tested on two real life data sets and it is clear from the statistical analysis that the proposed model offers a better fit.

Keywords: Ishita Distribution, Power Series Distribution, Compounding and Order Statistics.

1 Introduction

Lifetime distributions are of great importance in several applications for theoretical research and applied fields such as insurance, medical science, engineering, biological sciences, communications and life testing. In recent years, attempts have been made by researchers to generate new lifetime probability distributions that provide an adequate fit and greater flexibility in modelling data. Many continuous probability distributions such as gamma, exponential and Weibull were used in statistical literature to analyse the lifetime data but these distributions cannot be efficiently employed to model the lifetime data that is bathtub in configuration besides having unimodal failure rates. To overcome this problem, researchers have shifted their attention to compounding mechanism which provides suitable, flexible and alternative models to fit the lifetime data of different types. For instance, Adamidis and Loukas [1], Rashid et al. [2,3,4,5,6,7] and Alizadeh et al. [8,9] obtained several lifetime distributions through compounding mechanism that proved very efficacious in modelling lifetime data with different features. A lot of work has been done by researchers in this field. Alkarni et al. [10], Nasir et al. [11], Bagheri et al. [12] and Chahkandi and Ganjali [13] introduced many distributions by compounding some useful lifetime distributions with power series distributions. Some other distributions obtained recently are: complementary exponentiated inverted weibull power series family and compound family of generalized inverse weibull power series distributions by Hassan et al. [14,15]: compound class of linear failure rate power series and exponentiated Weibull –logarithmic distribution by Mahmoudi et al. [16,17]; compound class of extended Weibull power series distributions by Silva et al. [18] and complementary weibull geometric distribution by Tojeiro et al. [19]. Consider a series system with N components, where N, the number of components is itself a discrete random variable with domain (N =1,2,…,). The lifespan of i^{th} component in this set up can be depicted by any one of the appropriate lifetime distributions viz; Ram Awadh, exponential, gamma, Weibull, Lindley, Ishita, Pranav, etc. And N, the discrete r.v may have any of the distribution such as geometric, zero truncated Poisson or power series distribution in general. The lifespan for this kind of system in series combination will be denoted by $Y = \min\left\{X_i\right\}_{i=1}^{N}$. In this paper we will contemplate the

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lifetime of i^{th} component to be distributed as Ishita distribution and the index N itself as powers series distribution. The new lifetime distribution that is acquired by compounding Ishita distribution with that of power series distribution will be named as Ishita power series distribution.

The present paper is organized as follows: In section (2), we present the construction of the proposed lifetime distribution. Density, survival, hazard rate functions and some of the characteristics of the proposed family are discussed in section (3). Moment generating function of proposed distribution is discussed in section (4). Order statistics, their moments and parameter estimation are respectively discussed in section (5) and (6). Special cases that comprise new lifetime distributions have been obtained in section (7). Finally, real application and inference about new models are respectively discussed in section (8) and (9).

2 Proposition of Class

Suppose $X_1, X_2, ...X_N$ be independent and identically distributed (i.i.d) random variables following one parameter Ishita distribution whose density is given by

$$g(x;\lambda)=\frac{\lambda^3}{\lambda^3+2}\left(\lambda+x^2\right)e^{-\lambda x}, x>0, \lambda>0 \quad (1)$$

Here we are considering a problem in which the sample size i.e, N is itself a random variable following zero truncated power series distribution with probability function given by

$$P(N=n)=\frac{a_n\theta^n}{C(\theta)},\ n=1,2,...$$

where a_n depends only on n, $C(\theta)=\sum_{n=1}^{\infty}a_n\theta^n$ and $\theta>0$ is such that $C(\theta)$ is finite. The below table shows some useful quantities of zero truncated power series distribution such as poisson, logarithmic, geometric and binomial distribution.

Table 2.1: Useful quantities of Some Power Series Distribution.

Distribution	a_n	$C(\theta)$	$C'(\theta)$	$C''(\theta)$	$C^{-1}(\theta)$	θ
Poisson	$n!^{-1}$	$e^{\theta}-1$	e^{θ}	e^{θ}	$\log(\theta+1)$	$\theta\in(0,\infty)$
Logarithmic	n^{-1}	$-\log(1-\theta)$	$(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$1-e^{-\theta}$	$\theta\in(0,1)$
Geometric	1	$\theta(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$2(1-\theta)^{-3}$	$\theta(\theta+1)^{-1}$	$\theta\in(0,1)$
Binomial	$\binom{m}{n}$	$(\theta+1)^m-1$	$m(\theta+1)^{m-1}$	$\frac{m(m-1)}{(\theta-1)^{2-m}}$	$(\theta-1)^{\frac{1}{m}}-1$	$\theta\in(0,1)$

Let $X_{(1)}=\min\{X_i\}_{i=1}^N$. The conditional cumulative distribution function of $X_{(1)}\mid N=n$ is given by

$$G_{X_{(1)|N=n}}(x)=1-\left[\overline{G}(x)\right]^n=1-\left[\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)\right]^n$$

and

$$P\left(X_{(1)}\le x, N=n\right)=\frac{a_n\theta^n}{C(\theta)}\left\{1-\left[\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)e^{-\lambda x}\right]^n\right\}, x>0, n\ge 1$$

The Ishita power series family of distributions is defined by the marginal cumulative distribution function of $X_{(1)}$:

$$F(x)=\sum_{n=1}^{\infty}\left\{1-\left[\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)e^{-\lambda x}\right]^n\right\}\frac{a_n\theta^n}{C(\theta)}$$

$$F(x)=1-\frac{C\left\{\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)\theta e^{-\lambda x}\right\}}{C(\theta)},x>0 \qquad (2)$$

3 Density, Survival and Hazard Rate Function

The probability density function of a family of Ishita power series (IPS) distribution can be obtained by differentiating (2) both sides with respect to x.

$$f(x)=\frac{dF(x)}{dx}$$

$$f(x)=\frac{\lambda^3}{\lambda^3+2}\theta\left(\lambda+x^2\right)e^{-\lambda x}\frac{C'\left(\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)\right)}{C(\theta)},x>0 \qquad (3)$$

$$S(x)=1-F(x)$$

$$S(x)=\frac{C\left(\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)\theta e^{-\lambda x}\right)}{C(\theta)},x>0$$

And the hazard function is

$$h(x)=\frac{f(x)}{S(x)}$$

$$h(x)=\frac{\lambda^3}{\lambda^3+2}\theta\left(\lambda+x^2\right)e^{-\lambda x}\frac{C'\left(\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)\right)}{C\left(\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)\right)}$$

We will now study some properties of family of IPS distribution in the form of following propositions.

Proposition 3. 1: The Ishita distribution is a limiting case of the IPS distribution when $\theta\to 0^+$.

Proof: From the cumulative distribution function of IPS distribution, we have

$$\lim_{\theta\to 0^+}F(x)=1-\lim_{\theta\to 0^+}\frac{C\left(\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)\right)}{C(\theta)}$$

Since, $C(\theta)=\sum_{n=1}^{\infty}a_n\theta^n$ (4)

$$\lim_{\theta\to 0^+}F(x)=1-\lim_{\theta\to 0^+}\frac{\sum_{n=1}^{\infty}a_n\left\{\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)\right\}^n}{\sum_{n=1}^{\infty}a_n\theta^n}$$

Using the L' Hospital's rule, we have

$$\lim_{\theta\to 0^+} F(x) = 1 - \frac{a_1\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)e^{-\lambda x} + \sum_{n=2}^{\infty} a_n n\theta^{n-1}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)e^{-n\lambda x}}{a_1 + \sum_{n=2} n a_n \theta^{n-1}}$$

$$= 1 - \left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)e^{-\lambda x}$$

Hence proved

Proposition 3.2: The densities of the IPS family of distribution can be expressed as an infinite linear combination of densities of the 1^{st} order statistics of Ishita distribution.

$$f(x) = \sum_{n=1}^{\infty} P(N=n) g_1(x,n)$$

Where $g_1(x,n) = \min(X_1, X_2, \ldots X_n)$ is the 1^{st} order statistics of Ishita distribution.
Proof: We know that

$$C(\theta) = \sum_{n=1}^{\infty} n a_n \theta^{n-1}$$

The pdf of IPS distribution takes the form after using the above result. Therefore it follows

$$f(x) = \frac{\lambda^3}{\lambda^3+2}\left(\lambda + x^2\right)e^{-\lambda x}\sum_{n=1}^{\infty} n\frac{a_n\theta^n}{C(\theta)}\left[e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)\right]^{n-1}$$

$$f(x) = \sum_{n=1}^{\infty} P(N=n) g_1(x,n) \tag{5}$$

Where $g_1(x,n) = n\frac{\lambda^3}{\lambda^3+2}\left(\lambda + x^2\right)e^{-\lambda x}\left[1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right]^{n-1}$ is the 1^{st} order statistics of Ishita distribution. Therefore the densities of IPS distribution can be expressed as an infinite linear combination of the 1^{st} order statistics of Ishita distribution. Hence it is obvious that properties of IPS distribution can be obtained from the 1^{st} order statistics $g_1(x,n) = \min(X_1, X_2, \ldots, X_n)$ of the Ishita distribution.

4 Moment Generating Function

The moment generating function of IPS distribution can be obtained from (4)

$$M_X(t) = \sum_{n=1}^{\infty} P(N=n) M_{X_{(1)}}(t)$$

Where $M_{X_{(1)}}(t)$ is the moment generating function of 1^{st} order statistics of Ishita distribution

$$M_{X_{(1)}}(t) = \int_0^{\infty} e^{tx} n\frac{\lambda^3}{\lambda^3+2}(\lambda + x^2)e^{-\lambda x}\left[\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)e^{-\lambda x}\right]^{n-1} dx$$

$$= \frac{n\lambda^3}{\lambda^3+2}\sum_{j=0}^{n-1}\binom{n-1}{j}\int_0^{\infty}\left(\lambda + x^2\right)\left[\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right]^{n-1-j} e^{-(n\lambda - t)x} dx$$

$$=\sum_{j=0}^{n-1}\sum_{k=o}^{n-1-j}\binom{n-1}{j}\binom{n-1-j}{k}\frac{2^k n\lambda^{2n-2j-k+1}}{\left(\lambda^3+2\right)^{n-j}}\left[\lambda\int_0^\infty x^{(2n-2j-k-1)-1}e^{-(n\lambda-t)x}dx+\int_0^\infty x^{(2n-2j-k+1)-1}e^{-(n\lambda-t)x}dx\right]$$

$$=\sum_{j=0}^{n-1}\sum_{k=0}^{n-1-j}\binom{n-1}{j}\binom{n-1-j}{k}\frac{2^k n\lambda^{2n-2j-k+1}}{(\lambda^3+2)^{n-j}}\left[\frac{\lambda(n\lambda-t)^2\,\Gamma(2n-2j-k-1)+\Gamma(2n-2j-k+1)}{(n\lambda-t)^{2n-2j-k+1}}\right]$$

And it follows that

$$M_X(t)=\sum_{n=1}^{\infty}\frac{a_n\theta^n}{C(\theta)}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1-j}\binom{n-1}{j}\binom{n-1-j}{k}\frac{2^k n\lambda^{2n-2j-k+1}}{\left(\lambda^3+2\right)^{n-j}}\left[\frac{\left\{\lambda\left(n\lambda-t\right)^2\right\}\Gamma\left(2n-2j-k-1\right)+\Gamma\left(2n-2j-k+1\right)}{\left(n\lambda-t\right)^{2n-2j-k+1}}\right]$$

Hence we get

$$E\left(X^r\right)=\sum_{n=1}^{\infty}\frac{a_n\theta^n}{C(\theta)}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1-j}\binom{n-1}{j}\binom{n-1-j}{k}\frac{2^k n\lambda^{2n-2j-k+1}}{\left(\lambda^3+2\right)^{n-j}}\left[\frac{\lambda\left(n\lambda\right)^2\Gamma\left(r+2n-2j-k-1\right)+\Gamma\left(r+2n-2j-k+1\right)}{\left(n\lambda\right)^{2n-2j-k+r+1}}\right] \tag{6}$$

5 Order Statistics and Their Moments

Let $X_1, X_2, \ldots X_n$ be a random sample from IPS distribution and $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ denote the corresponding order statistics. The pdf of i^{th} order statistics say $X_{i:n}$ is given by

$$f_{i:n}(x)=\frac{n!}{(n-i)!(i-1)!}\left[F(x)\right]^{i-1}\left[1-F(x)\right]^{n-i}f(x) \tag{7}$$

$$f_{i:n}(x)=\frac{n!\,f(x)}{(n-i)!(i-1)!}\left[1-\frac{C\left\{\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)\right\}}{C(\theta)}\right]^{i-1}\times\left[\frac{C\left(\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)\right)}{C(\theta)}\right]^{n-i}$$

From

$$f(x)\left[F(x)\right]^{k+i-1}=\left(\frac{1}{k+i}\right)\frac{d}{dx}\left[F(x)\right]^{k+i}$$

The associated CDF of $f_{i:n}(x)$ denoted by $F_{i:n}(x)$ becomes

$$F_{i:n}(x)=\frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{n-i}\frac{\binom{n-i}{k}(-1)^k}{(k+i)}\left[1-\frac{C\left(\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)\right)}{C(\theta)}\right]^{k+i} \tag{8}$$

alternatively, expression (8) can be written as

$$F_{i:n}(x)=1-\frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{i-1}\frac{\binom{i-1}{k}(-1)^k}{(k+n-i+1)}\left[\frac{C\left(\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)\right)}{C(\theta)}\right]^{k+n-i+1}$$

The expression for the r^{th} moment of i^{th} order statistic $X_{i:n}$ with CDF (2) can be obtained, using a eminent result due to Barakat and Abdelkadir (2004) as follows

$$E\left(X_{i:n}^r\right)=r\sum_{k=n-i+1}^{n}(-1)^{k-n+i}\binom{k-1}{n-i}\binom{n}{k}\int_0^\infty x^{r-1}S(x)^k\,dx$$

where S(x) is the survival function of IPS distribution. Hence

$$E\left(X_{i:n}^r\right)=r\sum_{k=n-i+1}^{n}\frac{(-1)^{k-n+i-1}}{C(\theta)^k}\binom{k-1}{n-i}\binom{n}{k}\int_0^\infty x^{r-1}C\left(\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)\right)^k dx$$

Where r=1, 2, 3…and i=1, 2,…, n

6 Parameter Estimation

Let $X_1, X_2,\ldots X_N$ be a random sample with observed values $x_1, x_2,\ldots x_n$ from IPS(λ,θ) distribution and let $\Theta=(\lambda,\theta)^T$ be the unknown parameter vector in the rest of the paper. The log-likelihood function is given by

$$l_n=l_n(x,\Theta)=3n\log\lambda+n\log\theta+\sum_{i=1}^{n}\log\left(\lambda+x_i^2\right)-\lambda\sum_{i=1}^{n}x_i-n\log\left(\lambda^3+2\right)-n\log C(\theta)$$

$$+\sum_{i=1}^{n}\log C'\left(\theta e^{-\lambda x_i}\left(1+\frac{\lambda x_i(\lambda x_i+2)}{\lambda^3+2}\right)\right)$$

The corresponding score functions are

$$\frac{\partial l_n}{\partial\lambda}=\frac{3n}{\lambda}+\sum_{i=1}^{n}\left(\frac{1}{\lambda+x_i^2}\right)-\sum_{i=1}^{n}x_i-\frac{3n\lambda^2}{\lambda^3+2}+\theta\sum_{i=1}^{n}\frac{C''\left(\theta e^{-\lambda x_i}\left(1+\frac{\lambda x_i(\lambda x_i+2)}{\lambda^3+2}\right)\right)}{C'\left(\theta e^{-\lambda x_i}\left(1+\frac{\lambda x_i(\lambda x_i+2)}{\lambda^3+2}\right)\right)}$$

$$\times\left(\frac{4-\left(\lambda^3+2\right)^2-\lambda^2\left(\lambda^3x_i^2+6\lambda x_i+2x_i^2-6\lambda+6\right)}{\left(\lambda^3+2\right)^2}\right)\left(x_ie^{-\lambda x_i}\right)$$

$$\frac{\partial l_n}{\partial\theta}=\frac{n}{\theta}-\frac{nC'(\theta)}{C(\theta)}+\sum_{i=1}^{n}\frac{C''\left(\theta e^{-\lambda x_i}\left(1+\frac{\lambda x_i(\lambda x_i+2)}{\lambda^3+2}\right)\right)}{C'\left(\theta e^{-\lambda x_i}\left(1+\frac{\lambda x_i(\lambda x_i+2)}{\lambda^3+2}\right)\right)}\left(1+\frac{\lambda x_i(\lambda x_i+2)}{\lambda^3+2}\right)e^{-\lambda x_i}$$

The maximum likelihood estimate of Θ say $\hat{\Theta}$ is obtained by solving the non-linear system of equations $U_n(\Theta)=\left(\frac{\partial l_n}{\partial\lambda},\frac{\partial l_n}{\partial\theta}\right)^T=0$. The solution of this non- linear system of equation can be found numerically using softwares such as R.

7 Some Special Sub Models

7.1 Ishita Poisson Distribution (IPD)

Poisson distribution is a special case of power series distribution for $C(\theta)=e^{\theta}-1$ and $C'(\theta)=e^{\theta}$. Hence cdf and pdf of a compound of Ishita Poisson (IP) distribution is

$$F(x)=1-\frac{e^{\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)}-1}{e^{\theta}-1}, x>0$$

The associated pdf, hazard and survival functions are respectively given by

$$f(x)=\frac{\lambda^3}{\lambda^3+2}\theta(\lambda+x^2)e^{-\lambda x}e^{\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)}(e^{\theta}-1)^{-1}$$

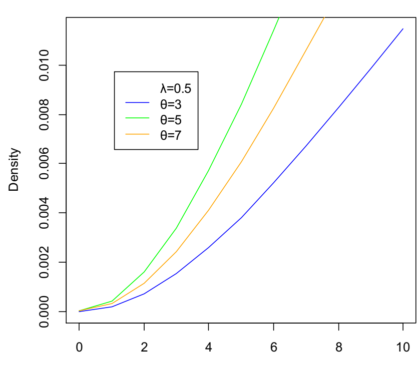
$$h(x)=\frac{\lambda^3\theta(\lambda+x^2)e^{-\lambda x}e^{\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)}}{(\lambda^3+2)\left(e^{\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)}-1\right)}$$

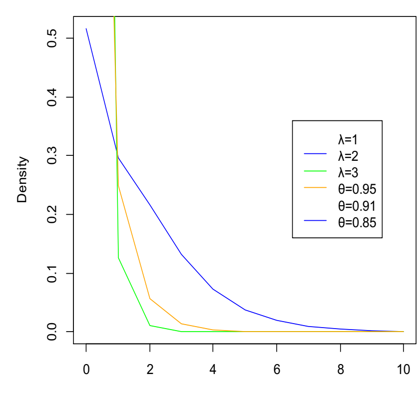
$$S(x)=\frac{e^{\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)}-1}{e^{\theta}-1}$$

for $x,\lambda>0$ and $0<\theta<\infty$, respectively. The expression for the r^{th} moment of a random variable following the Ishita Poisson distribution becomes , by taking $a_n=n!^{-1}$ and $C(\theta)=e^{\theta}-1$ in (6)

$$E(X^r)=(e^{\theta}-1)^{-1}\sum_{n=1}^{\infty}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1-j}\frac{\theta^n}{n!}\binom{n-1}{j}\binom{n-1-j}{k}\frac{n2^k\lambda^{2n-2j-k+1}}{(\lambda^3+2)^{n-j}}$$
$$\times\left[\frac{(n^2\lambda^3)\Gamma(r+2n-2j-k-1)+\Gamma(r+2n-2j-k+1)}{(n\lambda)^{r+2n-2j-k+1}}\right]$$

for $\lambda>0$ and $0<\theta<\infty$





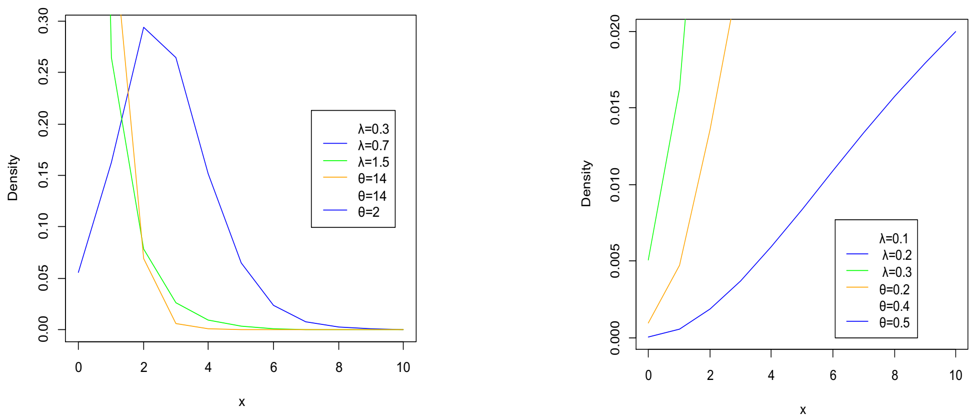


Fig 1: Graphs showing the flexibility of density of IPD for some selected values of parameters λ and θ.

7.2 Ishita Logarithmic Distribution (ILD)

The logarithmic distribution is a special case of the PSD when $C(\theta)=-\log(1-\theta)$ and $C'(\theta)=(1-\theta)^{-1}$.
Hence a compound of Ishita Logarithmic (IL) distribution is obtained from (2) using the above result

$$F(x)=1-\frac{e^{\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)}-1}{e^{\theta}-1}, x>0$$

The associated pdf, hazard rate and survival functions are

$$f(x)=\frac{\lambda^3\theta(\lambda+x^2)e^{-\lambda x}}{(\lambda^3+2)\left[\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)-1\right]\log(1-\theta)}$$

$$h(x)=\frac{\lambda^3\theta(\lambda+x^2)e^{-\lambda x}}{(\lambda^3+2)\left[\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)-1\right]\log\left[1-\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)\right]}$$

$$S(x)=\frac{\log\left[1-\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)\theta e^{-\lambda x}\right]}{\log(1-\theta)}$$

For $x,\lambda>0$ and $0<\theta<0$ respectively.

The expression for the r^{th} moment of a random variable following Ishita Logarithmic distribution becomes by taking $a_n=n^{-1}$ and $C(\theta)=-\log(1-\theta)$ in (6)

$$E(X^r)=\frac{1}{\log(1-\theta)^{-1}}\sum_{n=1}^{\infty}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1-j}\frac{\theta^n}{n}\binom{n-1}{j}\binom{n-1-j}{k}\frac{n2^k\lambda^{2n-2j-k+1}}{(\lambda^3+2)^{n-j}}\left[\frac{(n^2\lambda^3)\Gamma(r+2n-2j-k-1)+\Gamma(r+2n-2j-k+1)}{(n\lambda)^{r+2n-2j-k+1}}\right]$$

For $\lambda>o$ and $0<\theta<1$ respectively.

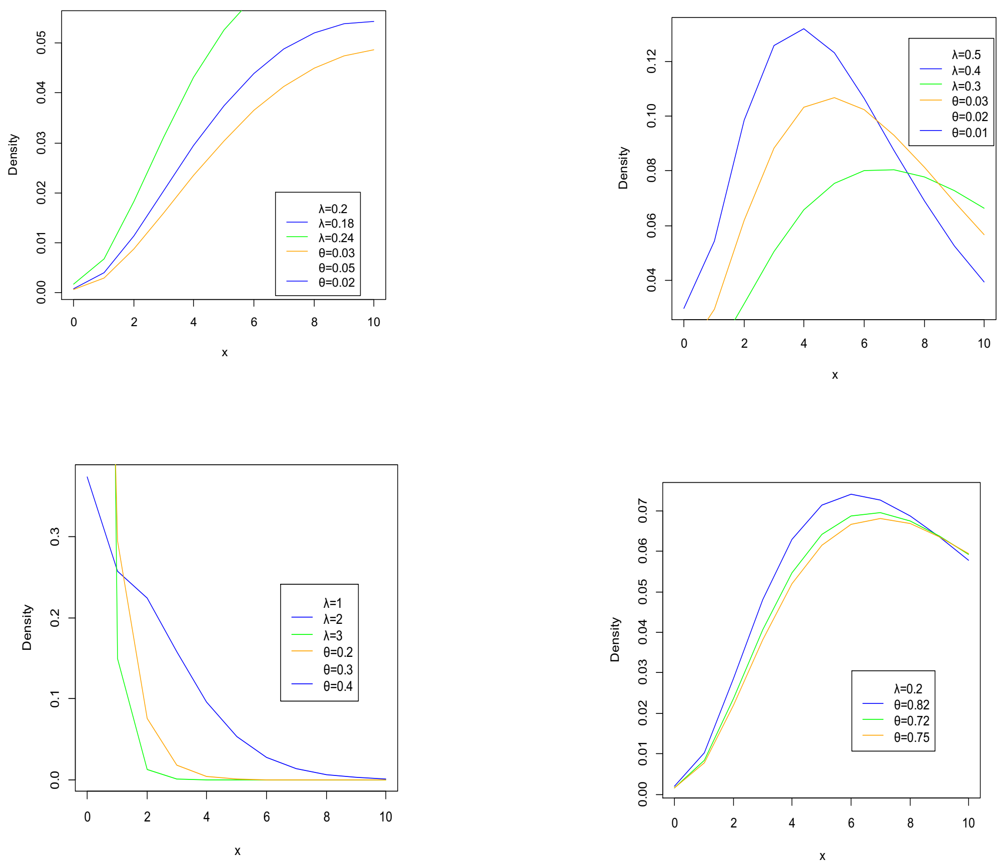


Fig 2: Graphs showing the flexibility of density of ILD for some selected values of parameters λ and θ

7.3 Ishita Geometric Distribution (IGD)

Geometric distribution is a particular case of PSD when $C(\theta)=\theta(1-\theta)^{-1}$ and a compound of Ishita Geometric distribution is followed from () after using $C(\theta)=\theta(1-\theta)^{-1}$ in it.

$$F(x)=\frac{1-\left[1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right]e^{-\lambda x}}{1-\theta e^{-\lambda x}\left[1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right]},x>0$$

The associated pdf, hazard rate and survival function are

$$f(x)=\frac{\lambda^3(1-\theta)(\lambda+x^2)e^{-\lambda x}}{\lambda^3+2}\left[1-\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)\right]^{-2}$$

$$h(x)=\frac{\lambda^3(\lambda+x^2)}{\lambda^3+\lambda x(\lambda x+2)+2}\left[1-\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)\right]^{-1}$$

$$S(x)=\frac{(1-\theta)\left[1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right]e^{-\lambda x}}{1-\theta e^{-\lambda x}\left[1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right]}$$

For $x,\lambda>0$ and $0<\theta<1$ respectively.

The expression for the r^{th} moment of a random variable following Ishita Geometric distribution is obtained by taking $a_n=1$ and $C(\theta)=\theta(1-\theta)^{-1}$ in (6)

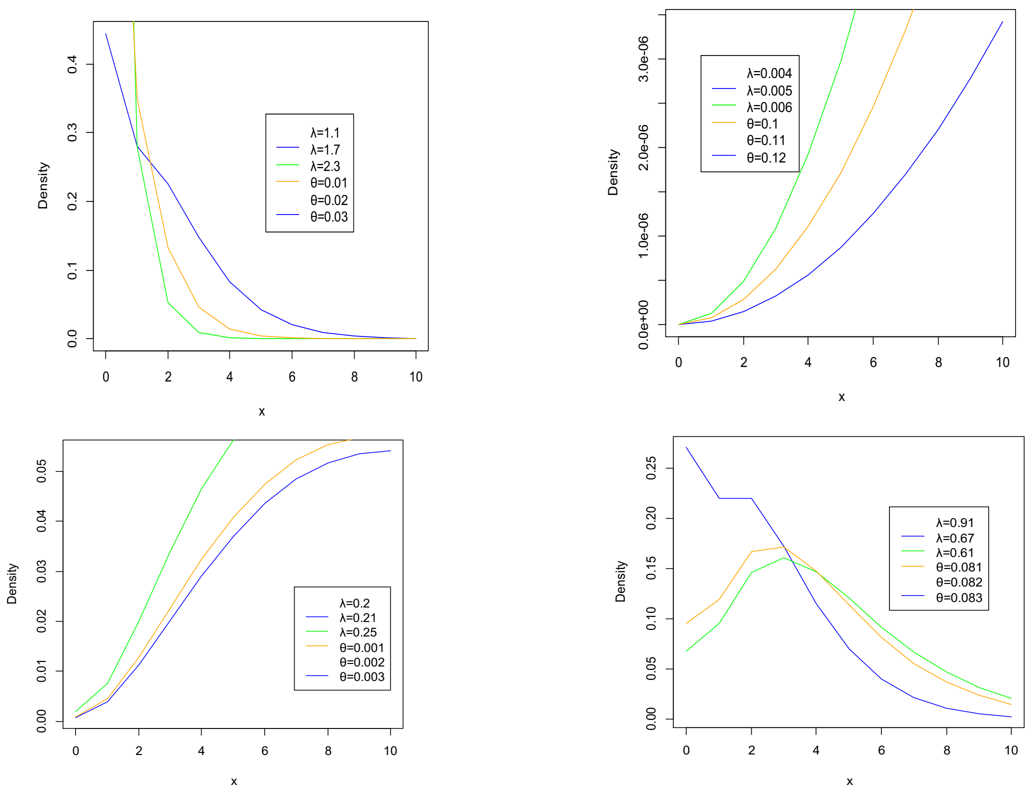


Fig 3: Graphs showing the flexibility of density of IGD for some selected values of parameters λ and θ .

7.4 Ishita Binomial Distribution (IBD)

Binomial distribution is a particular case of PSD for $C(\theta)=(\theta+1)^m+1$ and a compound of Ishita Binomial distribution is followed from (2) using $C(\theta)=(\theta+1)^m+1$ in it.

$$F(x)=1-\frac{\left[\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)+1\right]^m-1}{(\theta+1)^m-1},x>0$$

Where m is a positive integer. The associated pdf, hazard rate and survival function are

$$f(x)=\frac{\lambda^3 m}{\lambda^3+2}\left(\lambda+x^2\right)\theta e^{-\lambda x}\left[\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)\right]^{m-1}\left[(\theta+1)^m-1\right]^{-1}$$

$$h(x)=\frac{\lambda^3 m}{\lambda^3+2}\left(\lambda+x^2\right)\theta e^{-\lambda x}\left[\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)\right]^{m-1}\left[\left(\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)\right)^m\right]^{-1}$$

$$S(x)=\frac{\left[\theta e^{-\lambda x}\left(1+\frac{\lambda x(\lambda x+2)}{\lambda^3+2}\right)\right]^m-1}{(\theta+1)^m-1}$$

For $x,\lambda>0$ and $0<\theta<\infty$, respectively. The expression for the r^{th} moment of a random variable following Ishita Binomial

Distribution becomes, by taking $a_n=\binom{m}{n}$ and $C(\theta)=(\theta+1)^m+1$ in (6)

$$E\left(X^r\right)=\left[(\theta+1)^m-1\right]^{-1}\sum_{n=1}^{\infty}\binom{m}{n}\theta^n\sum_{j=o}^{n-1}\sum_{k=0}^{n-1-j}\binom{n-1}{j}\binom{n-1-j}{k}\frac{n2^k\lambda^{2n-2j-k+1}}{\left(\lambda^3+2\right)^{n-j}}\left[\frac{\left(n^2\lambda^3\right)\Gamma(r+2n-2j-k-1)+\Gamma(r+2n-2j-k+1)}{(n\lambda)^{r+2n-2j-k+1}}\right]$$

for $n\le m,\lambda>0$ and $0<\theta<\infty$

8 Applications

In this section, the potentiality of the proposed model namely Ishita Power Series distribution will be explored and comparison among its sub-models will also be studied on two real life data sets.

Data set 1: Lifetime of fatigue of Kevlar 373/epoxy [11], that are subjected to constant pressure at the 90% stress level until all had failed. The data set is

Table 8.1: Lifetime of fatigue fracture of Kevlar 373/epoxy, subject to constant pressure at the 90% stress level.

0.0251	0.886	0.0891	0.2501	0.3113	0.3451	0.4763	0.565	0.5671	0.6566	0.6748
0.6751	0.6753	0.7696	0.8375	0.8391	0.8425	0.8645	0.8851	0.9113	0.912	0.9836
1.0483	1.0596	1.0773	1.1733	1.257	1.2766	1.2985	1.3211	1.3503	1.3551	1.4595
1.488	1.5728	1.5733	1.7083	1.7263	1.746	1.763	1.7746	1.8275	1.8375	1.8503
1.8808	1.8878	1.8881	1.9316	1.9558	2.0048	2.0408	2.0903	2.1093	2.133	2.21
2.246	2.2878	2.3203	2.347	2.3513	2.4951	2.526	2.9911	3.0256	3.2678	3.4045
3.4846	3.7433	3.7455	3.9143	4.8073	5.4005	5.4435	5.5295	6.5541	9.096	

Our motive is to fit this data by the proposed family of compound distributions. The MLE of unknown parameters and Akaike information criterion (AIC) and Bayesian information criterion (BIC) of the fitted distribution is given in table: 8.2

Table 8.2: Analysis of model fitting.

MODEL	MLE	AIC	BIC
IG	$\hat{\lambda}=0.42, \hat{\theta}=0.95$	247.32	250.38
IP	$\hat{\lambda}=0.60, \hat{\theta}=4.98$	246.65	249.70
IL	$\hat{\lambda}=1.11, \hat{\theta}=0.00000234$	253.6	256.74

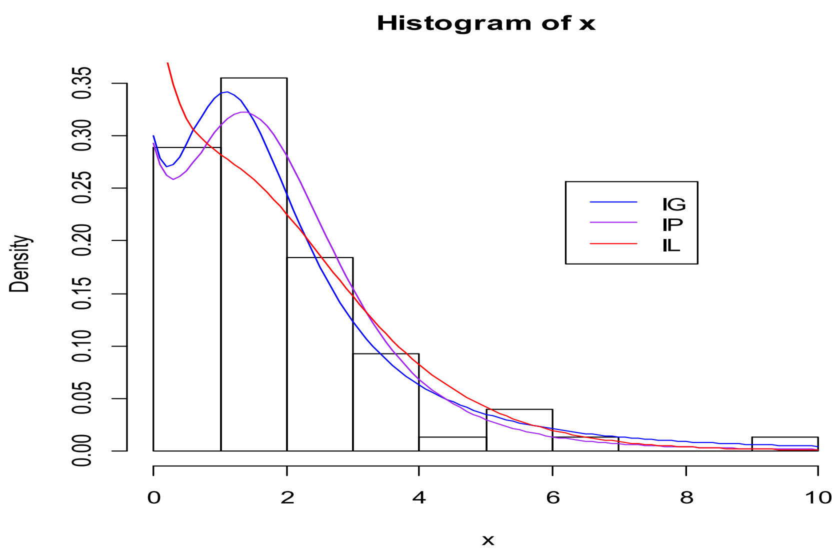


Fig 4: Fitting of IG, IP, IL to Lifetime fatigue.

Data set 2: The data set represents the failure times (in minutes) for a sample of 15 electronic components in an accelerated life test Lawless JF (2003).

Table 8.3: Failure times data in an accelerated life test.

1.4	5.1	6.3	10.8	12.1
18.5	19.7	22.2	23.0	30.6
37.3	46.3	53.9	59.8	66.2

Table 8.4: Analysis of model fitting.

MODEL	MLE	AIC	BIC
IP	$\hat{\lambda}=0.098, \hat{\theta}=0.788$	138.07	141.12
IG	$\hat{\lambda}=0.086, \hat{\theta}=0.576$	137.69	140.75
IL	$\hat{\lambda}=0.055, \hat{\theta}=0.993$	135.57	138.62

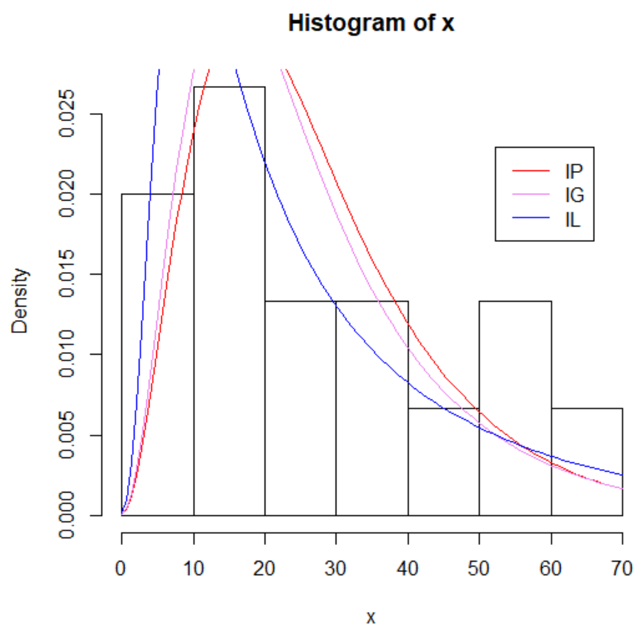


Fig 5: Fitting of IP, IG, IL to failure data.

9 Conclusion

A new class of compound lifetime distribution is constructed by compounding Ishita Distribution with that of Power Series Distribution. Moreover, it has been shown that its sub-models are very flexible in terms of density and hazard rate functions. Furthermore, special properties like moments, order statistic, moment generating function and parameter estimation through MLE have been also discussed. Ultimately, the applicability of the proposed model has been explained by fitting it to two real life data sets. It is apparent from the statistical analysis that the IP family from data set 1 and IL family from data set 2 of compound distributions perform exquisitely well which is also displayed graphically. So practitioners are recommended to exploit one of our models to obtain efficacious consequences when it comes to fit lifetime data.

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